NOTES ON THE DEGREE FORMULA

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The major aim of this text is to provide a proof of Remark 10.4 in [1]. I am indebted to A. Suslin for helpful and encouraging comments.

1. The invariants ρ^i

Let p be a prime. We work over a ground field k of characteristic different from p. Let $G=\mu_p$.

Let X be a smooth variety over k, equidimensional of dimension d, together with a G-action. Let $Y\subset X$ be the fix-point subscheme. We assume that Y is smooth (this is always the case, but I don't know a reference at the moment). Let $U=X\setminus Y$.

Let $\overline{X}=X/G$ be the quotient, let $\overline{U}\subset\overline{X}$ be the image of U and let $\overline{Y}=\overline{X}\setminus\overline{U}$ with the reduced subscheme structure. Certainly $\overline{U}$ is smooth. Since $\operatorname{char} k\neq p$, the projection $\pi\colon X\to\overline{X}$ induces an isomorphism $Y\to\overline{Y}$.

In the following we will identify $\overline{Y}$ with Y.

The morphism $\pi'=\pi|U\colon U\to\overline{U}$ is a μ_p-torsor. Let

$$\alpha\in A^0(\overline{U},K_1/p)$$

be the corresponding unramified p-th power class (locally π' is presented as $R=\overline{R}[t]/(t^p-f)$ for some regular function f, and α is then the p-th power class of f). Let further L be the line bundle on $\overline{U}$ induced from π' via the inclusion $\mu_p\to\mathbf{G}_{\mathrm{m}}$ and let

$$\beta\in\mathrm{CH}^1(\overline{U})=A^1(\overline{U},K_1)$$

be the first Chern class of L.

We define a series of classes $\rho^i\in\mathrm{CH}_{d-i}(Y)/p$ by

$$\rho^i(X)=\rho^i=\partial_Y^{\overline{U}}(\alpha\beta^{i-1})$$

where

$$\partial_Y^{\overline{U}}\colon A^{i-1}(\overline{U},K_i/p)\to\mathrm{CH}_{d-i}(Y)/p$$

is the boundary map, cf. [2].

Proposition 1. *(Degree formula) Let $\varphi\colon X'\to X$ be a morphism of smooth G-varieties of the same dimension with X/G irreducible. Let Y, Y' be the G-fix-point subschemes and let $\psi=\varphi|Y'\colon Y'\to Y$. Then*

$$\psi_*\big(\rho^i(X')\big)=(\deg\varphi)\rho^i(X)$$

Proof. Let $Z=\varphi^{-1}(Y)$. The restriction of α' resp. β' to $(X'\setminus Z)/G$ is equal to the pull back of α resp. β. Therefore

$$\varphi^*\big(\rho^i(X)\big)=i_*\big(\rho^i(X')\big)$$

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where $i\colon Z \to X'$ is the inclusion. Thus

$$\varphi_*\big(\rho^i(X')\big) = \varphi_*\big(\varphi^*\big(\rho^i(X)\big)\big) = (\deg\varphi)\rho^i(X).$$

$\square$

Remark 1. It is very likely that one can extend the considerations to not necessarily smooth varieties. More generally one may consider also arbitrary branched coverings $X \to \overline{X}$ of degree p which are not necessarily cyclic. Everything should work in arbitrary characteristic (this means essentially to establish the Steenrod operations in all characteristics) . It would be interesting to understand the situation also for arbitrary finite morphisms $X \to \overline{X}$.

2. The invariant ρ

Let us assume that Y is proper. We write $I(Y)$ for the image of the degree map $\deg_Y\colon \operatorname{CH}_0(Y) \to \mathbf{Z}$. We have then a degree map

$$\widetilde{\deg}_Y\colon \operatorname{CH}_0(Y)/p \to \mathbf{Z}/pI(Y)$$

We define

$$\rho = \widetilde{\deg}_Y(\rho^d)$$

Note that the image of ρ^i in $\operatorname{CH}_{d-i}(\overline{X})/p$ is trivial. Therefore, if X is proper, then the image of ρ in $\mathbf{Z}/p$ is trivial.

3. The invariant η

In this section we assume that X is proper.

The degree map $\deg_{\overline{X}}$ defines a degree map

$$\deg_{\overline{U}}\colon \operatorname{CH}_0(\overline{U}) = \operatorname{CH}_0(\overline{X})/\operatorname{CH}_0(Y) \to \mathbf{Z}/I(Y)$$

We define

$$\eta = \deg(\beta^d) \in \mathbf{Z}/I(Y).$$

Lemma 1. *The image of η under the inclusion*

$$\mathbf{Z}/I(Y) \xrightarrow{p} \mathbf{Z}/pI(Y)$$

is equal to $-\rho$.

