On the dimension and other numerical invariants of algebras and vector products

Larissa Cadorin, Max-Albert Knus, and Markus Rost

Abstract. Tensor categorical and diagrammatic techniques can be used to compute the dimension and other numerical invariants for algebras defined by tensor identities. These techniques are described and applied to symmetric compositions and 3-vector products.

Introduction

Many algebraic structures defined on vector spaces occur only in specific dimensions, independent of the base field. One classical example is given by Hurwitz's theorem for quadratic composition algebras, which implies that such algebras exist only in dimension 1, 2 (quadratic extensions), 4 (quaternions) and 8 (octonions). Recall that quadratic composition algebras are (not necessarily associative) algebras V with unit 1 over a field F with a nonsingular quadratic form $q : V \to F$, which is multiplicative, i.e., $q(xy) = q(x)q(y)$ for all x, $y \in V$. The usual proofs of Hurwitz's theorem are based on an explicit construction of the composition law and its classification. (See for example Jacobson [**Jac58**], or the book [**KMRT98**]). Moreover over algebraically closed fields the dimension is a numerical invariant which classifies quadratic composition algebras.

Vector products are also examples of algebraic structures which occur only in specific dimensions. 2-vector products are generalizations of the classical vector product in the 3-dimensional euclidian space. It is known that they can only occur in dimension 3 and 7 (see [**Eck91**] or [**BG67**]). Let $b\colon V \times V \to F$ be a nondegenerate, symmetric bilinear

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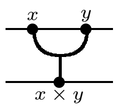
form on a vector space V. A 2-vector product is a skew-symmetric bilinear map $V\times V \to V$, $(x,\, y) \mapsto x\times y$, such that $b(x\times y, x) = b(x\times y, y) = 0$ and

$$b(x \times y, x \times y) = \det \begin{vmatrix} b(x,x) & b(x,y) \\ b(y,x) & b(y,y) \end{vmatrix}$$

for all $x,\, y \in V$.

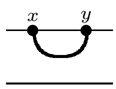
Recently a new, tensor theoretical approach was given in [**Ros96**] and [**Boo98**] for the calculation of the dimension of spaces admitting 2-vector products (see also [**Ros04**]). A variation on the argument of [**Ros96**] is given in [**Mey02**]. The linearization of the relations satisfied by 2-vector products gives tensor identities and a universal tensor category can be constructed for these relations. A canonical ring of numerical invariants is attached to this category. One proves that the dimension d of the 2-vector product algebra generates this ring of numerical invariants and satisfies the identity $d(d-1)(d-3)(d-7) = 0$.

Inspired by knot theory and quantum field theory (see for example [**BN95**], [**BA02**] or [**Tur94**]), a graphical incarnation of the universal tensor category associated with 2-vector products is used in [**Boo98**]. This graphical representation allows to do linear algebra and tensor calculus in a handy way. The bilinear product $V \times V \to V$, $(x, y) \mapsto x \times y$ is represented as a graph connecting two horizontal lines:



$$\begin{array}{c} V \otimes V \\ \downarrow \\ V \end{array} \quad .$$

A single point on a horizontal line corresponds to one copy of V and a finite set of n points to the tensor product $V^{\otimes n}$ of the same number of copies of V. Since $V^{\otimes 0}$ is identified with the base field F, the symmetric bilinear form $b : V \times V \to F$ is represented by the graph



$$\begin{array}{c} V \otimes V \\ \downarrow \\ F \end{array} \quad .$$

Any multilinear map occurring in the tensor identities satisfied by the 2-vector product can accordingly be represented by a graph connecting points on the two horizontal lines and composition of maps corresponds to gluing of diagrams. Closed graphs (i.e., without points on the two horizontal lines) represent numerical invariants. The tensor identities defining the 2-vector product translate into relations between formal sums of graphs and the above identity for the dimensions of vector products can be obtained by pure graph computations.

In this report we describe these diagrammatic techniques for a class of algebras and apply them for two specific algebraic structures: symmetric compositions and 3-vector products.

Symmetric composition algebras are algebras (without 1) with a multiplicative, nonsingular, quadratic form q such that the associated symmetric bilinear form $b(x,y) = \frac{1}{2}\big(q(x+y) - q(x) - q(y)\big)$ is associative:

$$b(xy, z) = b(x, yz) \quad \text{for all } x,\, y,\, z \in V.$$

They arise in a natural way from degree 3 alternating algebras and their structure is known (see [**EM93**] or[**KMRT98**]). In Section 5 we prove that the ring of numerical invariants of the corresponding universal tensor category is generated by two numerical invariants: the dimension d and the norm $e = q(c)$ of a Casimir element $c \in V$ (see Section 1 for the definition of the Casimir element). These two invariants satisfy the identities

$$(e - (d-2)^2)e = 0 \quad \text{and} \quad (d-2)(d-8)(d-e) = 0$$

with possible values

$$(d, e) = (0,0),\ (1,1),\ (2,0),\ (4,4),\ (8,0) \quad \text{and} \quad (8,36).$$

For each pair (d,e) there exists a symmetric composition algebra with invariants (d,e); further, over an algebraically closed field the pair (d,e) determines the symmetric composition algebra up to isomorphism. In particular the appearance of the second invariant e permits to distinguish the two known types of symmetric compositions in dimension 8. The computation of the ring of numerical invariants of the universal tensor category of symmetric compositions given here is due to the third named author.

A 3-vector product is an alternating trilinear product

$$V \times V \times V \to V,\ (x,y,z) \mapsto xyz,$$

such that

$$b(xyz, xyz) = \det \begin{vmatrix} b(x,x) & b(x,y) & b(x,z) \\ b(y,x) & b(y,y) & b(y,z) \\ b(z,x) & b(z,y) & b(z,z) \end{vmatrix}$$

for all $x,\, y,\, z \in V$. The structure of 3-vector products is known (or more generally r-vector products, see [**Eck91**] or [**BG67**]). In particular they only occur in dimension 4 and 8. The tensor categorical interpretation of 3-vector products given in Section 6 is new and will be part of the PhD thesis [**Cad05**] of the first author. More applications of tensor and diagrammatic techniques to algebraic structures, including a diagrammatic proof of Hurwitz's theorem, will be given in the thesis.

1. Algebraic structures and tensors

Let F be a field of characteristic different from 2 and let V be a finite dimensional vector space over F. Let $q : V \to F$ be a quadratic form on V with polar form $b(x,y) = \frac{1}{2}\big(q(x+y)-q(x)-q(y)\big)$. We assume that b is nonsingular, i.e., b induces an isomorphism $h : V \xrightarrow{\sim} V^* = \operatorname{Hom}_F(V,F)$, $x \mapsto b(x,-)$.

Let $V^{\otimes m} = V \otimes \cdots \otimes V$ (m copies), $V^{\otimes 0} = F$. We identify $\big(V^{\otimes m}\big)^*$ with $(V^*)^{\otimes m}$ through the canonical map $(V^*)^{\otimes m} \to \big(V^{\otimes m}\big)^*$given by

$$(f_1 \otimes \cdots \otimes f_m)(v_1 \otimes \cdots \otimes v_m) = f_1(v_1)\cdots f_m(v_m)$$

for $f_i \in V^*$ and $v_i \in V$. The dual $\varphi^* : (V^{\otimes n})^* \to (V^{\otimes m})^*$ of any linear map $\varphi : V^{\otimes m} \to V^{\otimes n}$ then can be viewed as a linear map $\varphi^* : (V^*)^{\otimes n} \to (V^*)^{\otimes m}$. We denote by $\varphi^t : V^{\otimes n} \to V^{\otimes m}$ the linear map $(h^{\otimes m})^{-1} \circ \varphi^* \circ h^{\otimes n}$ and call it the *transpose* of φ. The bilinear form b can be viewed as a *contraction* $b : V \otimes V \to F$. The fact that b is symmetric can be expressed as $b = b \circ \tau$, where $\tau : V \otimes V \to V \otimes V$ is the *switch* $\tau(x \otimes y) = y \otimes x$, x, $y \in V$. We extend b to $b_k \colon V^{\otimes k} \otimes V^{\otimes k} \to F$, $k \geq 1$:

$$b_k\big((u_1 \otimes \cdots \otimes u_k) \otimes (v_1 \otimes \cdots \otimes v_k)\big) = \prod_{i=1}^{k} b(u_i, v_i).$$

For $m \geq k$, we have isomorphisms

$$\operatorname{Hom}_F(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \operatorname{Hom}_F(V^{\otimes(n+k)}, V^{\otimes(m-k)}),\ \theta \mapsto \theta^{[-k]} \tag{1.1}$$

where $\theta^{[-k]}$ is the composite

$$\underbrace{V \otimes \cdots \otimes V}_{n+k \text{ factors}} \xrightarrow{\theta \otimes 1_{V^{\otimes k}}} \underbrace{V \otimes \cdots \otimes V}_{m+k \text{ factors}} \xrightarrow{1_{V^{\otimes(m-k)}} \otimes b_k} \underbrace{V \otimes \cdots \otimes V}_{m-k \text{ factors}}.$$

We assume that b admits an orthonormal basis $(e_1, \ldots, e_d)$, $b(e_i, e_j) = \delta_{ij}$, i, $j = 1, \ldots, d$, extending the base field if necessary. We fix $f_i = b(e_i, -)$, $i = 1, \ldots, d$, as a dual basis of V^*, so that $h : V \to V^*$ is given by $e_i \mapsto f_i$. The map $b^t \colon F \to V \otimes V$ is given by

$$b^t(1_F) = \sum_i e_i \otimes e_i. \tag{1.2}$$

It follows that $(b \circ b^t)(1_F) = \sum_i b(e_i, e_i) = d$, the dimension of V.

