COMPUTATIONS IN THE MOD 2 LAZARD RING

MARKUS ROST

`preliminary version`

Contents

Introduction

This text contains purely algebraic considerations in the mod 2 Lazard ring L. We define operations

$$\mathrm{Sq}^k\colon L \to L.$$

and prove a vanishing property for them (Theorem 2). We apply this to obtain the canonical logarithm of the universal mod 2 Lazard formal group law. This approach does not involve the usual game with binomial coefficients.

This text is preliminary in a manifold sense: We only define the logarithm, but do not describe it in more detail. Missing are also the Landweber-Novikov operations. The geometric analogies of the material are only mentioned partially in some side remarks. No attempt has been made on the mod p-analogies.

1. Basic definitions

Let R be a $\mathbf{F}_2$-algebra. By a mod 2 *formal group law* we understand a power series

$$F(x,y) \in R[[x,y]]$$

such that

$$F(x,0) = F(0,x) = x, \tag{1}$$

$$F\big(x,F(y,z)\big) = F\big(F(x,y),z\big), \tag{2}$$

$$F(y,x) = F(x,y), \tag{3}$$

$$F(x,x) = 0. \tag{4}$$

Date: March 10, 2000.

These equations are understood in the rings $R[[x]]$, $R[[x,y,z]]$, etc.

It follows from (1) that

$$F(x,y) = x + y + \sum_{i,j\geq 1} a_{i,j} x^i y^j$$

with $a_{i,j} \in R$. Sometimes we write for simplicity

$$F(x,y) = \sum_{i,j} a_{i,j} x^i y^j,$$

thereby understanding $i,j \geq 0$, $i+j \geq 1$, $a_{1,0} = a_{0,1} = 1$, and $a_{i,0} = a_{0,j} = 0$ for $i,j > 1$.

Let

$$\widetilde{L} = \mathbf{F}_2[u_{i,j}]$$

be the polynomial ring in the variables $u_{i,j}$, $i,j \geq 1$ and let

$$I \subset \widetilde{L}$$

be the smallest ideal such that

$$F_{\mathrm{univ}}(x,y) = x + y + \sum_{i,j\geq 1} u_{i,j} x^i y^j$$

becomes a mod 2 formal group law over the quotient

$$L = \widetilde{L}/I.$$

The ring L is called the mod 2 *Lazard ring* and F_{univ}, considered as element of $L[[x,y]]$, is called the *universal* mod 2 *formal group law*.

For any mod 2 formal group law

$$F(x,y) = x + y + \sum_{i,j\geq 1} a_{i,j} x^i y^j \in R[[x,y]]$$

there exist a unique ring homomorphism

$$\rho_F \colon L \to R$$

with

$$\rho_F(u_{i,j}) = a_{i,j}.$$

All formal group laws considered in this text are mod 2 formal group laws. We will call them simply “formal group laws” and L will be called “the Lazard ring”.

We consider $\widetilde{L}$ as a $\mathbf{Z}$-graded ring with

$$\deg u_{i,j} = 1 - i - j.$$

For a $\mathbf{Z}$-graded ring R we extend the grading to any power series ring $R[[x_1,\dots,x_r]]$ by associating the degree 1 to each of variables x_i.

This way $F_{\mathrm{univ}}(x,y) \in \widetilde{L}[[x,y]]$ is homogeneous of degree 1. Also, the equations (1)–(4) are homogeneous. It follows that the ideal I is a homogeneous ideal. Therefore the Lazard ring carries a $\mathbf{Z}$-grading

$$L = \bigoplus_{k\leq 0} L^k$$

with

$$u_{i,j} \in L^{1-i-j}.$$

Remark. The cobordism ring of a space X can be described as

$$\mathcal{N}^*(X) = \bigoplus_{k\in\mathbf{Z}} \mathcal{N}^k(X),$$
$$\mathcal{N}^k(X) = [X_+, S^k \wedge \mathrm{MO}].$$

One has $\mathcal{N}^k(X) = 0$ for $k > \dim X$. For nonempty X the groups $\mathcal{N}^k(X)$ are nonzero in non-positive degrees. The Lazard ring L is isomorphic to the unoriented cobordism ring $\mathcal{N}^* = \mathcal{N}^*(\text{point})$.

Once in a while we will refer to these facts for some explanations. We will certainly not make use of them, because this text is supposed as a preparation to establish the isomorphism $L = \mathcal{N}^*$.

