Exercises to Probability Theory II

Sheet 12
Submission before: Friday, 01.07.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 30. (Characterisation of Brownian motion)
Let $(X_t)_{t\in[0,1]}$ be $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable and $X_0 = 0$. Show that the following are equivalent:

(a) $(X_t)_{t\in[0,1]}$ is a Brownian motion.

(b) $(X_t)_{t\in[0,1]}$ is a centered Gaussian family with covariance

$$\mathbb{E}[X_s X_t] = s \wedge t, \quad s,t \in [0,1].$$

Hint: For (a) $\Rightarrow$ (b), Exercise 28 might be helpful. For (b) $\Rightarrow$ (a) you might want to use Proposition 9.1.3 and Corollary 9.1.4. (8 points)

Problem 31. (A representation of the Borel σ-algebra of $C[0,1]$)
Let $C[0,1]$ be the space of continuous functions $w\colon [0,1] \to \mathbb{R}$. Then $C[0,1]$ is Banach space with the supremum (or maximum) norm $\|\cdot\|_\infty$. Let $\mathcal{B}$ be the associated Borel σ-algebra. Show that

$$\mathcal{B} = \sigma\left(\pi_t \mid 0 \leqslant t \leqslant 1\right),$$

where $\pi_t\colon C[0,1] \to \mathbb{R}$, $t \in [0,1]$ are given by $\pi_t(w) := w(t)$, $w \in C[0,1]$.

Hint: $C[0,1]$ is a separable metric space, so for "$\subset$" it is enough to show that every ball from $C[0,1]$ is contained in $\sigma\left(\pi_t \mid 0 \leqslant t \leqslant 1\right)$, as we saw last semester in Problem 50 on Sheet 12.
For "$\supset$": How is $\sigma\left(\pi_t \mid 0 \leqslant t \leqslant 1\right)$ defined? (4 points)