A bilinear *multiplication* on V is a linear map $m : V \otimes V \to V$, $x \otimes y \mapsto xy$. We call the pair (V, m) an *algebra* (over F). Let $(e_1, \ldots, e_d)$ be an orthonormal basis of V and let $m(e_i \otimes e_j) = \sum_k m_{ijk} e_k$. It follows that

$m^t(e_k) = \sum_{i,j} m_{ijk} e_i \otimes e_j$. If the algebra (V, m) admits an *associative (or invariant)* nonsingular bilinear form, i.e.,

$$b(xy, z) = b(x, yz) \quad \text{for all } x,\ y,\ z \in V, \tag{1.3}$$

then

$$m^t(v) = \sum_i e_i v \otimes e_i = \sum_i e_i \otimes v e_i. \tag{1.4}$$

For any $x \in V$, let $\ell_x \colon V \to V$ be the linear map $y \mapsto xy$ and let $\mathrm{Tr}(\ell_x)$ be the trace of ℓ_x.

LEMMA 1.5. *Let (V, b, m) be an algebra with a nonsingular associative bilinear form and let $(e_1, \ldots, e_d)$ be an orthonormal basis. The element $c = \sum_i e_i e_i \in V$ is uniquely determined by the property $\mathrm{Tr}(\ell_x) = b(c, x)$. In particular it is independent of the choice of the orthonormal basis.*

PROOF. Uniqueness follows from the fact that b is nonsingular. Let $xe_i = \sum_j x_{ij} e_j$, so that $\mathrm{Tr}(\ell_x) = \sum_i x_{ii}$. We have

$$\sum_i b(x, e_i e_i) = \sum_i b(xe_i, e_i) = \sum_{i,j} x_{ij} b(e_j, e_i) = \sum_i x_{ii}.$$

$\square$

The element c is usually called a *Casimir element*. The dimension and the value $e = b(c, c) \in F$ are numerical invariants attached to the algebra (V, b, m) (where b is associative with respect to m).

For simplicity we call elements of $\mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m})$, for arbitrary n and m, *tensors*. For any triple (V, b, m), b a nonsingular symmetric bilinear form and m a multiplication, we have an *alphabet* of basic tensors consisting of 1_V, b, m, the switch τ and the transposed b^t, m^t. Let $\mathfrak{C} = \{c_i \in \mathrm{Hom}_F(V^{\otimes n_i}, V^{\otimes m_i})\}$ be a set of tensors, usually generated (through compositions and tensor products) by elements from the basic alphabet. We say that an algebra (V, b, m) is *of tensor type* $\mathfrak{C}$ if the c_i are identities for (V, b, m), i.e., if

$$c_i(x_1 \otimes \ldots \otimes x_{n_i}) = 0$$

for all i and all $x_k \in V$.

EXAMPLE 1.6. The vector product in $\mathbb{R}^3$, $m\colon\ x \otimes y \mapsto x \times y$, satisfies the identities

$$\begin{aligned} x \times y + y \times x &= 0 \\ x \times (y \times z) - \langle x, z \rangle y + \langle x, y \rangle z &= 0 \end{aligned} \tag{1.7}$$

for $x,\, y,\, z \in V$, where $\langle x, y\rangle = b(x, y)$ is the scalar product. The tensors are $m + m \circ \tau$ and $m \circ (1_V \otimes m) - (b \otimes 1_V) \circ (1_V \otimes \tau) + b \otimes 1_V$ in $\mathrm{Hom}_F(V^{\otimes k}, V)$, $k = 2,\, 3$.

2. Graphs and tangles

A *graph* G is a finite 1-dimensional cell space. It consists of a finite set V of *vertices* (0-cells) and a finite set E of *edges* (1-cells) and is denoted by $G = (V, E)$. Circles are admitted. Every edge e connects two vertices $\{v_1, v_2\}$, which can possibly coincide. The two endpoints determine the edge and we write $e = v_1v_2$. Graphs are drawn as 2-dimensional projections with the convention that braidings are symmetric:

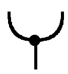
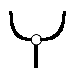
τ =  =  .

A *path* P is a graph with vertices $\{v_0, \dots, v_k\}$ and edges

$$\{v_0v_1,\, v_1v_2, \dots, v_{k-1}v_k\}$$

where the v_i are pairwise different, with the possible exception $v_0 = v_k$. We say that P is a path from v_0 to v_k, denote it $P = v_0v_1 \dots v_k$ and call k the *length* of the path. A *refinement* of a graph G is a graph obtained from G by dividing some of the edges, i.e., by replacing these edges with paths having the same endpoints and so that the intermediate points are not vertices of G. Given a graph $G = (V, E)$, let $G' = (V', E')$ be the refinement obtained by adding a vertex in the "middle" of each edge. The *star* $\mathrm{St}(v)$ of a vertex v of G is the set of edges $e' \in E'$ which are incident with v and the *valency* $\mathrm{val}(v)$ of v is the cardinality of $\mathrm{St}(v)$.

The set of vertices of a graph G having valency 1 is the *boundary* ∂G of G. We consider graphs whose vertices have valency ≤ 3. Let v be a trivalent vertex. Any bijective map Φ_v of $\{1, 2, 3\}$ to the star $\mathrm{St}(v)$ of v is an *ordering* of the vertex v. We use the ordering to equip v with an *orientation*. The orientation is *positive* if the image of Φ_v is (clockwise) an even permutation and is *negative* otherwise. Orientations are indicated by black or white colorings:

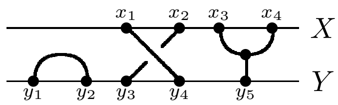
(2.1)  (positive) ,  (negative) .

If no orientation is indicated we understand some orientation.

Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_\ell\}$ be finite sets (the empty set is not excluded). A *m-tangle* [1] $G : X \to Y$ is a graph such that ∂G is

[1]Classical tangles do not have trivalent edges. The "m" in the definition of a m-tangle recalls that trivalent vertices represent multiplications.

the disjoint union $X \sqcup Y$. The sets X and Y are represented by points on two superposed horizontal lines and G connects the two sets. The set X on the upper line is the *source* of G and Y on the lower line is the *target* of G. For example



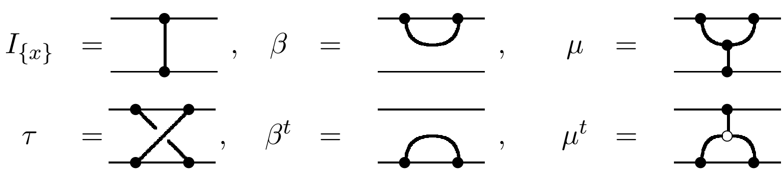
is a m-tangle from $X = \{x_1, \ldots, x_4\}$ to $Y = \{y_1, \ldots, y_5\}$. For any X there is an obvious identity m-tangle $I_X\colon X \to X$.

Exchanging source and target of G through a reflection along the target line we get the *transposed m-tangle* $G^t\colon Y \to X$ of G. The disjoint union $G \sqcup G'\colon X \sqcup X' \to Y \sqcup Y'$ of two m-tangles $G\colon X \to Y$, $G'\colon X' \to Y'$ is a m-tangle. There is an obvious composition of m-tangles

$$(G\colon Y \to Z) \times (H\colon X \to Y) \mapsto G \circ H\colon X \to Z$$

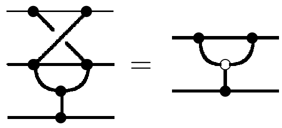
defined by gluing the source of H with the target of G. Two m-tangles with identical refinements are called *isomorphic* and the isomorphism class of a m-tangle G is denoted by $[G]$. Open paths are isomorphic to edges and closed ones to circles. Composition passes over to isomorphism classes and we may define the *m-tangle category* $\mathcal{T}_1$: the objects of $\mathcal{T}_1$ are finite sets X and a morphism $\gamma\colon X \to Y \in \mathrm{Mor}_{\mathcal{T}_1}(X, Y)$ is the isomorphism class $[G]$ of a m-tangle $G\colon X \to Y$. The composition of two morphisms is induced by the corresponding composition of m-tangles and $[I_X]$ is the identity element of $\mathrm{Mor}_{\mathcal{T}_1}(X, X)$. The disjoint union $\sqcup$ induces a *tensor product* $\square\colon \mathcal{T}_1 \times \mathcal{T}_1 \to \mathcal{T}_1$. The tensor product is associative and has the unit object $[\emptyset]$. For pairs of objects (X, Y), the transpose induces a map $[G] \mapsto [G^t]$, $\mathrm{Mor}_{\mathcal{T}_1}(X, Y) \to \mathrm{Mor}_{\mathcal{T}_1}(Y, X)$ such that $([G] \circ [H])^t = [H]^t \circ [G]^t$. Note that transposing changes the orientation of 3-valent vertices.

The following morphisms



are called *basic*. They constitute the *alphabet* of $\mathcal{T}_1$.

Remark 2.2. Composing μ with τ we can change the orientation of the trivalent vertex of μ. We have



up to isomorphisms.

Proposition 2.3. *The morphisms of the category $\mathcal{T}_1$ are generated up to isomorphisms through composition and disjoint union by basic morphisms.*

Remark that refinements are needed in (2.3), for example to separate adjacent trivalent vertices. Moreover, the decomposition of a morphism in basic ones is not unique. In fact the alphabet is redundant. The morphism μ^t can be deduced from the other basic morphisms.

The next step is to enrich the category $\mathcal{T}_1$ to a category $\mathcal{T}_2$ in such a way that, for every pair of objects X and Y, the set of morphisms $\mathrm{Mor}(X, Y)$ has a linear structure. Let $R = \mathbb{Z}[\frac{1}{2}]$. The objects of $\mathcal{T}_2$ are finite sets, as for $\mathcal{T}_1$. For two finite sets X and Y, $\mathrm{Mor}_{\mathcal{T}_2}(X, Y)$ is the free R-module generated by the set $\mathrm{Mor}_{\mathcal{T}_1}(X, Y)$ and composition is extended by linearity: for $\gamma = \sum_i a_i[G_i]$, $a_i \in R$, $[G_i] \in \mathrm{Mor}_{\mathcal{T}_1}(X, Y)$ and $\rho = \sum_j b_j[H_j]$, $b_i \in R$, $[H_j] \in \mathrm{Mor}_{\mathcal{T}_1}(Y, Z)$, we have $\rho\circ\gamma = \sum_{i,j} a_i b_j [H_j \circ G_i] \in \mathrm{Mor}_{\mathcal{T}_2}(X, Z)$. The identity of X is by definition the element $1\cdot[I_X]$. The tensor product $\square$ also extends by linearity:

$$H \,\square\, G = \sum_{i,j} a_i b_j [H_j \,\square\, G_i]$$

and the extension is again associative with unit element $1 \cdot [\emptyset]$. Composition and tensor product coincide on $\mathrm{Mor}_{\mathcal{T}_2}(\emptyset, \emptyset)$ and endow $\mathrm{Mor}_{\mathcal{T}_2}(\emptyset, \emptyset)$ with the structure of a free commutative R-algebra with 1.