The negative grading on L introduced above coincides with the natural grading on $\mathcal{N}^*$. For $k > 0$ one has $L^k = \mathcal{N}^k = 0$ and for $k \geq 0$ the group $L^{-k} = \mathcal{N}^{-k}$ is the group of bordism classes of k-dimensional smooth compact manifolds.

2. A preliminary computation

Let $F(x, y)$ be a (mod 2) formal group law over R. We consider the (continuous) homomorphism over $R[[t]]$

$$\tau\colon R[[t, x]] \to R[[t, x]],$$
$$x \mapsto F(t, x).$$

Note that τ is an involution:

$$\tau^2(x) = F\big(t, F(t, x)\big) = F\big(F(t, t), x\big) = F(0, x) = x.$$

Lemma 1. *Let $f \in R[[t, x]]$ with $\tau(f) = f$. Then there exist a unique element $g \in R[[t, u]]$ such that*

$$f(t, x) = g\big(t, xF(t, x)\big).$$

Proof. Let $u = x\tau(x) = xF(t, x)$. Then

$$u = x^2 + t\alpha$$

for some $\alpha \in R[[t, x]]$. Using standard arguments for power series rings, it follows that

$$R[[t, x]] = R[[t, u]] \oplus xR[[t, u]].$$

Note that $\tau(u) = u$, so every element of the subring $R[[t, u]]$ is τ-invariant. Write f as

$$f(t, x) = g(t, u) + xh(t, u).$$

Then $\tau(f) = f$ implies

$$F(t, x)h(t, u) = xh(t, u).$$

However

$$F(t, x) - x = t + xt \sum_{i,j\geq 1} a_{i,j} x^{i-1} t^{j-1}$$

is not a zero divisor. Thus $h = 0$. □

3. Operations defined by power series

Let

$$F(x,y)=\sum_{i,j} a_{i,j}x^iy^j$$

be a (mod 2) formal group law over R and let

$$q(x)=\sum_{i\geq 0} c_ix^{i+1}\in R[[x]]$$

be a power series with c_0 invertible in R. Then

$$\begin{aligned} R[[x]]&\to R[[x]],\\ x&\mapsto q(x)\end{aligned}$$

defines an automorphism of $R[[x]]$ over R. In particular there exist the inverse power series

$$q^{-1}(x)\in R[[x]]$$

with

$$q\big(q^{-1}(x)\big)=q^{-1}\big(q(x)\big)=x.$$

We put

$$F_q(x,y)=q\big(F\big(q^{-1}(x),q^{-1}(y)\big)\big)\in R[[x,y]].$$

This is a formal group law, obtained from the formal group law F by means of the coordinate change $x\mapsto q(x)$. Therefore there exist a ring homomorphism

$$\theta_q=\rho_{F_q}\colon L\to R$$

such that

$$q\big(F(x,y)\big)=\sum_{i,j}\theta_q(u_{i,j})q(x)^iq(y)^j. \tag{5}$$

We extend θ_q to the continuous homomorphism

$$\begin{aligned}\bar\theta_q\colon L[[x_1,\dots,x_r]]&\to R[[x_1,\dots,x_r]],\\ x_i&\mapsto q(x_i).\end{aligned}$$

Then

$$\bar\theta_q\big(F_{\mathrm{univ}}(x_k,x_l)\big)=q\big(F(x_k,x_l)\big).$$

by (5).

This means for instance that the following diagram is commutative:

$$\begin{array}{ccc} L[[z]] & \xrightarrow{\bar\theta_q} & R[[z]]\\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow\\ L[[x,y]] & \xrightarrow{\bar\theta_q} & R[[x,y]]. \end{array} \tag{6}$$

Here the maps α are the identity on L resp. R and map z to $F(x,y)$. The homomorphisms $\bar\theta_q$ are understood as above: they extend $\theta_q\colon L\to R$ by $u\mapsto q(u)$ for $u=x$, y, z.

Remark. These considerations are the formal group analogies of a construction in (unoriented) cobordism theory, see [1]. Given the power series $q(x)$, one may define for topological spaces X a natural transformation

$$\Theta_q\colon \mathcal{N}^*(X) \to \mathcal{N}^*(X) \otimes_L R.$$

Here the tensor product is understood via the isomorphism $L \to \mathcal{N}^*$, given by the canonical formal group law in $\mathcal{N}^*(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \mathcal{N}^*[[x, y]]$, and via $\rho_F\colon L \to R$, given by the formal group law F.