Proof. Suppose first that X is a proper smooth curve. Let $f \in k(\overline{X})^*$ be a function representing α. Then

$$d(f) = pA + B$$

for a zero cycle A on $\overline{X}$ and a zero cycle B on $\overline{Y}$. Since $\sqrt[p]{f}$ is a section of the line bundle L, on has

$$\beta = [A]$$

On the other hand

$$\partial^{\overline{U}}_{\overline{Y}}(\alpha) = [B]$$

by the very definition of $\partial^{\overline{U}}_{\overline{Y}}$. Since

$$0 = \deg_{\overline{X}}\big(d(f)\big) = p\deg_{\overline{X}}(A) + \deg_{\overline{X}}(B)$$

one has

$$p\deg_{\overline{X}}(\beta) = p\deg_{\overline{X}}(A) = -\deg_{\overline{X}}(B) = -\deg_{\overline{Y}}(B) = -\deg_{\overline{Y}}\big(\partial^{\overline{U}}_{\overline{Y}}(\alpha)\big)$$

This proves the claim for $d = 1$.

For the general case one chooses a proper smooth 1-dimensional variety $\overline{C}$ and a morphism $f\colon \overline{C} \to \overline{X}$ such that $\beta^{d-1} \in \mathrm{CH}_1(\overline{U})$ is represented by the image of the fundamental cycle of $\overline{C}$. This way one easily reduces the problem to the case of a curve. □

4. Relations with η_p

Let Y be a smooth proper equidimensional variety, let $X = Y^p$ and let $G = \mathbf{Z}/p = \mu_p$ act on X by cyclic permutations. We then have the invariant

$$\eta \in \mathbf{Z}/I(Y)$$

as above. This class is denoted by η_p in [1].

The computation of η_p stated in Remark 10.4 in [1] will follow from Lemma 1 and the computations of the classes ρ^i below. The technical advantage of the more general setting of the classes ρ^i is that one does not need to assume that X is proper.

5. Reduction to the case of G-vector bundles

Lemma 2. *Let V be the normal bundle of Y in X. Then*

$$\rho^i(X) = \rho^i(V)$$

Proof. This follows easily by applying our construction to the deformation to the normal cone. To be a bit more specific, let $M \to \mathbf{A}^1$ be the deformation to the normal cone of Y in X with fibre V over $0 \in \mathbf{A}^1$ and fibre X otherwise. Consider the class

$$\rho^i(M) \in \mathrm{CH}_{d+1-i}(Y \times \mathbf{A}^1)/p$$

It specializes to $\rho^i(V)$ at $Y \times \{0\}$ and to $\rho^i(X)$ at $Y \times \{1\}$. □

6. Preliminaries on Chern classes

Let $R = \mathbf{Z}/p[[c_1, c_2, \ldots]]$ be the completed ring of mod p Chern classes.

For each $r \in \mathbf{Z}/p$, $r \neq 0$ define the series

$$b^{[r]} = 1 + b_1^{[r]} + b_2^{[r]} + \cdots \in R$$

as follows. Write formally

$$1 + c_1 + \cdots + c_n = \prod_{i=1}^{n}(1 + x_i)$$

and then define

$$b^{[r]} = \prod_{i=1}^{n}(1 + r^{-1}x_i) \quad \text{mod terms of degree} > n$$

Remark 2. Note that $b^{[1]} = 1 + c_1 + c_2 + \cdots$.

Remark 3. Put

$$b = \prod_{r} b^{[r]}$$

Since

$$b = \prod_{i=1}^{n}(1 + x_i^{p-1})^{-1} \quad \text{mod terms of degree} > n$$

the series b lies in the subring of R of elements of degree divisible by $p-1$. Thus b is of the form

$$b = 1 + b_{p-1} + b_{2(p-1)} + \cdots$$

with b_n of degree n. The classes $b_{k(p-1)}$ are closely related to the Steenrod powers $P^{2k(p-1)}$.

