Winter term 2021/22
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Exercises to Probability Theory I

Sheet 3
Submission before: Friday, 05.11.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 9. (cf. Bemerkung 1.3.2 (iii))
Let $(\Omega_i, \mathcal{A}_i)$ for $i = 1, 2, 3$ be measurable spaces and $T_i : \Omega_i \to \Omega_{i+1}$ for $i = 1, 2$ measurable maps. Show that $T_2 \circ T_1$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable. (2 points)

Problem 10. (Permutations)
Let $n \in \mathbb{N}$, $\Omega := \{\omega : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \mid \omega \text{ bijective}\}$ and let $P : \mathcal{P}(\Omega) \to [0, 1]$ be the uniform distribution on $(\Omega, \mathcal{P}(\Omega))$. Let a random variable $X : \Omega \to \{1, 2, \ldots, n\}$ be given by

$$\omega \mapsto X(\omega) := \sum_{i=1}^{n} 1_{\{\omega(i)\}}(i) \quad \forall \omega \in \Omega.$$

Calculate (a) the expectation $\mathbb{E}[X]$ and (b) the variance $\mathrm{var}(X)$. (6 points)

Problem 11. (Repetition of the construction of the integral)
Proof the following proposition (step by step as in the construction of the integral):

Satz 1. Let X be a random variable[1] on $(S, \mathcal{S})$ with $\mu(A) := P[X \in A]$, i.e. X is a measurable map $X : (\Omega, \mathcal{A}, P) \to (S, \mathcal{S})$. If $f : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable function with $f \geqslant 0$ or $\mathbb{E}[|f(X)|] < \infty$, then

$$\mathbb{E}[f(X)] = \int_S f(y)\mu(dy).$$

(4 points)

Problem 12. (Factorisation lemma)
Let Ω be a set and let $(\tilde{\Omega}, \tilde{\mathcal{A}})$ be a measurable space. Further let $T : \Omega \to \tilde{\Omega}$ and $f : \Omega \to \bar{\mathbb{R}}$ be **arbitrary** maps. Show that f is $\sigma(T)/\mathcal{B}(\bar{\mathbb{R}})$-measurable if and only if there is a map $\varphi : \tilde{\Omega} \to \bar{\mathbb{R}}$ which is $\tilde{\mathcal{A}}/\mathcal{B}(\bar{\mathbb{R}})$-measurable with $f = \varphi \circ T$. (4 points)

[1] Here we consider the general case on a measurable space $(S, \mathcal{S})$. The case $S = \mathbb{R}$ can be considered an example.