The homomorphisms $\bar{\theta}_q\colon L[[x_1, \dots, x_r]] \to R[[x_1, \dots, x_r]]$ are the formal analogies of the operations Θ_q for $X = \mathbf{P}^\infty \times \cdots \times \mathbf{P}^\infty$.

The commutative diagram (6) corresponds to the functoriality of Θ_q with respect to the map $\mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$, the sum map for the Eilenberg-MacLane space $\mathbf{P}^\infty = H(\mathbf{Z}/2, 1)$.

The operations considered in the next section are the formal analogies of the Steenrod squares in cobordism theory.

4. Steenrod squares

Now let

$$R = L[[t]][t^{-1}]$$

be the ring of Laurent series over L, let

$$F(x, y) = F_{\mathrm{univ}}(x, y) = \sum_{i,j} u_{i,j} x^i y^j \in R[[x, y]]$$

be the universal formal group law considered as formal group law over R (so that $\rho_F\colon L \to R$ is the inclusion), and let

$$q(x) = \frac{xF(x, t)}{t} \in L[[t, x]][t^{-1}] \subset R[[x]].$$

Then

$$q(x) = x + \frac{x^2}{t} + x^2 \sum_{i,j \geq 1} u_{i,j} x^{i-1} t^{j-1},$$

and $x \to q(x)$ defines an invertible endomorphism of $R[[x]]$ over R. Thus we may apply the construction of the previous section and get a ring homomorphism $\theta_q\colon L \to R$. We write $\mathrm{Sq} = \theta_q$. Thus Sq is the ring homomorphism

$$\mathrm{Sq}\colon L \to L[[t]][t^{-1}]$$

such that

$$q\Big(\sum_{i,j} u_{i,j} x^i y^j\Big) = \sum_{i,j} \mathrm{Sq}(u_{i,j}) q(x)^i q(y)^j. \tag{7}$$

Note that q is homogeneous of degree 1, and therefore Sq is homogeneous of degree 0 (with respect to the $\mathbf{Z}$-gradings given by the grading on L and by $\deg x = \deg t = 1$). We write

$$\mathrm{Sq} = \sum_{k \in \mathbf{Z}} t^{-k} \mathrm{Sq}^k$$

with additive homomorphisms

$$\mathrm{Sq}^k\colon L \to L.$$

Then Sq^k is homogeneous of degree k, that is

$$\mathrm{Sq}^k(L^n) \subset L^{n+k}.$$

The maps Sq^k are called *Steenrod squares*. In the following we establish the properties to be expected from operations with this name.

The Cartan formula

$$\mathrm{Sq}^k(\alpha\beta) = \sum_{h+l=k} \mathrm{Sq}^h(\alpha)\,\mathrm{Sq}^l(\beta)$$

follows from the multiplicativity of the total Steenrod square Sq.

Since L is concentrated in non-positive degrees, it follows that

$$\mathrm{Sq}^k(\alpha) = 0 \quad \text{for } \alpha \in L^n,\ k > -n.$$

The next theorem sharpens this a priori vanishing property.

Theorem 2. *Let $\alpha \in L^n$. Then*

$$\mathrm{Sq}^n(\alpha) = \alpha^2, \tag{8}$$

$$\mathrm{Sq}^k(\alpha) = 0 \quad \textit{for } k > n. \tag{9}$$

Proof. By (7) we have

$$\frac{F(x,y)F\big(F(x,y),t\big)}{t} = \sum_{i,j} \mathrm{Sq}(u_{i,j}) \left(\frac{xF(x,t)}{t}\right)^i \left(\frac{yF(y,t)}{t}\right)^j,$$

or

$$F(x,y)F\big(F(x,y),t\big) = \sum_{i,j} \frac{\mathrm{Sq}(u_{i,j})}{t^{i+j-1}} \big(xF(x,t)\big)^i \big(yF(y,t)\big)^j.$$

The left hand side is an element of $L[[t,x,y]]$ and is invariant under the involutions given by

$$\begin{aligned} \tau_x\colon &\quad t \mapsto t, \quad x \mapsto F(t,x), \quad y \mapsto y,\\ \tau_y\colon &\quad t \mapsto t, \quad x \mapsto x, \qquad\qquad y \mapsto F(t,y). \end{aligned}$$

Indeed, we have

$$\begin{aligned} \tau_x\Big(F(x,y)F\big(F(x,y),t\big)\Big) &= F\big(F(t,x),y\big)F\Big(F\big(F(t,x),y\big),t\Big)\\ &= F\big(F(x,y),t\big)F(x,y). \end{aligned}$$

Similarly for τ_y.