We have to consider also integral representatives of the $b^{[r]}$. For each $r \in \mathbf{Z}/p$, $r \neq 0$ choose an integer r' with $r'r = 1 \mod p$. Then define

$$\tilde{b}^{[r]} \in \mathbf{Z}[[c_1, c_2, \ldots]]$$

as follows. Write formally

$$1 + c_1 + \cdots + c_n = \prod_{i=1}^{n} (1 + x_i)$$

and then define

$$\tilde{b}^{[r]} = \prod_{i=1}^{n} (1 + r' x_i) \quad \text{mod terms of degree} > n$$

7. The fundamental identity for the class β

Let $X = V$ be a G-vector bundle of rank e over a smooth variety Y. We assume that $V^G = 0$ and identify Y with the zero section. Write

$$V = \bigoplus_r V_r$$

for the decomposition of V into G-eigenspaces with G acting on V_r by $(\zeta, v) \mapsto \zeta^r v$ with $r \in \mathbf{Z}/p$, $r \neq 0$. Let $q\colon \overline{U} = (V \setminus 0)/G \to Y$ be the projection.

Note that the class

$$\beta \in \mathrm{CH}^1(\overline{U})$$

is p-torsion, since the line bundle L becomes trivial after pull back to U and since $U \to \overline{U}$ is of degree p.

Put

$$\tilde{b}(V) = \prod_r \tilde{b}^{[r]}(V_r) \in \mathrm{CH}^*(Y)$$

and write

$$\tilde{b}(V) = 1 + \tilde{b}_1(V) + \tilde{b}_2(V) + \cdots$$

with $\tilde{b}_i(V) \in \mathrm{CH}^i(Y)$. One has $\tilde{b}_e(V) = N c_e(V)$ for some integer N with $N \mod p$ not depending on the choices of the r'.

Further let

$$b(V) = q^*\big(\tilde{b}(V)\big) \in \mathrm{CH}^*(\overline{U})$$

and write

$$b(V) = 1 + b_1(V) + b_2(V) + \cdots$$

with $b_i(V) = q^*\big(\tilde{b}_i(V)\big) \in \mathrm{CH}^i(\overline{U})$. Note that the vector bundle V admits a nonvanishing section after pull back to U. Since $U \to \overline{U}$ is of degree p, the Euler class $q^*\big(c_e(V)\big)$ is p-torsion. Therefore the classes

$$\beta^e,\ b_1(V)\beta^{e-1},\ \ldots,\ b_e(V)$$

do not depend on the choices of the r'.

Theorem 1. *One has*

$$\beta^e + b_1(V)\beta^{e-1} + \cdots + b_e(V) = 0 \tag{1}$$

in $\mathrm{CH}^e(\overline{U})$.

Proof. Consider first the case $V = V_1$. Let $\overline{U} \to \mathbf{P}(V) = (V \setminus 0)/\mathbf{G}_{\mathrm{m}}$ be the projection. Then $\beta \in \mathrm{CH}^1(\overline{U})$ is the pull back of the first Chern class ξ of the tautological line bundle on $\mathbf{P}(V)$ (up to a sign, about which I don't care in this exposition). Equation (1) follows then from the standard identity for ξ.

Consider next the case $V = V_r$. This case follows similarly as for $r = 1$. Let $\overline{U} \to \mathbf{P}(V) = (V \setminus 0)/\mathbf{G}_{\mathrm{m}}$ be the projection. Then $\beta \in \mathrm{CH}^1(\overline{U})$ is r-times the pull back of the first Chern class ξ of the tautological line bundle on $\mathbf{P}(V)$. Equation (1) follows then again from the standard identity for ξ.

For the general case one uses a standard Mayer-Vietoris argument. Write

$$\tilde{b}_V(t) = t^e + \tilde{b}_1(V)t^{e-1} + \cdots + \tilde{b}_e(V) \in \mathrm{CH}^*(Y)[t]$$

and put

$$b_V(t) = q^*(\tilde{b}_V)(t) \in \mathrm{CH}^*(\overline{U})[t]$$

We further write $\beta = \beta_V$ to indicate the dependence on the bundle V.

By induction we may suppose that there exists a decomposition

$$V = W_1 \oplus W_2$$

of G-vector bundles such that equation (1) holds for the bundles W_1, W_2. One has

$$\tilde{b}_V(t) = \tilde{b}_{W_1}(t)\tilde{b}_{W_2}(t)$$

Consider the exact sequence

$$\mathrm{CH}^*\big((W_1 \setminus 0)/G\big) \xrightarrow{i_*} \mathrm{CH}^*\big((V \setminus 0)/G\big) \xrightarrow{j_*} \mathrm{CH}^*\big((V \setminus W_1)/G\big) \to 0$$

of Chow groups induced from the closed immersion $i\colon (W_1 \setminus 0)/G \to (V \setminus 0)/G$ and the open immersion $j\colon (V \setminus W_1)/G \to (V \setminus 0)/G$ of its complement. The projection $s\colon (V \setminus W_1)/G \to (W_2 \setminus 0)/G$ is a vector bundle. One has

$$j^*(\beta_V) = s^*(\beta_{W_2})$$

Hence

$$j^*\big(q^*\big(\tilde{b}_{W_2}\big)(\beta_V)\big) = s^*\big(b_{W_2}(\beta_{W_2})\big) = 0$$

