Winter term 2021/22
Prof. Dr. Michael Röckner and Dr. Andre Schenke

# Exercises to Probability Theory I

Sheet 6
Submission before: Friday, 26.11.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

**Problem 21.** (Corollary 1.10.7, Proof of (i) ⇒ (ii)) (4 points)
Consider the probability measures $\mu_n, \mu$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let the sequence $(\mu_n)_{n\in\mathbb{N}}$ converge vaguely to $\mu$. Show that this implies that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to $\mu$.

To this end, show first that for $f \in C_b(\mathbb{R})$, $f \geqslant 0$ the following holds:

$$\liminf_{n\to\infty} \int f d\mu_n \geqslant \int f d\mu.$$

**Problem 22.** (2+2 points)
Let $X, X_n, U_n$ for $n \in \mathbb{N}$ be random variables with values in $\mathbb{R}$. Assume that the distribution of $X_n$ converges weakly to the distribution of $X$ for $n \to \infty$. Assume further that the distribution of $U_n$ converges weakly to the Dirac-measure $\delta_u$ for $n \to \infty$, for a $u \in \mathbb{R}$. Prove the following:

(a) $U_n$ converges stochastically to $u$ for $n \to \infty$.

(b) For $n \to \infty$, the distribution of the sum $U_n + X_n$ converges weakly to the distribution of $u + X$.

**Problem 23.** (2+2 points)
Let $S_n$ for $n = 0, 1, 2, \dots, 2N$ be the "random walk" of Problem 19, where now $2N$ steps are considered instead of $N$. We define the first return time to 0 via

$$T_0(\omega) := \min\{n > 0 \mid S_n(\omega) = 0\},$$

and the time of the last visit to 0 by

$$L(\omega) := \max\{0 \leqslant n \leqslant 2N \mid S_n(\omega) = 0\}.$$

You may use that

$$P[L = 2n] = P[S_{2n} = 0] \cdot P[S_{2N-2n} = 0] = 2^{-2N} \binom{2n}{n} \binom{2(N-n)}{N-n}$$

holds. Please show that:

(a) For all $0 < a < b < 1$, the following holds[1]

$$P\left[\frac{L}{2N} \in ]a,b]\right] \overset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} 1_{]a,b]}(x)\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}dx.$$

(b) Conclude using (a) that the distribution of $\frac{L}{2N}$ for $N \to \infty$ converges weakly to the distribution with the following density:

$$f(x) = \begin{cases} \frac{1}{\pi\sqrt{x(1-x)}}, & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Problem 24.** (cf. Proposition 1.11.11) (2+2 points)

Let $S$ be a metric space with Borel-$\sigma$ algebra $\mathcal{S}$. According to Example 1.11.13, we know that this $\sigma$-algebra equals $\sigma(C_b(S))$. Consider a probability measure $\mu$ on $(S, \mathcal{S})$, and $1 \leqslant p < \infty$.

(a) Show that for every Borel-measurable function $f$ on $S$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of functions from $C_b(S)$ such that

$$\|g_n - f\|_p \overset{n\to\infty}{\longrightarrow} 0,$$

where $\|h\|_p := \left(\int_S |h|^p d\mu\right)^{1/p}$.

(b) Show that the same statement holds[2] even for $f \in \mathcal{L}^p$. What does this mean?

[1]Hint: Use **Stirling's formula** (for a proof of it, cf. e.g. Amann, Escher: Analysis II, English edition, Theorem 9.10, p. 109):

$$m! = C_m \cdot \sqrt{2\pi m} \cdot m^m \cdot e^{-m}, \quad \text{with } \lim_{m\to\infty} C_m = 1.$$

[2]You may use the following property (sometimes called "**inner regularity**") of probability meaasures on the Borel-$\sigma$ algebra $\mathcal{S}$ of a metric space $S$: For every $A \in \mathcal{S}$ it holds that

$$\mu(A) = \sup\{\mu(C) \mid C \subset A,\ C \text{ closed}\}.$$