In addition to generators we also can impose relations: Let

$$\Gamma = \{\gamma_i \in \mathrm{Mor}_{\mathcal{T}_2}(X_i, Y_i),\ i \in I\}$$

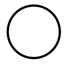
be a set of morphisms of $\mathcal{T}_2$. For each pair of objects (X, Y) of $\mathcal{T}_2$, the set Γ generates through compositions and tensor products a submodule $M(X, Y)$ of $\mathrm{Mor}_{\mathcal{T}_2}(X, Y)$. We define a new category $\mathcal{T}_\Gamma$ by setting

$$\mathrm{Obj}(\mathcal{T}_\Gamma) = \mathrm{Obj}(\mathcal{T}_2) \quad \text{and} \quad \mathrm{Mor}_{\mathcal{T}_\Gamma}(X, Y) = \mathrm{Mor}_{\mathcal{T}_2}(X, Y)/M(X, Y).$$

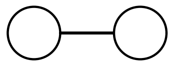
For any $\rho \in \mathrm{Mor}_{\mathcal{T}_2}(X, Y)$, let $[\rho]$ be its class in $\mathrm{Mor}_{\mathcal{T}_\Gamma}(X, Y)$. We get a projection functor $\mathcal{P}\colon \mathcal{T}_2 \to \mathcal{T}_\Gamma$ such that $\mathcal{P}(X) = X$ for $X \in \mathrm{Obj}(\mathcal{T}_2)$ and $\mathcal{P}(\rho) = [\rho]$ for ρ any morphism. In particular we have $\mathcal{P}(\gamma_i) = 0$ for each $\gamma_i \in \Gamma$. We say that the γ_i are *relations defining* $\mathcal{T}_\Gamma$ and we call $\mathcal{T}_\Gamma$ the *m-tangle category associated with the set of relations* Γ. The algebra

$\mathrm{Mor}_{\mathcal{T}_2}(\emptyset,\emptyset)$ projects on $E_\Gamma = \mathrm{Mor}_{\mathcal{T}_\Gamma}(\emptyset,\emptyset)$. We call E_Γ the *algebra of numerical invariants* of the category $\mathcal{T}_\Gamma$. To compute E_Γ is the main feature of the theory. We call a m-tangle category associated with some set of tensor relations a *m-tangle category of type* $\mathcal{T}_3$.

EXAMPLE 2.4. The simplest nontrivial element of E_Γ is the circle

$$d = \beta \circ \beta^t =$$  .

Another interesting element of E_Γ is the element

$$e = \beta \circ (\mu \Box \mu) \circ (\beta^t \Box \beta^t) =$$  .

(As we shall see the orientations do not play a rôle.)

A simple example is given by $\Gamma = \{\mu\}$, so that $\mu = 0$ in $\mathcal{T}_\Gamma$ is the only relation. Then d generates E_Γ and E_Γ is isomorphic to the polynomial ring $R[d]$.

For any of the categories $\mathcal{T}$ and any object $X = \{x_1, \dots x_n\}$, we have a contraction $\beta_X\colon\ X \,\Box\, X \to \emptyset$:

$$\beta_X\big(\{x_1,\dots,x_n\},\ \{y_1,\dots,y_n\}\big) = \beta\big(\{x_1,y_1\}\big)\Box\dots\Box\beta\big(\{x_n,y_n\}\big) \in \emptyset,$$

where $\{y_1,\dots,y_n\}$ is another copy of X. As for vector spaces, the contraction induces functorial isomorphisms

$$\mathrm{Mor}_{\mathcal{T}}(X, Y_1 \Box Y_2) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{T}}(X \Box Y_2, Y_1) \tag{2.5}$$

given by $\theta \mapsto \theta' = (I_{Y_1} \,\Box\, \beta_{Y_2}) \circ (\theta \,\Box\, I_{Y_2})$. For example the image of the isomorphism class of $\mu^t \in \mathrm{Mor}_{\mathcal{T}}\big(\{x\}, \{y_1,y_2\}\big)$ in $\mathrm{Mor}_{\mathcal{T}}\big(\{x,y_2\},\{y_1\}\big)$ is the isomorphism class of $\mu \circ \tau$.

3. Representations of m-tangle categories

Let F be a field of characteristic different from 2 and let $\mathfrak{V} = (V, b, m)$ be an F-algebra. The bilinear form is supposed to be associative with respect to m. Let $\mathcal{V}$ be the category of finite dimensional vector spaces over F. Enlarging F if necessary (which will not change the tensor type of $\mathfrak{V}$), we may assume that V has an orthonormal basis $(e_1,\dots,e_d)$. Let $I_{\{x\}}$, β, β^t, μ, μ^t, and τ be the basic generators (the alphabet) of the m-tangle categories $\mathcal{T}_1$ and $\mathcal{T}_2$ introduced in Section 2.

THEOREM 3.1. (Boos, [**Boo98**]) *There exists a unique functor*

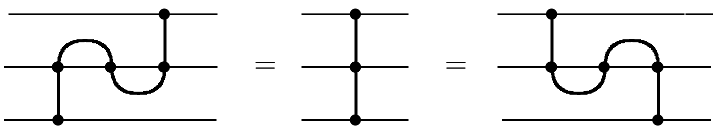
$$\mathcal{R}_{\mathfrak{V}}\colon\ \mathcal{T}_2 \to \mathcal{V}$$

such that

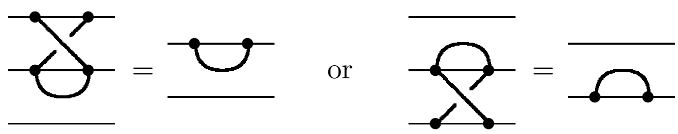
(1) *$\mathcal{R}_{\mathfrak{V}}(\emptyset) = F$, $\mathcal{R}_{\mathfrak{V}}(X) = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ for $X = (x_1, x_2, \ldots, x_n)$ and V_i denotes a copy of V to which x_i is mapped,*
(2) *$\mathcal{R}_{\mathfrak{V}}(I_{\{x\}}) = 1_V$, $\mathcal{R}_{\mathfrak{V}}(\tau) = \tau$,*
(3) *$\mathcal{R}_{\mathfrak{V}}(\beta) = b$, $\mathcal{R}_{\mathfrak{V}}(\mu) = m$, $\mathcal{R}_{\mathfrak{V}}(\gamma^t) = \mathcal{R}_{\mathfrak{V}}(\gamma)^t$ and $\mathcal{R}_{\mathfrak{V}}(\gamma \,\Box\, \rho) = \mathcal{R}_{\mathfrak{V}}(\gamma) \otimes \mathcal{R}_{\mathfrak{V}}(\rho)$ for morphisms γ and ρ in $\mathcal{T}_2$.*

PROOF. It suffices to check that $\mathcal{R}_{\mathfrak{V}}$ is uniquely defined on the category $\mathcal{T}_1$. The extension to $\mathcal{T}_2$ follows by linearity. The functor is clearly uniquely defined on basic morphisms (the alphabet). By Proposition 2.3 an extension to $\mathcal{T}_1$ will be unique. However, since there is no unique decomposition of morphisms in basic morphisms, the existence of an extension is not obvious. In particular there is only one type of open paths up to isomorphisms, but open paths may have a priori different realizations. We follow the clever argument given in [**Boo98**]. We first consider the case where $\mathcal{T}_1$ has no 3-valented vertices. Connected components are paths and it suffices to check that $\mathcal{R}_{\mathfrak{V}}$ is well defined on paths. Paths are composed of elementary morphisms of type $I_{\{x\}}$, β, β^t and τ. We have

(3.2)



up to isomorphism, since open paths are isomorphic. The image of the left diagram under the functor $\mathcal{R}_{\mathfrak{V}}$ is the linear map $v \mapsto \sum_i e_i b(e_i, v)$ and the image of the right diagram is $v \mapsto \sum_i b(v, e_i) e_i$. Both expressions are equal to v since $(e_1, \ldots, e_d)$ is an orthonormal basis[2]. By induction on the length of a path, this implies that $\mathcal{R}_{\mathfrak{V}}$ is well defined on isomorphism classes of paths which do not contain braidings. In presence of braidings we have



in $\mathcal{T}_1$. This corresponds in $\mathcal{V}$ to the fact that the bilinear form b is symmetric. Again by induction $\mathcal{R}_{\mathfrak{V}}$ is well defined. For the general situation it is convenient to reduce to the case where the target of any morphism ρ is empty. By (2.5) we have a canonical isomorphism

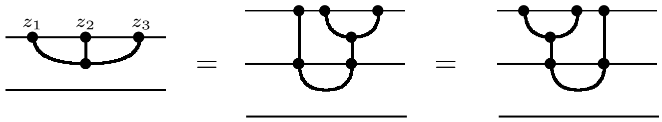
$$\mathrm{Mor}_{\mathcal{T}_2}(X, Y) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{T}_2}(X \Box Y, \emptyset) \tag{3.3}$$

[2] In a more abstract setting, the two parts of (3.2) corresponds to the two so-called "rigidity axioms".

which maps to

$$\mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \mathrm{Hom}_F(V^{\otimes n} \otimes V^{\otimes m}, F)$$

in $\mathcal{V}$ through the functor $\mathcal{R}_{\mathfrak{V}}$ (see (2.5)). Thus we are reduced to the case $Y = \emptyset$ in ρ. Assume that the morphism $\rho\colon X \to \emptyset$ contains k trivalent vertices $P_1, \dots P_k$. Refining ρ if necessary, we may assume that all P_i are centers of morphisms μ_i of type μ. Replacing the negative oriented ones by $(\mu_i \circ \tau) \circ \tau$ we get graphs $\mu_i \circ \tau$ which are positively oriented and we can assume that all orientations are positive. Let $\overline{\rho}$ be the graph obtained from ρ by deleting all μ_i. Let (z_i^1, z_i^2, z_i^3) be the boundary of μ_i (in a positive orientation) and let Z be the finite ordered set $Z = (z_1^1, z_1^2, z_1^3, \dots, z_k^1, z_k^2, z_k^3)$. The graph $\overline{\rho}$ defines uniquely a morphism $\overline{\rho}\colon X \to Z$ which does not contain any μ_i (up to isomorphism). Thus $\mathcal{R}_{\mathfrak{V}}$ can be uniquely extended to $\overline{\rho}\colon X \to Z$. Let

(3.4) $\overline{\mu} =$ 

.