The involutions τ_x, τ_y commute, so we may apply Lemma 1 to them successively and find that

$$F(x,y)F\big(F(x,y),t\big) = \sum_{i,j} Q_{i,j}(t) u_x^i u_y^j \tag{10}$$

with $u_x = xF(t,x)$, $u_y = yF(t,y)$, and $Q_{i,j} \in L[[t]]$. Comparing coefficients we get

$$\frac{\mathrm{Sq}(u_{i,j})}{t^{i+j-1}} = Q_{i,j}(t).$$

This proves (9) for $\alpha = u_{i,j}$.

Moreover, setting $t = 0$ in (10), we get

$$F(x,y)^2 = \sum_{i,j} Q_{i,j}(0) x^{2i} y^{2j}.$$

Hence $Q_{i,j}(0) = u_{i,j}^2$. This proves (8) for $\alpha = u_{i,j}$.

Since L is generated by the $u_{i,j}$, the claims follow from the Cartan formula. □

Remark. The vanishing properties

$$\mathrm{Sq}^k(\alpha) = 0 \quad \text{for } \alpha \in L^n,\ -n \geq k > n. \tag{11}$$

have the following geometric analogue:

Proposition 3. *Let M be a compact n-manifold. The pair $\big(\mathbf{P}(TM), \mathbf{L}(TM)\big)$ consisting of the projective tangent bundle of M and its tautological line bundle is bordant.*

Proof. By the *strict blow up* of a manifold Y in a submanifold Z we understand the manifold $X \to Y$ obtained from Y by "cutting along Z", that is by replacing Z by the sphere bundle $\mathbf{S}(N)$ of the normal bundle of Z in Y. If Y has no boundary, then X is a manifold with boundary $\mathbf{S}(N)$. The usual (real) blow up of Y in Z is the quotient of the strict blow up by the involution $v \mapsto -v$ on $\mathbf{S}(N)$. See [6, p. 56–57] for details.

Now let $X \to M \times M$ be the strict blow up in the diagonal. The switch involution on $M \times M$ lifts to an involution σ on X. This involution is fixed point free. The double cover $X \to X/\sigma$ has as boundary the double cover $\mathbf{S}(TM) \to \mathbf{P}(TM)$. The latter is the sphere bundle of the line bundle $\mathbf{L}(TM)$ which therefore extends to a line bundle on X/σ. □

One may represent the pair $\big(\mathbf{P}(TM), \mathbf{L}(TM)\big)$ by a map $f\colon \mathbf{P}(TM) \to \mathbf{P}^\infty$. Then the proposition means that f is bordant.

Since $\dim \mathbf{P}(TM) = 2n-1$, we may represent the pair by a map $f\colon \mathbf{P}(TM) \to \mathbf{P}^{2n-1}$. The bordism group of $\mathbf{P}^r$ injects into the bordism group of $\mathbf{P}^\infty$. Therefore the map f will be bordant also as map to $\mathbf{P}^{2n-1}$. Thus we get a relation

$$0 = [f] \in \mathcal{N}^0\big(\mathbf{P}^{2n-1}\big) = \bigoplus_{i=0}^{2n-1} \mathcal{N}^i.$$

The $2n$ relations of (11) are the formal analogies of the vanishing of the $2n$ components of $[f]$.

All the Wu-relations among Stiefel-Whitney numbers are encoded in these relations. For instance the 0-th component of $[f]$ is just $w_n(-TM)[M] \in \mathbf{F}_2$. The formal analogue of this is $\mathrm{Sq}^n(\alpha) = 0$ for $\alpha \in L^{-n}$.

These relations appear in some form in the approaches to the cobordism ring by Quillen [7], [1] and Buonchristiano and Hacon [2], [3], [4], [5].

