Winter term 2021/22
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory I

Sheet 9
Submission before: Friday, 17.12.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 33. (Remark 2.5.6) (4 points)
Prove the direction " $\Leftarrow$" of the remark: a collection $X_1, \ldots, X_n$ of random variables is independent if and only if

$$\hat{P}_{(X_1,\ldots,X_n)}(u_1, \ldots, u_n) = \prod_{j=1}^{n} \hat{P}_{X_j}(u_j). \tag{*}$$

Problem 34. (CLT for Example 2.1.5) (4 points)
Determine with detailed arguments an approximate value for the probability that with 3600 rolls of a fair (new) dice at least 630 times a

is rolled. Tables for the distribution function can be found on the internet.

Problem 35. (Proposition 2.6.8) (1 + 1,5 + 1,5 points)
Let $\lambda \in \mathbb{R}$ and $(X_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables on the probability space $(\Omega, \mathcal{A}, P)$ with

$$P[X_k = k^\lambda] = P[X_k = -k^\lambda] = \frac{1}{2}.$$

Define

$$s_n := \left(\sum_{k=1}^{n} \operatorname{var}(X_k) \right)^{1/2}.$$

Show that

(a) If $\lambda < -\frac{1}{2}$, then $(s_n)_{n\in\mathbb{N}}$ is bounded.

(b) If $\lambda \geqslant 0$, then

$$s_n^2 \geqslant \int_0^n x^{2\lambda} dx \quad \underline{\text{and}} \quad (X_k)_{k\in\mathbb{N}} \text{ has the CLP.}$$

(c) If $\lambda \in [-\frac{1}{2}, 0]$, then

$$s_n^2 \geqslant \int_1^{n+1} x^{2\lambda} dx \quad \underline{\text{and}} \quad (X_k)_{k\in\mathbb{N}} \text{ has the CLP.}$$

Problem 36. (Properties of the convolution) (1+2+1 points)
Let $f \in C_0(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R})$. Show that the convolution $f \star g$ has the following properties:

(a) $f \star g$ has a compact support,

(b) $f \star g$ is differentiable with $(f \star g)' = f \star (g')$,

(c) $f \star g \in C_0^\infty$.

Problem* 37. (Relation to the classical Fourier transform) (1+2 points)
The "classical" Fourier transform for functions $f \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$ is defined by

$$\hat{f}(u) := \int_{\mathbb{R}^n} e^{ix \cdot u} f(x) dx.$$

Let μ be a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)$ with $\mu \ll \lambda^n$ and density $f \in L^1(\mathbb{R}^n)$.

(a) How are the Fourier transform of the measure μ and the Fourier transform of f related?

(b) Let $X, Y \colon \Omega \to \mathbb{R}$ be independent random variables with absolutely continuous distributions with densities $f, g \colon \mathbb{R} \to \mathbb{R}_+$. Show that then for the distribution of the sum $X + Y$ the following holds:

$$\hat{P}_{(X+Y)} = \hat{f} \cdot \hat{g}.$$