Exercises to Probability Theory I

Sheet 4
Submission before: Friday, 12.11.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 13. (cf. Proposition 1.6.7) (3 points)
Show for $X_1, \dots, X_n \in \mathcal{L}^2$ pairwise uncorrelated random variables the following identity of Bienaymé:

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i).$$

Problem 14. (Law of large numbers) (5 points)
Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X_n \colon \Omega \to \mathbb{R}$ with $X_n \in \mathcal{L}^2$ for $n \in \mathbb{N}$ be a sequence of random variables with the property that

$$\mathbb{E}[X_n] = \mathbb{E}[X_1] \quad \forall n \in \mathbb{N}. \tag{1}$$

Furthermore, let $r \colon \{0, 1, 2, \dots\} \to [0, \infty)$ be a function such that

$$|\operatorname{cov}(X_n, X_m)| \leqslant r(|n-m|) \quad \forall n, m \in \mathbb{N}, \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{r(k)}{n}\left(1 - \frac{k}{n}\right) = 0. \tag{3}$$

Show that under these conditions

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mathbb{E}[X_1]\right| \geqslant \varepsilon\right) = 0.$$

Problem 15. (cf. Remark 1.8.3)
Let $\Omega = [0, 1[$ and P be the Lebesgue measure restricted to Ω, defined on the Borel σ-algebra $\mathcal{A} = \mathcal{B}([0, 1[)$. Define further

$$A_{2^m+k} := \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right[\quad \text{for } k \in \{0, \dots, 2^m - 1\}, m \in \mathbb{N}_0 \text{ and } Y_n := 1_{A_n} \text{ for } n \in \mathbb{N}.$$

1. Show that Y_n converges in $\mathcal{L}^1$, but **not** P-a.s. (2 points)
2. Give an example of a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables on $(\Omega, \mathcal{A}, P)$ which are P-a.s. convergent, but **not** in $\mathcal{L}^1$. (2 points)

Problem 16. (Proposition 1.8.4 (i) $\Rightarrow$ (ii) + Lemma 1.8.8) (4 points)
Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1$ be a sequence of random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $\lim_n X_n = X$ in $\mathcal{L}^1$ for a $X \in \mathcal{L}^1$. Show that under these conditions the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.