Composing $\overline{\rho}$ with k copies of the morphism $\overline{\mu}$ we get back ρ. Thus, to conclude, we only have to show that the images under $\mathcal{R}_{\mathfrak{V}}$ of the two left morphisms in (3.4) define the same multilinear maps in $\mathcal{V}$. This follows from the associativity of b, $b(x, yz) = b(xy, z)$. $\square$

Corollary 3.5. *Let $\mathfrak{V} = (V, b, m)$ be an algebra of tensor type $\mathfrak{C} = \{c_i\}$ for tensors c_i expressible through the alphabet 1_V, m, m^t, b, b^t and τ. Let $\Gamma = \{\gamma_i\}$ be a chosen set of graphs in $\mathcal{T}_2$ such that $\mathcal{R}_{\mathfrak{V}}(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$ and let $\mathcal{T}_\Gamma$ be the corresponding category of type Γ. There is a unique functor $\mathcal{R}_\Gamma\colon \mathcal{T}_\Gamma \to \mathcal{V}$ such that $\mathcal{R}_{\mathfrak{V}} = \mathcal{R}_\Gamma \circ \mathcal{P}$.*

Corollary 3.6. *A necessary condition for the existence of an algebra of tensor type, for a set of tensors $\mathfrak{C} = \{c_i\}$, is the existence of an R-algebra homomorphism $\Phi\colon E_\Gamma \to F$, where $\Gamma = \{\gamma_i\}$ is a set of relations in $\mathcal{T}_2$ such that $\mathcal{R}_{\mathfrak{V}}(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$.*

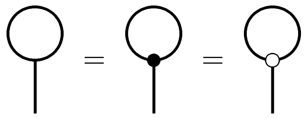
Proof. If such an algebra exists, the functor $\mathcal{R}_\Gamma$ induces a homomorphism $E_\Gamma = \mathrm{Mor}_{\mathcal{T}_\Gamma}(\emptyset, \emptyset) \to \mathrm{Hom}_F(F, F) = F$. $\square$

Since

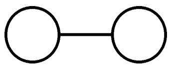
$$\mathcal{R}_{\mathfrak{V}}(\beta \circ \beta^t)(1_F) = (b \circ b^t)(1_F) = \sum_i b(e_i, e_i) = \dim_F V \cdot 1_F$$

the image of the circle d under the functor $\mathcal{R}_{\mathfrak{V}}$ is the dimension of V. (It is convenient to use the same symbol d for the circle and the dimension.)

Rotation around the vertical symmetry axis gives the following identities

(3.7) $\mu \circ \beta^t =$  .

The image under $\mathcal{R}_\Gamma$ of $\mu \circ \beta^t$ is the Casimir element $c = \sum_i e_i e_i$. Thus the element

$e =$ 

of E_Γ is mapped under $\mathcal{R}_\Gamma$ to $b(c,c) \cdot 1_F$, which we denoted also e in Section 1.

EXAMPLE 3.8. Let (V, b, m) be an algebra with $m = 0$, as in Example 2.4. The algebra E_Γ is the polynomial ring $R[d]$, where d is the circle, and $\mathcal{R}_\Gamma(d) = \dim_F \cdot 1_F$. Thus there are no obstruction to the existence of such an algebra in any dimension d (which is of course obvious!).

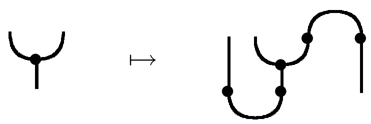
EXAMPLE 3.9. Let $(\mathbb{R}^3, \times)$ be the classical vector product with the identities (1.7). With the techniques described in the next section, it can be readily shown that for the corresponding set Γ of relations,

$$E_\Gamma \xrightarrow{\sim} R[d]/d(d-1)(d-3).$$

Thus a non trivial vector product satisfying the identities (1.7) can only exist in dimension 3.

4. Associative bilinear forms

It is clear from the last step of the proof of Theorem 3.1 that the associativity of the bilinear form plays an essential rôle. The associativity has a nice diagrammatic interpretation. A 120° degree clockwise rotation around the center of the diagram μ describing a multiplication does not change the m-tangle. The rotation can also be viewed as deformation:

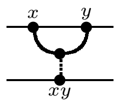
(4.1) 

and the two graphs in (4.1) are isomorphic. The image under the functor $\mathcal{R}_{\mathfrak{V}}$ of the graph on the right corresponds to the linear map

$$u \otimes v \mapsto \sum_i b(u, ve_i)e_i, \quad u,\, v \in V$$

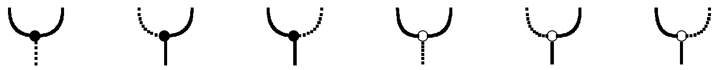
which is the multiplication map $u \otimes v \mapsto uv$ if the bilinear form b is associative. Thus the associativity of the bilinear form corresponds to the invariance of a trivalent vertex under a cyclic permutation of the edges.

If the bilinear form is not associative with respect to the multiplication, we need to consider 6 types of trivalent vertices. Taking the two orientations in account, this can be done by distinguishing one edge (which gives the output of the multiplication under the realization functor $\mathcal{R}_{\mathfrak{V}}$):



.

The 6 different trivalent vertices are:

(4.2) 

and we get a much more complicated alphabet of basic m-tangles. This is the case for example for quadratic compositions with unity ([**Cad05**]). Observe that the second m-tangle in (4.2) is obtained from the first one through a deformation of type (4.1). A similar situation occurs for 3-vector products, see Section 6.

5. Symmetric compositions

A *symmetric composition algebra* is an algebra (V, m) with a non-singular associative symmetric bilinear form b such that

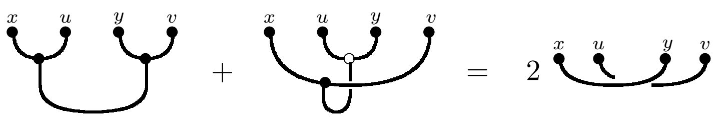
$$q(xy) = q(x)q(y), \quad q(x) = b(x,x) \quad \text{for all } x,\, y \in V. \tag{5.1}$$

Symmetric composition algebras were considered independently by Petersson [**Pet69**], Okubo [**Oku78**] and Faulkner [**Fau88**].

The linearization of (5.1) is the identity

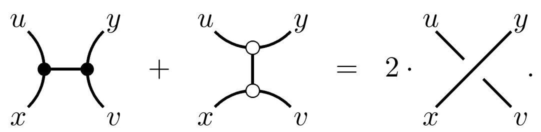
$$b(xu, yv) + b(xv, yu) = 2b(x,y)b(u,v) \tag{5.2}$$

for x, y, u, $v \in V$. Thus symmetric compositions are algebras of tensorial type and there is a corresponding m-tangle category $\mathcal{T}_{\rm sc}$ (Recall that the bilinear form is assumed to be associative). As in Section 2, trivalent vertices are cyclically oriented. The relation in the m-tangle category $\mathcal{T}_{\rm sc}$ corresponding to (5.2) is

(5.3) 

Applying the isomorphism (3.3) with $X = \{u, y\}$ and $Y = \{x, v\}$ to (5.3) we get the relation

(5.4)



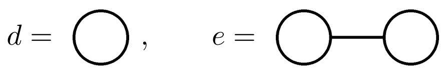
in $\mathcal{T}_{\rm sc}$.

Remark 5.5. It is curious to compare the relation (5.4) with the so-called IHX identity in the theory of knot invariants (see [**BN95**], p. 428). The IHX identity reflects the Jacobi identity of Lie algebras.

Expressed in terms of the basic alphabet, the relation (5.4) is

$$\big(1\square\beta\square 1\big)\circ\big((\tau\circ\mu^t)\square(\tau\circ\mu^t)\big)+\mu^t\circ\mu\circ\tau=2\tau\ .$$

Let $E_{\rm sc} = \mathrm{Mor}_{\mathcal{T}_{\rm sc}}(\emptyset,\emptyset)$ be the ring of numerical invariants of the category $\mathcal{T}_{\rm sc}$. As in Section 3 we denote



in $E_{\rm sc}$.

