Winter term 2021/22
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Exercises to Probability Theory I

Sheet 5
Submission before: Friday, 19.11.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 17. (cf. Remark 1.8.9 (ii)) (4 points)
Let $(\Omega, \mathcal{A}, P)$ be a probability space and I an index set. Consider $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$, two uniformly integrable families of random variables. Furthermore let $\alpha, \beta \in \mathbb{R}$. Show that then the linear combination $(\alpha X_i + \beta Y_i)_{i\in I}$ is also uniformly integrable.

Problem 18. (cf. Remark 1.9.5) (4 points)
Let $F\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing, bounded function. Show that F has at most countably many points of discontinuity.

Problem 19. (Random Walk) (2+2 points)
Let $\Omega = \{\omega = (x_1, \ldots, x_N) \mid x_i \in \{-1, 1\}\}$, P be the uniform distribution on Ω and $X_i\colon \Omega \to \mathbb{R}$ given by the projection $X_i(\omega) := x_i$ for $\omega = (x_1, \ldots, x_N) \in \Omega$. The sum

$$S_n = X_1 + \ldots + X_n, \quad \text{für } n = 0, \ldots, N$$

can be understood as a random motion of a particle on $\mathbb{Z}$ starting in 0, i.e. as a so-called "random walk". For $a \in \mathbb{Z}$ with $a > 0$ let T_a be the time of the first visit of the particle to the site a, i.e.

$$T_a := \min\{n > 0 \mid S_n = a\},$$

where for $\{n > 0 \mid S_n = a\} = \emptyset$ we have $T_a = \infty$. Show that

(a) For every $c > 0$:

$$P[S_n = a - c, T_a \leqslant n] = P[S_n = a + c].$$

(b) For the distribution of T_a the following holds true:

(i) $P[T_a \leqslant n] = P[S_n \notin [-a, a-1]]$,

(ii) $P[T_a = n] = P[S_n = a] - P[S_n = a, T_a \leqslant n-1] = \frac{1}{2}\left(P[S_{n-1} = a-1] - P[S_{n-1} = a+1]\right)$.

Problem 20. (cf. Proposition 1.9.9) (2+2 points)
Let $h\colon \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function and X, Y be random variables with distributions μ, ν. Let X be absolutely continuous with density f and let Y be discretely distributed with $\nu(S) = 1$ for a countable set $S \subset \mathbb{R}$. Show the following:

(a) $\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) dx$,

(b) $\mathbb{E}[h(Y)] = \sum_{y\in S} h(y)\nu(\{y\})$.