Therefore there exist $T \in \mathrm{CH}^0\big((W_1 \setminus 0)/G\big)$ with

$$i_*(T) = q^*\big(\tilde{b}_{W_2}\big)(\beta_V)$$

Thus

$$b_V(\beta_V) = q^*\big(\tilde{b}_{W_1}\big)(\beta_V)q^*\big(\tilde{b}_{W_2}\big)(\beta_V) = q^*\big(\tilde{b}_{W_1}\big)(\beta_V)i_*(T) = i_*\big(i^*\big(q^*\big(\tilde{b}_{W_1}\big)(\beta_V)\big)T\big)$$

But $i^*(\beta_V) = \beta_{W_1}$ and therefore $i^*\big(q^*\big(\tilde{b}_{W_1}\big)(\beta_V)\big) = b_{W_1}(\beta_{W_1}) = 0$.

□

8. Computation of ρ^i

We are now ready to proceed similary as in [1, Section 5]. Let $f = \dim Y$, so that $d = \dim X = f + e$.

Lemma 3. *One has*

$$\partial(\alpha\beta^h) = \begin{cases} 0 & \text{for } h < e-1 \\ u[Y] & \text{for } h = e-1 \end{cases}$$

for some $u \in \mathbf{Z}/p$, $u \neq 0$.

Remark 4. The number u can be easily determined from the ranks of the bundles V_r. We omit this.

Proof. One has $\partial(\alpha\beta^h) \in \mathrm{CH}_{f+e-(h+1)}(Y)$. Therefore $\partial(\alpha\beta^h) = 0$ for $h < e-1$ by dimension reasons. For $h = e-1$ we have $\partial(\alpha\beta^h) \in \mathrm{CH}^0(Y)$. This case follows from the case when $\dim Y = 0$. We may therefore assume that $Y = \operatorname{Spec} k$.

There exists a decomposition

$$V = W_1 \oplus W_2$$

of G-vector modules over k such that $\operatorname{rank} W_2 = 1$. Let r be the weight of W_2 as a G-module and choose r' with $rr' = 1 \mod p$. The line bundle L over $\overline{U} = (V \setminus 0)/G$ is given by

$$\big((W_1 \times W_2) \setminus 0\big) \times_G \mathbf{A}^1 \to \big((W_1 \times W_2) \setminus 0\big)/G$$

It has the section

$$[w_1, w_2] \mapsto [w_1, w_2, s(w_2)^{r'}]$$

where s is a trivialization of the 1-dimensional vector space W_2. Hence

$$\beta = c_1(L) = r' i_*([(W_1 \setminus 0)/G])$$

By induction we are reduced to the case $\operatorname{rank} V = 1$. In this case $V = \mathbf{A}^1$ and $\overline{U} = (V \setminus 0)/G \simeq \mathbf{A}^1 \setminus 0$ with α represented by t^r where t is the standard coordinate of $\mathbf{A}^1$ and where r is an integer representing the weight of V. Obviously

$$\partial_0^{\overline{U}}(\alpha) = r \mod p$$

and the claim follows. □

By equation (1) we have

$$\beta^{e-1+i} = \sum_{h=0}^{e-1} \beta^{e-1-h} q^*\big(g_{i,h}\big)$$

with $g_{i,h} \in \mathrm{CH}^{i+h}(Y)$. The $g_{i,h}$ are universal polynomials in the Chern classes of the bundles V_r. One finds in particular

$$g_{0,0} + g_{1,0} + g_{2,0} + \cdots = \tilde{b}(V)^{-1}$$

One has by Lemma 3

$$\rho^{e+i} = \partial(\alpha\beta^{e+i-1}) = \partial(\alpha\beta^{e-1})g_{i,0} = u g_{i,0} \mod p$$

Thus we have computed

$$\rho = u \deg(g_{\dim Y,0}) \in \mathbf{Z}/pI(Y)$$

in terms of characteristic numbers of the bundles V_r. It is easy to check that this yields the description of η_p in Remark 10.4 in [1], perhaps up to a sign.

References

[1] A. S. Merkurjev, *Degree formula*, notes, May 2000, http://www.math.ohio-state.edu/~rost/chain-lemma.html.
[2] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

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