Theorem 5.6. *Let $R = \mathbb{Z}[\frac{1}{2}]$ and let J be the ideal of the polynomial ring $R[\bar d, \bar e]$, generated by the elements*

$$p_1 = \bar e\big(\bar e - (2-\bar d)^2\big) \tag{5.7}$$

$$p_2 = (\bar d - 2)(\bar d - 8)(\bar d - \bar e). \tag{5.8}$$

The homomorphism $\Phi\colon\ R[\bar d,\bar e]\to E_{\rm sc}=\mathrm{Mor}_{\mathcal{T}_{\rm sc}}(\emptyset,\emptyset)$ given by $\phi(\bar d)=d$, $\phi(\bar e)=e$, induces a surjective homomorphism of rings

$$\overline{\Phi}\colon\ R[\bar d,\bar e]/J\to E_{\rm sc}.$$

In particular the only possible values of the pair (d, e) of invariants are

$$(d,e) = (0,0),\ (1,1),\ (2,0),\ (4,4),\ (8,0)\ \ \textit{and}\ \ (8,36). \tag{5.9}$$

Remark 5.10. We do not know whether the homomorphism $\overline{\Phi}$ in Theorem 5.6 is injective. There is the isomorphism

$$R[\bar d,\bar e]/J\otimes\mathbb{Q}\simeq\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}[\epsilon]/(\epsilon^2)\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$$

given by

$$\bar d\mapsto(0,1,2,4,8,8),\qquad \bar e\mapsto(0,1,\epsilon,4,0,36).$$

On the other hand, since the 6 pairs in (5.9) can be realized by a symmetric composition algebra over $\mathbb{Q}$ (see Proposition 5.33), there is a homomorphism

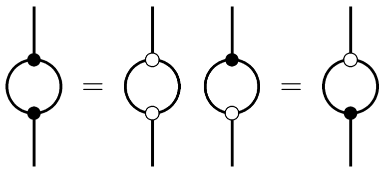
$$E_{\rm sc}\to\mathbb{Q}^6$$

with

$$d \mapsto (0,1,2,4,8,8), \qquad e \mapsto (0,1,0,4,0,36).$$

It follows that the kernel of $\overline{\Phi}_{\mathbb{Q}}$ is contained in the 1-dimensional radical of $R[\bar{d},\bar{e}]/J \otimes \mathbb{Q}$.

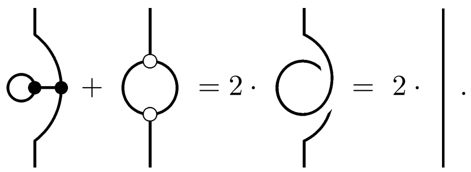
PROOF. We keep the convention that if, in a plane graph, no orientation is indicated we understand that the graph does not depend on the choice of the orientation. For oriented trivalent vertices we have the identities

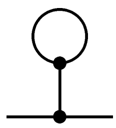


induced by rotation along the vertical symmetry axis.

To start with a proof, first consider the following consequence of (5.4):

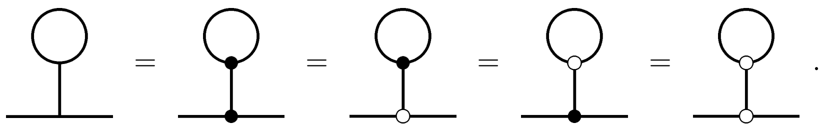
(5.11)



This and (5) show that  is invariant under a reflection.

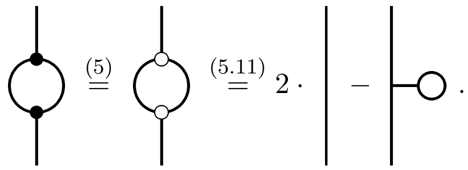
Together with (3.7) this means that in such a graph the orientation at the two vertices is inessential. So we may forget about them and we have

(5.12)

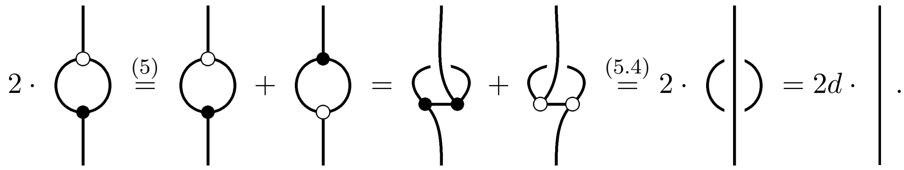


We have by (5) and (5.11)

(5.13)

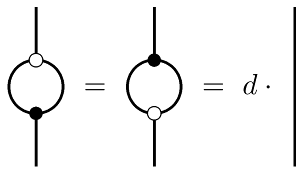


Identities (5) and (5.4) show



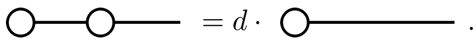
Hence we have

(5.14)

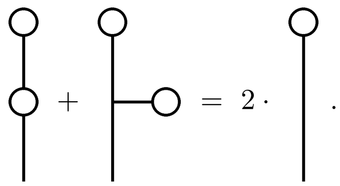


which implies

(5.15)

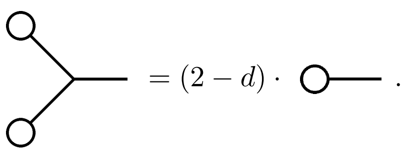


By (5.12) and (5.14) this equation is valid for any choice of orientation. Relation (5.11) implies similarly

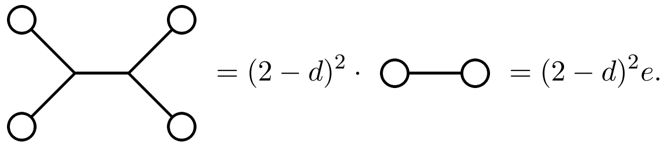


By (5.15) this gives

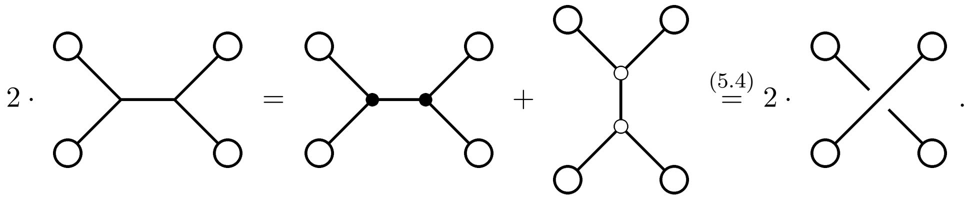
(5.16)



Composing both sides with themselves yields

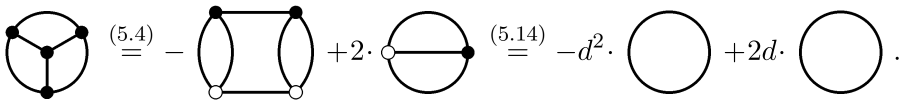


In the next computation the second equation follows from (5.4)



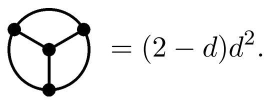
We conclude that $(2-d)^2 e = e^2$ and get the first relation $e(e-(2-d)^2) = 0$.

Going on, an application of (5.4) and of (5.14) (three times) gives

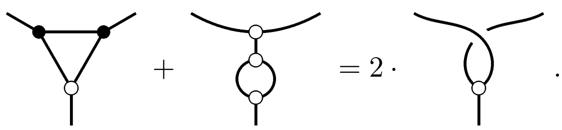


Hence

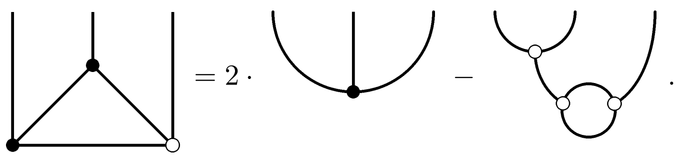
(5.17)



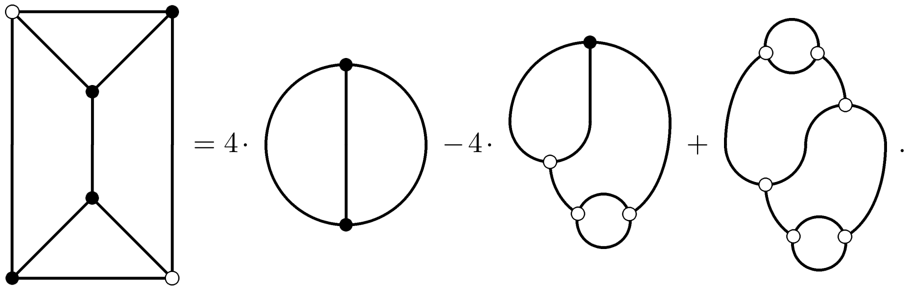
A further application of (5.4) gives



Hence

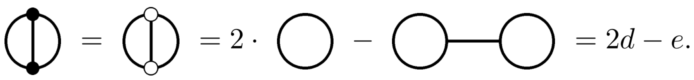


Composing this relation with itself shows

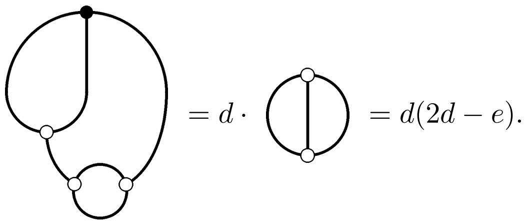


We compute the three terms on the right. (5.13) gives

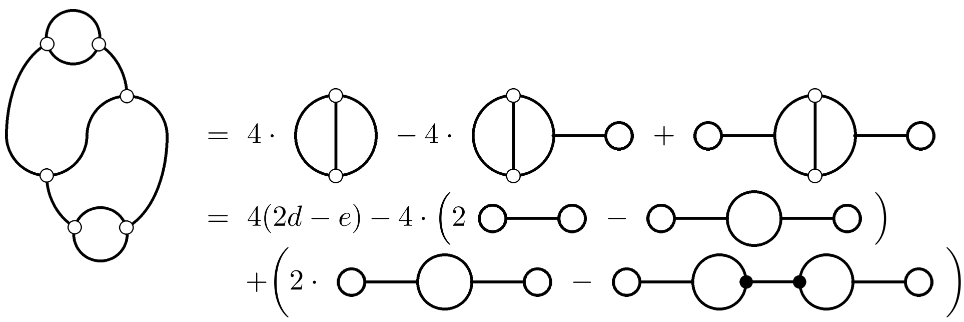
(5.18)



Then (5.14) and the last computation gives

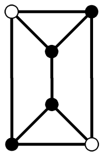


Applying (5.13) twice for the first equation, then (5.18),(5.13),(5.4) for the second equation and finally (5.15) for the third equation gives



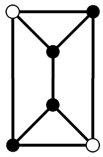
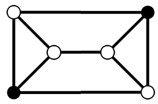
$$= 4(2d-e) - 4(2e-de) + (2de-d^2e).$$
$$= 4(2d-e) + (2-d)(-4e+de).$$

So far we have computed

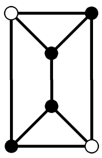
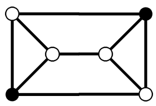
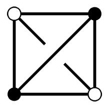
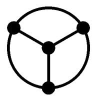


$$= 4(2d - e) - 4d(2d - e) + 4(2d - e) + (2 - d)(-4e + de)$$
$$= (2 - d)(8d - 8e + de).$$

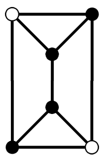
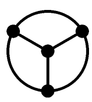
The right hand side is invariant under orientation reversing. Hence

(5.19)  $=$  $= (2 - d)(8d - 8e + de).$

Next note

(5.20)  $+$  $\overset{(5.4)}{=} 2 \cdot$  $= 2 \cdot$  .

