COHOMOLOGICAL INVARIANTS (mod 2) FOR TWISTED C_4

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Introduction

These are first notes. For now see "Essential dimension of twisted C_4" [2] for the set up and notations.

So let k be a field with $\operatorname{char} k \neq 2$ and let G be a Galois module over k with

$$G(\bar{k}) = \mathbf{Z}/4\mathbf{Z}$$

Let further

$$H(k) = H^*_{\mathrm{et}}(k, \mathbf{Z}/2\mathbf{Z})$$

denote the Galois cohomology ring mod 2.

The purpose of this text is to prove

Proposition 1. *Any normalized invariant for G in* mod 2 *Galois cohomology is a linear combination of η_1, η_2 with coefficients from $H(k)$.*

A proof is presented in the next section. Afterwards we present without proof an extension to more general coefficients together with a precise computation of the group of invariants with coefficients in $H(k)$.

1. Proof of Proposition 1

We start with the exact sequence (see [2])

$$K^* \xrightarrow{\pi} k^* \times \frac{K^*}{k^*} \xrightarrow{\delta_k} H^1(k, G) \to 0$$

Here

$$\pi(\lambda) = \big(N_{K/k}(\lambda), [\lambda^2]\big)$$

It follows that a generic parameter space for G-torsors is given by

$$k^* \times \frac{K^*}{k^*}$$

or rather

$$X = \mathbf{G}_{\mathrm{m}} \times (\mathbf{P}^1 \setminus \operatorname{Spec} K)$$

Hence a generic G-torsor lives over $k(x, y)$ where we use the (rational) coordinates

$$(x, [1 + y\sqrt{d}]) \in X$$

In order to determine all cohomological invariants (mod 2), as a first step one has to determine the unramified cohomology of X. The unramified cohomology of $\mathbf{G}_{\mathrm{m}}$ (with function field $k(x)$) is

$$H(k) \oplus (x)H(k) \subset H(k(x))$$

Date: January 15, 2017.

The unramified cohomology of the torus $T=\mathbf{P}^1\setminus\operatorname{Spec}K$ (with function field $k(y)$) is

$$H(k)\oplus(1-y^2d)H(k)\subset H(k(y))$$

or more precisely:

$$H(k)\oplus(1-y^2d)\big(H(k)/(d)H(k)\big)\subset H(k(y))$$

(Proof: Use the standard Milnor/Arason exact sequence for $H(k(t))$ and the fact that the kernel of the norm $H(K)\to H(k)$ is the image of $H(k)$ in $H(K)$.)

It follows that the unramified cohomology of X is

$$H(k)\oplus(x)H(k)\oplus(1-y^2d)H(k)\oplus(x,1-y^2d)H(k)\subset H(k(x,y))$$

(For any variety T the unramified cohomology of $\mathbf{G}_{\mathrm{m}}\times T$ is $U_T\oplus(x)U_T$ where U_T is the unramified cohomology of T.)

The final step is to determine all those classes in this group which are invariant under the action of $K^\times$ (the Weil-restriction of $\mathbf{G}_{\mathrm{m}}$ with respect to K/k) on X described by the group morphism π above. Writing $\lambda=s+t\sqrt{d}$, one gets that the square class (x) is changed by

$$(x)\mapsto(x)+(s^2-t^2d)$$

and that $1-y^2d$ is changed by the SQUARE of the norm of λ, so that the square class $(1-y^2d)$ is NOT CHANGED at all.

This shows already that the subgroup

$$H(k)\oplus(1-y^2d)H(k)$$

is invariant. It yields the constant invariant and the class η_1.