5. A contracting property of Sq^0

For our power series

$$q(x) = \frac{xF(x,t)}{t}$$

we consider now the associated homomorphisms $\bar{\theta}_q$ (see section 3), which we denote by Sq as well. Thus we have ring homomorphisms

$$\mathrm{Sq}\colon L[[x_1, \dots, x_r]] \to L[[t]][t^{-1}][[x_1, \dots, x_r]]$$

extending Sq on L and with $\mathrm{Sq}(x_i) = q(x_i)$.

Note that if we restrict to the polynomial rings, we get homomorphisms

$$\mathrm{Sq}\colon L[x_1,\dots,x_r] \to L[[t,x_1,\dots,x_r]][t^{-1}].$$

Again we write

$$\mathrm{Sq} = \sum_{k\in\mathbf{Z}} t^{-k}\,\mathrm{Sq}^k$$

with additive homomorphisms

$$\mathrm{Sq}^k\colon L[[x_1,\dots,x_r]] \to L[[x_1,\dots,x_r]]$$

of degree k.

Remark. These operations are the formal versions the Steenrod squaring operations on $\mathcal{N}^*(\mathbf{P}^\infty\times\cdots\times\mathbf{P}^\infty)$.

Let

$$I = \langle x_1,\dots,x_r\rangle L[[x_1,\dots,x_r]] \subset L[[x_1,\dots,x_r]]$$

be the ideal generated by the x_i.

We write

$$L[[x_1,\dots,x_r]] = \bigoplus_{k\in\mathbf{Z}} U^k$$

with U^k the homogeneous component of degree k.

Proposition 4. *One has*

$$\mathrm{Sq}^k(I^{n+k}\cap U^l) \subset I^{2n+l-k}\cap U^{k+l}.$$

We will need and prove this only in the following special case:

Proposition 5. *For $n\geq 0$ one has*

$$\mathrm{Sq}^0(I^{n+1}\cap U^1) \subset I^{2n+1}\cap U^1.$$

Proof. Since Sq^0 is homogeneous of degree 0, it suffices to show: Let $n\geq 0$, $a_{-n}\in L^{-n}$, let further $p=(p_1,\dots,p_r)$ be a multi-index with $\sum p_s = n+1$, and let $\alpha = a_{-n}x^p$. Then

$$\mathrm{Sq}^0(\alpha)\in I^{2n+1}.$$

We have

$$\begin{aligned}\mathrm{Sq}(\alpha) &= \mathrm{Sq}(a_{-n})\prod_s\Big(\frac{x_sF(x_s,t)}{t}\Big)^{p_s}\\ &= \sum_{k\leq -n} t^{-k}\,\mathrm{Sq}^k(a_{-n})t^{-(n+1)}\prod_s\big(x_sF(x_s,t)\big)^{p_s}.\end{aligned}$$

Here we have used Theorem 2 to get the upper bound for the index k.

Multiplying both sides by t we get

$$t\,\mathrm{Sq}(\alpha) = \sum_{h\geq 0} t^h\,\mathrm{Sq}^{-n-h}(a_{-n})\prod_s\big(x_sF(x_s,t)\big)^{p_s} \tag{12}$$

with $h=-k-n$.

On the other hand we have

$$t\,\mathrm{Sq}(\alpha) = \mathrm{Sq}^1(\alpha) + t\,\mathrm{Sq}^0(\alpha) + t^2\,\mathrm{Sq}^{-1}(\alpha) + \cdots. \tag{13}$$

Since

$$F(x,t) = x + t + xtP(x,t)$$

for some $P \in L[[t,x]]$, we have

$$\prod_s F(x_s,t)^{p_s} \in I^{n+1} + tI^n + t^2 L[[t,x_1,\dots,x_r]].$$

Therefore, by (12),

$$t\,\mathrm{Sq}(\alpha) \in I^{2n+2} + tI^{2n+1} + t^2 L[[t,x_1,\dots,x_r]]$$

and (13) yields

$$\mathrm{Sq}^1(\alpha) + t\,\mathrm{Sq}^0(\alpha) \in I^{2n+2} + tI^{2n+1}.$$

This proves the claim. $\square$

6. The logarithm

We apply this to the case $r = 1$ and write $z = x_1$. So let $I = zL[[z]]$ and

$$L[[z]] = \bigoplus_{k\in\mathbf{Z}} U^k.$$

Then

$$U^1 = z\mathbf{F}_2 \oplus z^2 L^{-1} \oplus z^3 L^{-2} \oplus \cdots .$$

Proposition 6.