For the second equality note that the two graphs are the same, they are just drawn in a different way. Relations (5.17), (5.19) and (5.20) give after dividing by 2

 $=$  $= (2 - d)d^2.$

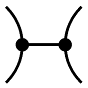
Putting things together:

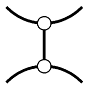
$$0 = (2 - d)(8d - 8e + de) - (2 - d)d^2 = (2 - d)(8 - d)(d - e).$$

This is the second desired equality.

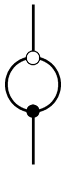
To complete the proof of Theorem 5.6 we next check that the two graphs d and e generate the ring of numerical invariants $E_{\mathrm{sc}} = \mathrm{Mor}_{\mathcal{T}_{\mathrm{sc}}}(\emptyset, \emptyset)$. We call the elements of E_{sc} *closed graphs*. Since they are defined up to isomorphism, we assume that closed graphs have only trivalent vertices. Observe that a closed graph is a linear combination of "real" closed graphs.

Definition 5.21. Two graphs are called *equivalent*, if one can be obtained from the other by a sequence of moves

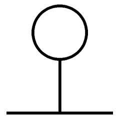




Proposition 5.22. *For any graph with at least one (trivalent) vertex there is an equivalent one which contains one of the portions*

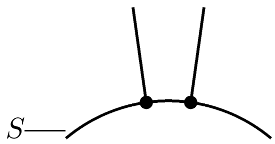


or

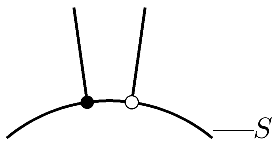


Proof. By a *cycle* S in a graph Γ we mean a simple closed path in Γ, where, in contrast with Section 2, all vertices have valence 3. By length(S) we mean the number of vertices on S.

Fix some direction on S. Then at each vertex P we have an incoming edge, an outgoing edge and the third edge not belonging to S. This triple determines an orientation at the vertex P which may be different from the given orientation in P. We now divide the vertices on S into two classes: the set of vertices for which this new orientation coincides with the given orientation of the trivalent vertex and the others. If two vertices belong to the same class, we say that they have the same orientation with respect to S. In the equivalence class of a given graph we choose a graph Γ which contains a cycle S of minimal length. If length$(S) = 1$, the proposition is clear. Suppose length$(S) \geq 2$. For an edge σ of S the two endpoints have either the same orientation or not:

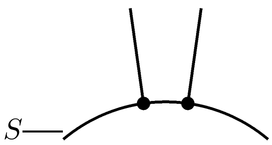


or

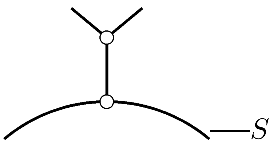


In the second case we say that σ is an alternating edge. In the first case we could make a move

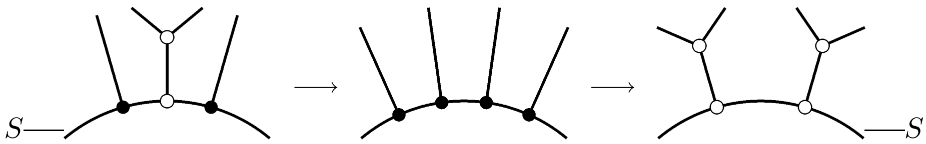
(5.23)



$\longrightarrow$



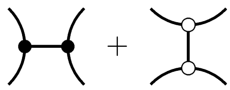
and find a cycle of smaller length in a graph equivalent to Γ. Since length(S) is minimal, any edge of S is alternating. Suppose that the endpoints P, Q of an edge σ of $\Gamma \setminus S$ lie both on S. Then the union of σ with one of the arcs on S between P and Q would have smaller length that S, except when each of these arcs have length 1. But then $\mathrm{length}(S) = 2$. The last two conclusions show that in case $\mathrm{length}(S) \geq 3$ there is a portion as indicated by the first of the following pictures:



Now the two indicated moves would again give rise to a cycle of smaller length. So we know $\mathrm{length}(S) \leq 2$ and in case $\mathrm{length}(S) = 2$ the edges of S are alternating. □

Now we consider graphs modulo the relation

(5.24)

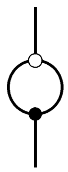


$= 0.$

Proposition 5.25. *The ring of closed graphs modulo the relation* (5.24) *is generated by the circle d and the closed graph e.*

Proof. Relation (5.24) has the following consequence (see the proof of (5.14)):

(5.26)

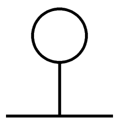


$= 0.$

Let Γ be a closed graph. We have to show that Γ is a polynomial in d and e. We may assume that Γ is connected and argue by induction on the number of vertices in Γ.

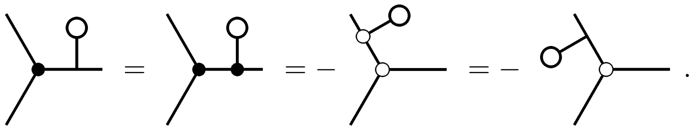
By Proposition 5.22 we have $\Gamma = \pm\Gamma'$, where Γ' contains one of the portions of Proposition 5.22. In the first case we may use (5.26) and reduce the number of vertices. Let us consider the second case, i.e., Γ contains a portion

$\eta =$

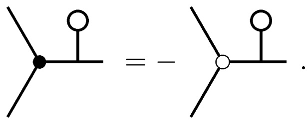


.

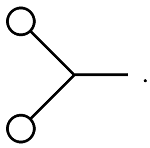
We may assume that Γ has more than two vertices, because otherwise $\Gamma = e$. Relation (5.24) gives

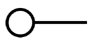
(5.27) 

Iterating this relation three times yields

(5.28) 

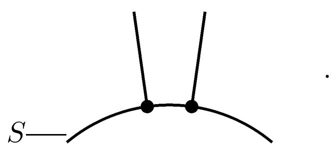
It follows that if we change the orientation of a vertex, we merely change the sign of Γ. This is clear if the vertex is near η as in (5.28). But in general we may use (5.27) to move η to the vertex, then change the orientation and move back again using (5.27). But if we don't have to care about orientations, then it is easy to use (5.24) to produce a further cycle of length 1. So $\Gamma = \pm\Gamma'$, where Γ' contains a portion



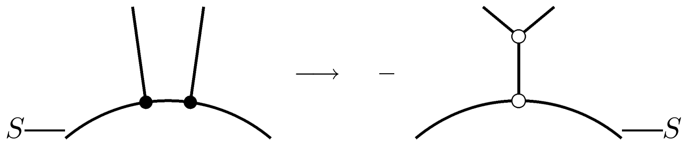
But this equals $(2-d)\cdot$  , cf. (5.16). $\square$

Corollary 5.29. *The ring E_{sc} of closed graphs is generated by d and e.*

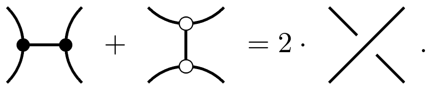
Proof. The claim follows from a filtration argument, where filtration is by the number of vertices. More precisely, consider for example the case of a cycle S in the closed graph Γ containing two black vertices:

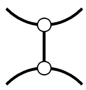
(5.30) 

Instead of relation (5.24) which leads to steps

(5.31) 

to reduce the length of S, we use the relation

(5.32) 

We claim that for S with minimal length, $\mathrm{length}(S) \leq 2$. Assume that $\mathrm{length}(S) \geq 3$. Let T be any cycle in Γ, different from S, but connected with S through the two marked black vertices in (5.30). We have $\mathrm{length}(T) \geq \mathrm{length}(S)$. For each move induced by (5.32), T is replaced by a linear combination of a cycle of length $\mathrm{length}(T)-1$ (corresponding to the term ) and a cycle of length $\mathrm{length}(T) + \mathrm{length}(S) - 4$ (corresponding to the term ). If $\mathrm{length}(T) = \mathrm{length}(S)$ we get in any case shorter cycles. If $\mathrm{length}(T) > \mathrm{length}(S)$, S is replaced by a linear combination of cycles which contains a shorter cycle. □

Theorem 5.6 follows now from Corollary 5.29. □

All values in Theorem 5.6 can be realized:

PROPOSITION 5.33. *Let F be an algebraically closed field of characteristic different from* 2 *and* 3*. For each value* $(0,0)$, $(1,1)$, $(2,0)$, $(4,4)$, $(8,0)$ *and* $(8,36)$ *of the pair* (d,e)*, there exists up to isomorphisms exactly one symmetric composition algebra with the given value.*

PROOF. Symmetric composition algebras are of two types (see for example [**OO81**], [**EM93**] or [**KMRT98**], Chapter VIII). The first one is related to quadratic composition algebras with identity ("Hurwitz algebras"), which, by Hurwitz' theorem, occur in dimension 1, 2, 4 and 8. Over an algebraically closed field, Hurwitz algebras are split and there is only one type up to isomorphism for each possible dimension. The bilinear form associated with the norm is not associative, but satisfies identities of the form

$$b(xy, z) = b(x, z\bar{y}), \tag{5.34}$$

where $x \mapsto \bar{x}$ it the conjugation map. It readily follows from (5.34) that, for the new multiplication $x \star y = \bar{x} \cdot \bar{y}$, the bilinear form is associative. Thus we can associate to every Hurwitz algebra, a symmetric composition algebra, which is usually called a *para-Hurwitz algebra*. Since F is an algebraically closed field, the standard basis $(1, i, j, \dots)$ of quadratic extensions, resp. quaternions or octonions, can be normalized so that $i^2 = j^2 = \dots = -1$ and is orthonormal. We get for the Casimir element

$$c = 1 + (d-1)(-1) = (2-d) \cdot 1_F.$$

Thus $e = b(c, c) = (d-2)^2$ and the numerical invariants for para-Hurwitz algebras are $(d, e) = (1, 1)$, $(2, 0)$, $(4, 4)$ and $(8, 36)$.