It remains to consider the invariant elements in

$$(x)H(k)\oplus(x,1-y^2d)H(k)$$

Take an element

$$\phi=(x)\alpha+(x,1-y^2d)\beta$$

and assume it is invariant. Invariance means that

$$0=(s^2-t^2d)\alpha+(s^2-t^2d,1-y^2d)\beta\in H(k(x,y,s,t))$$

Specializing at the place $y=0$ yields

$$0=(s^2-t^2d)\alpha\in H(k(x,s,t))$$

Looking here at the place $s=1$, $1-t^2d=0$ with residue class field $K(x)$, it follows that $\alpha_K=0$, hence

$$\alpha=(d)\alpha'$$

so that

$$\phi=(x,d)\alpha'+(x,1-y^2d)\beta$$

This leads to the second invariant

$$\eta_2=(x,d)$$

It remains to show that there are no more invariants, which means now that

$$(x,1-y^2d)\beta=0$$

What we know is

$$0=(s^2-t^2d,1-y^2d)\beta$$

Looking at the pace $s = 1$, $1 - t^2 d = 0$, $y = t$ with residue class field $K(x)$ one sees that $\beta_K = 0$, hence

$$\beta = (d)\beta'$$

Thus, indeed,

$$(x, 1 - y^2 d)\beta = (x, 1 - y^2 d)(d)\beta' = 0$$

since $(1 - y^2 d, d) = 0$.

2. More general coefficients

Let M be a cycle module over k (see [1]). A standard example for a cycle module in the context of cohomological invariants is the extension of the Brauer group

$$M_{\mathrm{Brauer}}(k) = \bigoplus_{n \geq 0} H^n(k, \mathbf{Q}/\mathbf{Z}(n-1))$$

with the 4-torsion and 2-torsion subgroups

$$M_4(k) = \bigoplus_{n \geq 0} H^n(k, \mu_4^{\otimes(n-1)})$$
$$M_2(k) = H(k)$$

For our G all cohomological invariants are killed by 4, so mod 4-cohomology ($M = M_4$) is a natural choice.

Proposition 2. *For the group of normalized invariants for G with coefficients in M one has the computation*

$$\mathrm{Inv}_0(G, M) \simeq \{ (\gamma, \delta) \in M(k) \oplus M(K) \mid \mathrm{res}_{K/k}(\gamma) = 2\delta,\ \mathrm{cor}_{K/k}(\delta) = 0 \}$$

A proof and an explicit description of this isomorphism is not given here.

However it is instructive to see how Proposition 1 fits in. So let us look at the case

$$M(k) = H(k)$$

Note that $2H(k) = 0$ and recall the exact sequence

$$H(K) \xrightarrow{\mathrm{cor}_{K/k}} H(k) \xrightarrow{(d)} H(k) \xrightarrow{\mathrm{res}_{K/k}} H(K) \xrightarrow{\mathrm{cor}_{K/k}} H(k)$$

Proposition 2 yields

$$\begin{aligned}\mathrm{Inv}_0(G, H) &= \{ (\gamma, \delta) \in H(k) \oplus H(K) \mid \mathrm{res}_{K/k}(\gamma) = 0,\ \mathrm{cor}_{K/k}(\delta) = 0 \} \\ &= H(k)/\mathrm{cor}_{K/k}\big(H(K)\big) \oplus H(k)/(d)H(k)\end{aligned}$$

The final result in the case $M = H$ is:

Proposition 3. *One has*

$$\mathrm{Inv}_0(G, H) = H(k)/\mathrm{cor}_{K/k}\big(H(K)\big) \oplus H(k)/(d)H(k)$$

Here a pair (α, β) with

$$\begin{aligned}\alpha &\in H(k) \quad \mathrm{mod}\ \mathrm{cor}_{K/k}\big(H(K)\big) \\ \beta &\in H(k) \quad \mathrm{mod}\ (d)H(k)\end{aligned}$$

corresponds to the invariant

$$\eta_1 \beta + \eta_2 \alpha$$

The presentation of η_1, η_2 in [2] shows that this correspondence is indeed well defined.

If $G = \mu_4$, then $K = k \times k$ and d is a square. In this case the norm $\mathrm{cor}_{K/k}$ is surjective, $(d) = 0$ and

$$H(k) \xrightarrow{\eta_1} \mathrm{Inv}_0(\mu_4, H)$$

is an isomorphism.

References

[1] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

[2] ———, *Essential dimension of twisted* C_4, Preprint, 2002, ⟨www.math.uni-bielefeld.de/~rost/ed.html#C4⟩.

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