Winter term 2021/22
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Exercises to Probability Theory I

Sheet 1
Abgabe: Friday, 22.10.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*(Exercises marked with "*" are additional exercises.*)

Problem 1. (4 points)

(a) Let $\Omega = \{1, 2, 3\}$. List all elements of the smallest σ-algebra on Ω that contains $\{1\}$.

(b) Give all possible σ-algebras of the set $\Omega = \{1, 2, 3\}$.

Problem 2. (4 points)
Let $\Omega \neq \emptyset$ and $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\Omega)$ be σ-algebras. Are then the following sets σ-algebras as well?

(a) $\mathcal{A} \cup \mathcal{B}$,

(b) $\mathcal{A} \cap \mathcal{B}$.

Problem 3. (4 points)
Let Ω be a countably infinite set.

$$\mathcal{A} = \{A \subset \Omega \mid A \text{ is finite or } A^c \text{ is finite}\}.$$

Check whether $\mathcal{A}$ is a σ-algebra. If not, give a counter example.

Problem 4. (4 points)
Let $\Omega \neq \emptyset$, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be an arbitrary σ-algebra, $\Omega_0 \subset \Omega$ a finite or countably infinite subset of Ω, $f\colon \Omega_0 \to [0, \infty]$ an arbitrary function and

$$\mu\colon \mathcal{A} \to [0, \infty] \quad \text{with} \quad A \mapsto \sum_{\omega \in A \cap \Omega_0} f(\omega).$$

Show that μ is a measure on $(\Omega, \mathcal{A})$. (A measure of this form is called a *discrete measure.*)
Remark: In the first tutorial we will show that a discrete measure μ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ exists with $\mu(A) > 0$ for all open subsets $A \neq \emptyset$ of $\mathbb{R}$.

Problem* 5. (4 points)
If Ω be a finite set, then the measure given by $f(\omega) = \frac{1}{|\Omega|}$, $\omega \in \Omega$ is a probability measure on $(\Omega, \mathcal{P}(\Omega))$, called *uniform distribution* on Ω. Explain how the classic lottery drawing "6 aus 49" [1] can be described by such a probability measure (and probability space). What is the probability that in a drawing the numbers $12, 24, 36, 48$ are drawn?

[1] Where six numbered balls are drawn without replacement from a set of 49 balls numbered from 1 to 49, similar to the UK National Lottery drawings before 2015.