Summer term 2022
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Exercises to Probability Theory II

Sheet 1
Submission before: Friday, 08.04.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

For the whole sheet, assume that $(\Omega, \mathcal{A}, P)$ is a probability space and $\mathcal{A}_0 \subset \mathcal{A}$ is a (sub) σ-algebra.

Problem 1. (Existence and uniqueness of conditional expectations)
Let $X \geqslant 0$ be a random variable on $(\Omega, \mathcal{A}, P)$. Prove using Theorem 3.1.11 (Radon-Nikodym) that there exists a random variable X_0 that satisfies the properties of Definition 5.1.1. (2 points)

Show further that any two such random variables X_0 and $\tilde{X}_0$ agree P-a.s. (2 points)

Problem 2. (Contraction property, Property 5.2 (c))
Let $u\colon \mathbb{R} \to \mathbb{R}$ be a concave function.

(a) Let M be the set of all functions $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) = ax + b$, where $a, b \in \mathbb{Q}$.
Show that if u is not a straight line, for every $x_0 \in \mathbb{R}$ we have

$$u(x_0) = \inf_{h \in M, h \geqslant u} h(x_0).$$

(2 points)

(b) Let X be an integrable random variable on $(\Omega, \mathcal{A}, P)$. Show using (a) that the following holds P-a.s.:

$$\mathbb{E}[u(X) \mid \mathcal{A}_0] \leqslant u\left(\mathbb{E}[X \mid \mathcal{A}_0]\right).$$

(if u is a straight line, this is trivial!) (2 points)

Problem 3. (Conditional uncorrelatedness, Proposition 5.2.2)
Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be two σ-algebras, and let $0 \leqslant X \in \mathcal{L}^1$ be a nonnegative random variable. Then the following statements are equivalent:

(a) $\mathbb{E}[X \mid \sigma(\mathcal{A}_1, \mathcal{A}_2)] = \mathbb{E}[X \mid \mathcal{A}_1]$.

(b) For all $\sigma(\mathcal{A}_1, \mathcal{A}_2)$-measurable random variables $Y \geqslant 0$, it holds that

$$\mathbb{E}\left[X \cdot Y \mid \mathcal{A}_1\right] = \mathbb{E}\left[X \mid \mathcal{A}_1\right] \cdot \mathbb{E}\left[Y \mid \mathcal{A}_1\right].$$

(c) For all $\mathcal{A}_2$-measurable random variables $X_2 \geqslant 0$, it holds that

$$\mathbb{E}\left[X \cdot X_2 \mid \mathcal{A}_1\right] = \mathbb{E}\left[X \mid \mathcal{A}_1\right] \cdot \mathbb{E}\left[X_2 \mid \mathcal{A}_1\right]$$

(4 points)

Hint for the proof of "(c) $\Rightarrow$ (a)": You need to show that $\mathbb{E}[XZ] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{A}_1]Z]$ for all $\sigma(\mathcal{A}_1, \mathcal{A}_2)$-measurable $Z \geqslant 0$. Consider first $Z = 1_{C_1 \cap C_2}$ with $C_1 \in \mathcal{A}_1$, $C_2 \in \mathcal{A}_2$. Then use arguments, for example from Section 1.11, to show the statement for more general Z.

Continue on the next page!

Problem* 4. (Conditional expectation with respect to a random variable, factorisation of conditional expectation, Remark 5.1.3)
Show the existence part of Remark 5.1.3 (v), i.e. the following statement: Let $(\Omega', \mathcal{A}')$ be a measurable space, and let $Y\colon \Omega \to \Omega'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. Furthermore, let $\mathcal{A}_0 := \sigma(Y)$, and let $X \geqslant 0$ be a random variable on Ω. Then there exists a function $f_X\colon \Omega' \to \bar{\mathbb{R}}_+$ such that the following holds P-a.s.:

$$\mathbb{E}[X \mid Y] := \mathbb{E}[X \mid \sigma(Y)] \stackrel{!}{=} f_X \circ Y.$$

More details can be found in the lecture notes. (2 points)