Winter term 2021/22
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory I

Sheet 10
Submission before: **Thursday**, 23.12.2021, **18:00**
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 38. (Proposition 2.6.8, Remark 2.6.9) (2 + 1 + 1 points)
Let X be a random variable with distribution

$$\mu = \frac{1}{Z} \sum_{j=3}^{\infty} \frac{\varepsilon_j}{j^3(\log(j))^2},$$

where Z is a normalising constant and ε_j ist the Dirac distribution in the point $\{j\}$.

(a) Show that $X \in \mathcal{L}^p$ for $p \in]0, 2]$.

(b) Show that $X \notin \mathcal{L}^p$ for $p > 2$.

(c) Use this to construct an example showing that the Lyapunov condition (Lya) is less general than the Lindeberg condition (L).

Problem 39. (CLP and Feller condition) (4 points)
Find an example of a sequence of independent random variables that possesses the CLP but does **not** satisfy the Feller condition and hence the Lindeberg condition.

Problem 40. (Monotone class theorem for trigonometric functions) (1 + 1,5 + 1,5 points)
Consider

$$\widetilde{\mathcal{M}} := \{f_u \mid u \in \mathbb{R}\} \cap \{g_u \mid u \in \mathbb{R}\}$$

with $f_u(x) := \cos(ux)$, $g_u(x) := \sin(ux)$. Let $\mathcal{M}$ be the linear span[1] of $\widetilde{\mathcal{M}}$.

(a) Show that the product of two elements of $\mathcal{M}$ lies again in $\mathcal{M}$.

(b) Show that $\Lambda_{[a,b]} = [a, b]$ holds, where

$$\Lambda_{[a,b]} := \{y \in \mathbb{R} \mid g_q(y) \in g_q([a, b]) \ \forall q \in \mathbb{Q}\}.$$

(c) Conclude using (b) that $\sigma(\mathcal{M}) = \mathcal{B}(\mathbb{R})$.

Problem 41. (Review of Proposition 1.11.11 and Proposition 2.5.3) (4 points)
Prove Proposition 2.5.3 in detail for the case $n = 1$ using the monotone class theorem and Problem 40.

[1] i.e. the set of all linear combinations of elements of $\widetilde{\mathcal{M}}$.