The other type of symmetric compositions is related to separable associative cubic algebras and were first considered by Okubo (see [**Oku78**] or [**KMRT98**]). Over an algebraically closed field we have three classes of separable associative cubic algebras; $A = F$, F^3 and $M_3(F)$. We denote the multiplication of A by $(x, y) \mapsto xy$. Let

$$p(X) = X^3 - T(x)X^2 + S(x)X - N(x) \cdot 1$$

be the generic polynomial. The form T is linear, S is quadratic and N cubic. Let $A^0 = \{x \in A \mid T(x) = 0\}$ be the vector space of trace zero elements in A. Let $q = -\frac{1}{3}S$ and let b be the associated symmetric bilinear form. Let ω be a primitive cubic root of 1. The bilinear form b is associative with respect to the multiplication defined on A^0 by $(x, y) \mapsto x \star y$ with

$$x \star y = \tfrac{1-\omega}{3}xy + \tfrac{2+\omega}{3}yx - \tfrac{1}{3}T(xy)$$

and $q(x \star y) = q(x)q(y)$ for all $x \in A^0$ (see for example [**KMRT98**]). Thus $(A^0, q, \star)$ is a symmetric composition algebra. One checks that in all three cases the Casimir element is trivial, so that symmetric composition algebras associated with cubic algebras have invariants $(d, e) = (0, 0)$, $(2, 0)$ and $(8, 0)$. The two algebras in dimension 2 (and for which $e = 0$) are isomorphic. In contrast, we have two types of symmetric compositions in dimension 8 which are not isomorphic, since they have different e-invariant. □

Remark 5.35. In dimension 8, one class of symmetric composition algebras ("type" G_2) is associated with octonions, and the second ("type" A_2) is associated with 3×3 matrices. The existence of these two types is related to the existence of two non conjugate trialitarian actions of S_3 on Spin_8 (see [**KMRT98**] or [**Hel78**], Chap. X, Exercise

E.2). Observe that relation (5.7) has two branches, one ($e = 0$) corresponding to type A_2 and the other one ($e = (d-2)^2$) to type G_2. The fact that the type of a symmetric algebra of dimension 8 can be easily detected through the Casimir element is new.

6. 3-Vector products

Let V be a d-dimensional vector space over a field F of characteristic zero. Let b be a nondegenerate, symmetric bilinear form on V.

An *r-vector product* in V is an r-linear map

$$P_r : V^r = V \times V \times \cdots \times V \longrightarrow V, \qquad 1 \leq r \leq n,$$

satisfying the following properties:

$$b\big(P_r(v_1, \ldots, v_r), v_i\big) = 0 \quad \text{for} \quad v_i \in V,\ i = 1, \ldots, r \tag{6.1}$$

$$b\big(P_r(v_1, \ldots, v_r), P_r(v_1, \ldots, v_r)\big) = \det\big(b(v_i, v_j)\big)_{i,j} \tag{6.2}$$

It follows from (6.1) that the multilinear form

$$f(v_1, \ldots, v_r,\, v_{r+1}) = b\big(P_r(v_1, \ldots, v_r), v_{r+1}\big) \tag{6.3}$$

and that the r-vector product P_r are alternating. Let $r = 3$ and denote $P_3(v_1, v_2, v_3) = v_1 v_2 v_3$. Relation (6.2) becomes

$$b(xyz, xyz) = \det \left| \begin{pmatrix} b(x,x) & b(x,y) & b(x,z) \\ b(y,x) & b(y,y) & b(y,z) \\ b(z,x) & b(z,y) & b(z,z) \end{pmatrix} \right|. \tag{6.4}$$

We have

$$v_{\pi(1)} v_{\pi(2)} v_{\pi(3)} = \mathrm{s}gn(\pi) v_1 v_2 v_3, \ \ v_i \in V \tag{6.5}$$

and

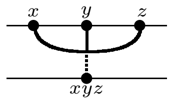
$$b(xyz, t) = -b(x, yzt) \quad \text{for } x,\ y,\ z \text{ and } t \in V. \tag{6.6}$$

Linearizing the above identity (6.4) we get:

$$\begin{aligned}
(6.7)\ \ & b(uxt, vys) + b(uxs, vyt) + b(uys, vxt) + b(uyt, vxs) = \\
&= 4b(x,y)b(u,v)b(s,t) - 2b(x,y)b(u,s)b(v,t) - 2b(x,y)b(u,t)b(v,s) \\
&\quad -2b(x,u)b(y,v)b(s,t) - 2b(x,v)b(y,u)b(s,t) - 2b(x,s)b(y,t)b(u,v) \\
&\quad -2b(x,t)b(y,s)b(u,v) + b(x,u)b(y,t)b(v,s) + b(x,v)b(u,t)b(y,s) \\
&\quad +b(x,u)b(y,s)b(v,t) + b(x,v)b(u,s)b(y,t) + b(x,s)b(y,u)b(t,v) \\
&\quad +b(x,s)b(y,v)b(t,u) + b(x,t)b(y,u)b(s,v) + b(x,t)b(y,v)b(s,u)
\end{aligned}$$

for all u, v, x, y, s and $t \in V$. In the terminology of Section 2, the data (V, P_3, b) is an algebra of tensor type, where the product m is replaced by the 3-vector product P_3. The tensor relations are (6.5), (6.6) and (6.7), and we shall associate to (V, P_3, b) a series of m-tangle categories $\mathcal{T}_i$,

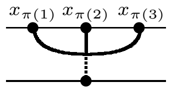
$i = 1, 2, 3$, as in Section 2. In particular we have again basic m-tangles $I_{\{x\}}$, τ, β and β^t. The main difference is that 3-valented vertices corresponding to products are replaced by 4-valented vertices corresponding to 3-vector products. Since there is no associativity for the bilinear form (a cyclic permutation of the variables in the multilinear form (6.3) changes the sign), we distinguish one edge out of every 4-valented vertex, which corresponds to the product xyz:

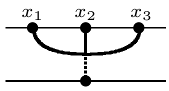
(6.8) $\mu =$ 

.

In a 4-valented vertex, the (clockwise) orientation of the three black edges will always be assumed to be positive, so that we do not mark 4-valented vertices. Composition of m-tangles is as in Section 2, with the rule that dotted edges can only be composed with dotted ones (because of (6.10) this can always be arranged).

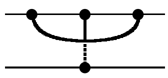
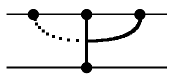
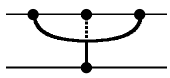
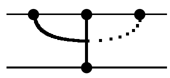
We next introduce the relations corresponding to a 3-vector product in the m-tangle category $\mathcal{T}_2$: the fact that the 3-product is alternating leads to the relation

(6.9) 

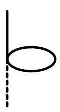
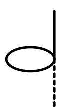
$= \operatorname{sgn}(\pi)$ 

for any permutation $\pi \in S_3$ of three objects. Relation (6.6) leads to

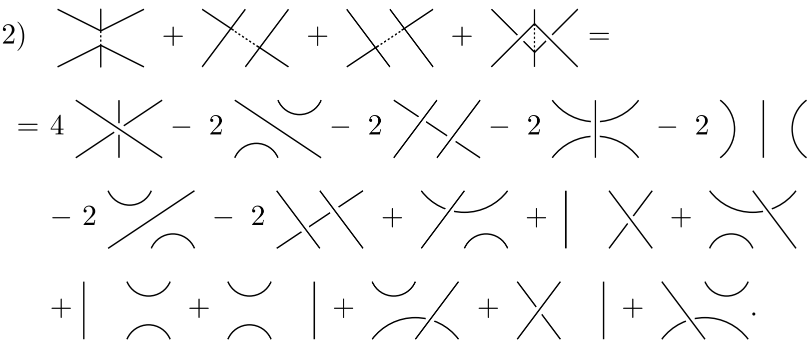
(6.10)

 $= -$  $=$  $= -$  ,

where the second m-tangle is obtained from the first through a deformation similar to the one defined in (4.1). Relation (6.9) implies that

(6.11)  $= 0$ and  $=$  $= 0.$

The morphisms of $\mathcal{T}_1$ consists of m-tangles generated by the alphabet $I_{\{x\}}$, τ, β, β^t and μ, those of $\mathcal{T}_2$ are linear combinations with coefficients in $R = \mathbb{Q}$. (We choose here the ring of coefficients to be $\mathbb{Q}$ to avoid computation difficulties.) The linearized formula (6.7) has the following diagrammatic translation:

(6.12) 

The step $\mathcal{T}_2 \to \mathcal{T}_3$ is as in Section 2: we define the category $\mathcal{T}_{3\text{vec}}$ of type $\mathcal{T}_3$ by going modulo the relations (6.9), (6.10) and (6.12).

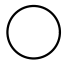
Let $\mathcal{V}$ be the category of finite dimensional vector spaces over F with morphisms linear maps. As for Theorem 3.1, we have:

Theorem 6.13. *Let V be a finite dimensional vector space over a field of characteristic zero. If there exists a structure (V, P_3, b) of 3-vector product on V, there is a unique functor $\mathcal{R}\colon \mathcal{T}_{3\text{vec}} \to \mathcal{V}$ sending a point to V, tensor products to tensor products, τ to τ, β to the symmetric bilinear form b and μ to the 3-multiplication P_3.*

The functor $\mathcal{R}$ induces a ring homomorphism

$$E_{3\text{vec}} = \mathrm{Mor}_{A_{3\text{vec}}}(\emptyset, \emptyset) \to F$$

and the main aim is to compute $E_{3\text{vec}}$ (or at least part of it). Here again a simple element of $E_{3\text{vec}}$ is the circle

$\beta^t \circ \beta =$  .