$$\mathrm{Sq}^0(z) - z \in I^2 \cap U^1, \tag{14}$$

and for $m \geq 0$ one has

$$(\mathrm{Sq}^0)^m (I^2 \cap U^1) \subset I^{2^m+1}. \tag{15}$$

Proof. We have

$$\mathrm{Sq}(z) = \frac{zF(z,t)}{t} = \frac{z^2}{t} + z + \sum_{i,j\geq 1} u_{i,j} z^{i+1} t^{j-1}$$

and therefore

$$\mathrm{Sq}^0(z) = z + z^2 \sum_{i\geq 0} u_{i+1,1} z^i.$$

This proves (14). Claim (15) follows from Proposition 5. $\square$

Corollary 7. *For any $\ell_0 \in U^1$, the series of elements*

$$\ell_m = (\mathrm{Sq}^0)^m(\ell_0) \in U^1$$

is convergent in the I-adic topology on U^1. The limit

$$\ell_\infty = \lim_{m\to\infty} \ell_m$$

depends alone on the class of ℓ_0 in $U^1/(I^2 \cap U^1)$.

Since $U^1/(I^2 \cap U^1) = \mathbf{F}_2$, there is only one nontrivial such limit. It is obtained for $\ell_0(z) = z$ and given by

$$\ell = \lim_{m\to\infty} (\mathrm{Sq}^0)^m(z).$$

It is called the *canonical logarithm* of the universal formal group law. In the following we show that ℓ is indeed a logarithm.

Obviously we have

$$\mathrm{Sq}^0(\ell) = \ell. \tag{16}$$

The commutative diagram (6) yields the commutative diagram

$$\begin{array}{ccc} L[[z]] & \xrightarrow{\mathrm{Sq}^0} & R[[z]] \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow \\ L[[x,y]] & \xrightarrow{\mathrm{Sq}^0} & R[[x,y]]. \end{array} \tag{17}$$

with $\alpha(z) = F(x,y)$. Let $\beta_x\colon L[[x]] \to L[[x,y]]$, $\beta_y\colon L[[y]] \to L[[x,y]]$ be the inclusions. Further let

$$I = \langle x, y\rangle L[[x,y]].$$

Lemma 8. *Let $\ell \in L[[z]]$ with $\deg \ell = 1$ and let $\ell' = \mathrm{Sq}^0(\ell)$.*

Let further $n \geq 0$ and suppose that

$$\alpha(\ell) = \beta_x(\ell) + \beta_y(\ell) \mod I^{n+1}.$$

Then

$$\alpha(\ell') = \beta_x(\ell') + \beta_y(\ell') \mod I^{2n+1}.$$

Proof. This follows from Proposition 5 and the fact that α, β_x, β_y are compatible with Sq^0. □

Corollary 9. $\ell\big(F(\ell^{-1}(x), \ell^{-1}(y)\big) = x + y$.

This clear from the Lemma and (16).

Remark. For the moment we have completely omitted a further description of the logarithm. It is desirable to give the formal analogy of the description of the coefficients of the logarithm in [8].

References

[1] T. Bröcker and T. tom Dieck, *Kobordismentheorie*, Springer-Verlag, Berlin, 1970, Lecture Notes in Mathematics, Vol. 178.

[2] S. Buoncristiano and D. Hacon, *An elementary geometric proof of two theorems of Thom*, Topology **20** (1981), no. 1, 97–99.

[3] ———, *The geometry of Chern numbers*, Ann. of Math. (2) **118** (1983), no. 1, 1–7.

[4] ———, *A geometrical approach to characteristic numbers*, Geometry today (Rome, 1984), Birkhäuser Boston, Boston, MA, 1985, Edited and with a preface by E. Arbarello, C. Procesi and E. Strickland, pp. 21–38.

[5] ———, *On the formal groups approach to cobordism*, Proc. Amer. Math. Soc. **97** (1986), no. 3, 559–564.

[6] K. Jänich, *On the classification of $O(n)$-manifolds*, Math. Ann. **176** (1968), 53–76.

[7] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56 (1971).

[8] K. Shibata, *A note on the formal group law of unoriented cobordism theory*, Osaka J. Math. **10** (1973), 33–42.

School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA

E-mail address: rost@mathematik.uni-regensburg.de

URL: http://www.physik.uni-regensburg.de/~rom03516