Summer term 2022
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Exercises to Probability Theory II

Sheet 9
Submission before: Friday, 10.06.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 21. (Missing step in proof of Doob's Upcrossing inequality)
Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be a right-continuous function and for $a < b$ and $T \in (0,\infty]$, let $U^\varphi(a,b;T)$ be the number of "upcrossings", defined in the lecture as

$$\begin{aligned} U^\varphi(a,b;T) &:= \inf\{n \geqslant 0 \mid \varphi \text{ crosses } [a,b] \text{ at most } n \text{ times in } [0,T)\} \\ &= \inf\{n \geqslant 0 \mid \varphi \text{ does not cross } [a,b]\ (n+1) \text{ times in } [0,T)\}. \end{aligned}$$

For $m \in \mathbb{N}$, we define the "discretisation" of φ by

$$\varphi_m(t) := \varphi\left(\frac{\lceil 2^m t \rceil}{2^m} \wedge \left(T - \frac{1}{2^m}\right)\right), \quad t \in [0,\infty),$$

and we write $U_m(a,b;T) := U^{\varphi_m}(a,b;T)$. Prove that $U_m(a,b;T) \uparrow U^\varphi(a,b;T)$ by showing

(a) $U_m(a,b;T) \leqslant U^\varphi(a,b;T)$, (2 points)

(b) $U_m(a,b;T) \leqslant U_{m+1}(a,b;T)$, (2 points)

(c) $U^\varphi(a,b;T) \geqslant K$ for a $K \in \mathbb{N} \quad \Rightarrow \quad \exists m \in \mathbb{N} :\ U_m(a,b;T) \geqslant K$. (2 points)

Problem 22. (Missing step in the proof of Corollary 8.5.3)
Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be a right-continuous function. Let $a < b$ and $T \in (0,\infty]$. Show that if $U(a,b;T) < \infty$ for all $a,b \in \mathbb{Q}$, then the left limit

$$\varphi(t-) := \lim_{s \uparrow t} \varphi(s) \in [-\infty,\infty] \quad \forall t \in (0,T]$$

exists. (3 points)

Hint: Proof by contraposition!

Problem 23. (Counterexample for the converse of Proposition 8.5.3)
Let $(X_i)_{i\in\mathbb{N}}$ be independent random variables on a probability space $(\Omega,\mathcal{A},P)$ with

$$\begin{aligned} P(X_i = 2^i) = P(X_i = -2^i) &= \frac{1}{2i^2}, \\ P(X_i = 0) &= 1 - \frac{1}{i^2}. \end{aligned}$$

We define $Y_n := X_1 + \ldots + X_n$, $n \in \mathbb{N}$. Prove the following:
$(Y_n)_{n\in\mathbb{N}}$ is a $(\sigma(X_1,\ldots,X_n))_{n\in\mathbb{N}}$-martingale such that the limit $Y_\infty := \lim_{n\to\infty} Y_n$ exists P-a.s., but $\sup_{n\in\mathbb{N}} \mathbb{E}[|Y_n|] = \infty$.

Hint: Use Lemma 1.1.13! (3 points)