Since $\mathcal{R}(\beta^t \circ \beta) = \dim_F V$, we denote the circle also by d.

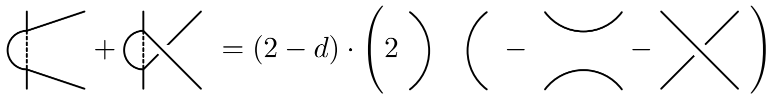
Theorem 6.14. *Let*

$$p(\bar{d}) = \bar{d}(\bar{d}-1)(\bar{d}-2)(\bar{d}+2)(\bar{d}-4)(\bar{d}+4)(\bar{d}-8) \in \mathbb{Q}[\bar{d}].$$

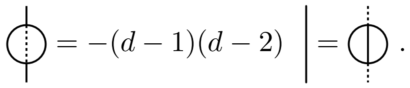
The map $\mathbb{Q}[\bar{d}] \to E_{3vec}$ induced by $\bar{d} \mapsto d$ induces a $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}[\bar{d}]/p(\bar{d}) \to E_{3vec}.$$

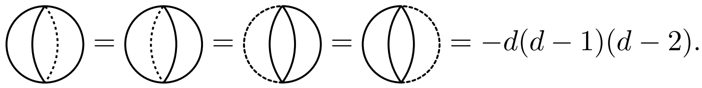
Proof. We verify that $p(d) = 0$ in E_{3vec}. Closing (6.12) on the left (and using (6.11)) gives

(6.15) 

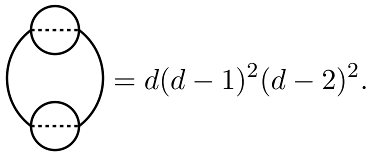
and closing on the right:

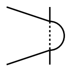
(6.16) 

Equation (6.16) yields immediately

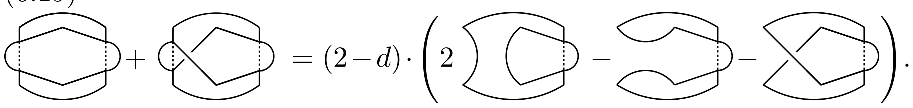
(6.17) 

Moreover one finds

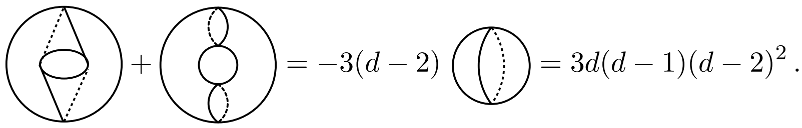
(6.18) 

We next close (6.15) with the graph  on the right as follows:

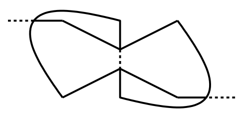
(6.19)



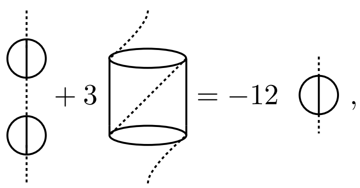
The fourth graph is zero by (6.11). The first and the second one can be redrawn and so we get the useful relation:

(6.20) 

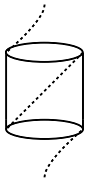
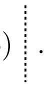
We now look for an equation which permits to compute one of the two graphs on the left side of (6.20). The strategy is to compose all graphs in (6.12) on the top left and on the bottom right with copies of μ. For example we get the graph



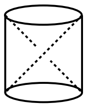
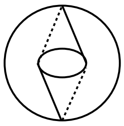
from the first graph of (6.12). This yields to



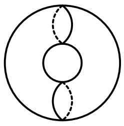
hence (with using (6.16))

3  $= -(d-1)(d-2)(d+2)(d-5)$  .

Connecting the dotted lines we get the first graph of (6.20):

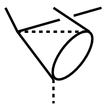
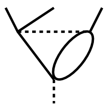
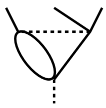
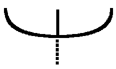
(6.21)  $=$  $= -\frac{1}{3}d(d-1)(d-2)(d+2)(d-5)\,,$

so that:

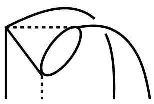
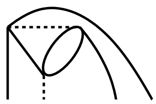
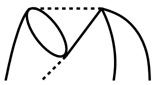
(6.22)  $= \frac{1}{3}d(d-1)(d-2)(d^2+6d-28).$

The graphs in (6.18), (6.21) and (6.22) play a central rôle in the next computations, since all other computed graphs will be linear combinations of these three.

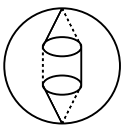
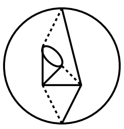
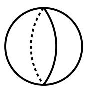
A further application of (6.12) (compose (6.12) with μ) gives

(6.23) $-$  $+$  $+$  $= (d-5)(d+2)$ 

Moving down the three points of the source, we obtain

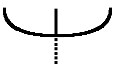
$-$  $+$  $+$  $= (d-5)(d+2)$  .

Then, composing this relation with itself:

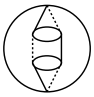
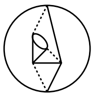
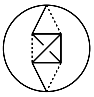
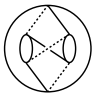
3  $+\,6$  $= (d-5)^2(d+2)^2$ 

$= -d(d-1)(d-2)(d+2)^2(d-5)^2.$

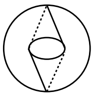
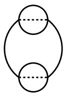
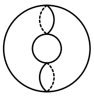
The graph relation (6.12) can be collated in other ways with graphs of type

 and  ,

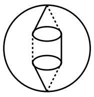
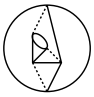
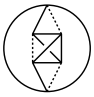
to get on the left side of (6.12) linear combinations of the diagrams

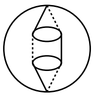
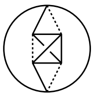
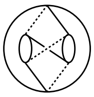
 ,  ,  and  ,

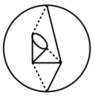
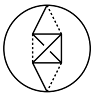
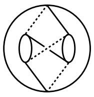
and on the right side linear combinations of the diagrams

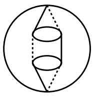
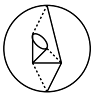
 ,  and 

which are already computed. We get a system of 4 equations

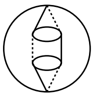
 $+\,2$  $+$  $= -d(d-1)(d-2)(d+2)(7d-26)$

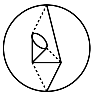
 $-\,2$  $+$  $= -6d(d-1)(d-2)^2(d-4)$

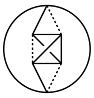
2  $+$  $+$  $= d(d-1)(d-2)(d^2+24d-100)$

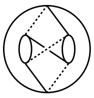
3  $+\,6$  $= -d(d-1)(d-2)(d+2)^2(d-5)^2$

which has the solution

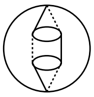
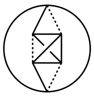
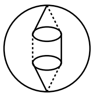
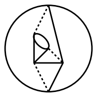
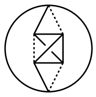
 $= \frac{1}{3}d(d-1)(d-2)(d+2)(d^3-8d^2-37d+206)$

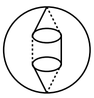
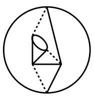
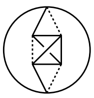
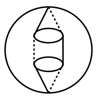
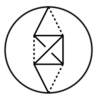
 $= -\frac{1}{3}d(d-1)(d-2)(d-4)(d+4)(d-8)(d+2)$

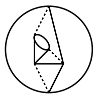
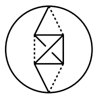
 $= \frac{1}{3}d(d-1)(d-2)(d-4)(d+4)(d-8)(d+2)$

 $= \frac{1}{3}d(d-1)(d-2)(d^4-6d^3-29d^2+168d-44)$

Two other ways of collating (6.12) yield to the two equations:

 $-$  $=$  $+\,2$  $+$  ,

 $+\,2$  $-$  $=$  $-$  ,

which implies  $= -$  $= 0$, hence

$$-\frac{1}{3}d(d-1)(d-2)(d+2)(d-4)(d+4)(d-8) = 0$$

in E. □

Corollary 6.24. *The possible values for the dimension of a vector space admitting a* 3*-vector product are* 0, 1, 2, 4 *and* 8. *In dimension* 0, 1 *and* 2 *the product is degenerate.*

Proof. The existence of a 3-vector product in dimension $d^* > 0$ implies the existence of an evaluation map $E_{3\mathrm{vec}} \to F$, sending d to d^*. Such a map induces in turn a homomorphism $\mathbb{Q}[\bar{d}]/p(\bar{d}) \to F$ (where $Q[\bar{d}]/p(\bar{d})$ is as in Theorem 6.14), sending again $\bar{d}$ to d^*. Thus d^* has to be equal to 0, 1, 2, 4 or 8. In dimensions 0, 1 and 2 the product is trivial. □

Remark 6.25. The polynomial $p(\bar{d})$ also admits the negative zeroes -2 and -4. We do not know if there is another relation in $E_{3\mathrm{vec}}$ which would exclude these values.

Acknowledgements

The first author thanks Professor B. Eckmann for suggesting to analyse 3-vector products with diagrammatic techniques. Her research is partially supported by the Swiss National Science Foundation, Grant 21-64712.01.

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Larissa Cadorin, Mathematik Departement, ETH Zentrum, CH-8092 Zürich, Switzerland

E-mail address: `cadorin@math.ethz.ch`

URL: `http://www.math.ethz.ch/~cadorin`

Max-Albert Knus, Mathematik Departement, ETH Zentrum, CH-8092 Zürich, Switzerland

E-mail address: `knus@math.ethz.ch`

URL: `http://www.math.ethz.ch/~knus`

Markus Rost, Fakultät für Mathematik Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

E-mail address: `rost@mathematik.uni-bielefeld.de`

URL: `http://www.mathematik.uni-bielefeld.de/~rost`