A (MOD 3) INVARIANT FOR EXCEPTIONAL JORDAN ALGEBRAS

MARKUS ROST

Abstract — We sketch the construction of an invariant in $H^3(F,\mathbf{Z}/3)$ for simple exceptional Jordan algebras over a field F with $\operatorname{Char} F \neq 2, 3$. This invariant vanishes if and only if the algebra has zero divisors. In case $\mu_3 \subset F^*$ the invariant can be lifted to Milnor's K-group $K_3^M F/3$.

Un invariant (mod 3) pour les algèbres de Jordan exceptionelles

Résumé — *Nous décrivons la construction d'une invariante dans* $H^3(F,\mathbf{Z}/3)$ *pour des algèbres de Jordan simples exceptionelles sur un corps* F *de caracteristique* $\neq 2, 3$. *Cette invariante est* 0 *si et seulement si l'algèbre a des diviseurs des zero. Lorsque* F *contient les racine cubiques de l'unité, cet invariante provient d'un invariant à valeurs dans le groupe de Milnor* $K_3^M F/3$.

Version Française Abrégée — Soit F un corps de caractéristique $\neq 2, 3$. Soit J une F-algèbre de Jordan centrale simple exceptionnelle, de dimension 27. Nous décrivons la construction d'un *invariant* (à isomorphisme près) $g(J) \in H^3(F, Z/3)$. Cet invariant est 0 si et seulement si J a des diviseurs de zéro. Le foncteur $J \mapsto g(J)$ définit une application

$$H^1(F,G) \to H^3(F,\mathbf{Z}/3),$$

oú G est le groupe des automorphismes de l'algèbre de Jordan exceptionnelle décomposée (*i.e.* un groupe de type F_4 déployé). L'éxistence d'une telle application avait été conjecturée par J.-P. Serre [3].

Cet invariant a des relations intéressantes avec le symbole galoisien

$$h\colon K_3^M F/3 \to H^3(F,\mu_3^{\otimes 3}).$$

Lorsque F^* contient μ_3 on peut relever g à $K_3^M F/3$, et en déduire que l'application h ci-dessus est *injective* sur le sous-ensemble des symboles de $K_3^M F/3$.

Dans ce qui suit, nous donnons le principe de la construction de g ; un exposé plus détaillé est en préparation.

On considère d'abord des couples (J, B), oú B est une sous-algèbre de Jordan de J, de dimension 9, isomorphe à $M_3(F)^+$ sur une extension convenable de F. Par une analyse de la ≪ première construction de Tits ≫, on obtient un invariant de couples à isomorphisme près :

$$g(J,B) \in H^3(F,\mathbf{Z}/3).$$

Soit alors

$$g(J) = g(J \otimes H, B_0) \in H^3(H,\mathbf{Z}/3),$$

The text appeared as: C. R. Acad. Sci. Paris Sér. I Math. 313 (1991), no. 12, 823–827.

oú B_0 est une sous-algèbre générique de dimension 9 ayant le corps H pour corps de définition. On montre que $\partial_v\big(g(J)\big) = 0$ pour tous les homomorphismes bords ∂_v attachés aux valuations discrètes v de H triviales sur F. Comme on peut supposer que H est une extension transcendante pure de F, cette propriété entraîne que $g(J)$ appartient au sous-groupe $H^3(F, \mathbf{Z}/3)$ de $H^3(H, \mathbf{Z}/3)$. Par spécialisation, on en déduit que $g(J) = g(J, B)$ pour tout B, d'oú l'invariant cherché.

1. Introduction

Let F be a field of characteristic $\neq 2, 3$. Let J be an exceptional Jordan algebra over F (we assume throughout J to be central simple and finite dimensional, so that $\dim J = 27$). We sketch the construction of an invariant up to isomorphism:

$$g(J) \in H^3(F, \mathbf{Z}/3).$$

Its basic property is that $g(J) = 0$ if and only if J has zero divisors. g gives rise to a functorial map

$$H^1(F, G) \to H^3(F, \mathbf{Z}/3)$$

where G is the automorphism group of the split exceptional Jordan algebra. G is of type F_4 and is isomorphic to the automorphism group of the cubic form in 26 variables

$$\widetilde{N}(x, y, z) = \det x + \det y + \det z - \operatorname{trace} xyz$$

where x, y, z are 3×3 matrices with $\operatorname{trace} x = 0$ ([1], Chapt. VI, Theorem 7). The invariant is also of interest for questions related with the Galois symbol

$$h\colon K_3^M F/3 \to H^3(F, \mu_3^{\otimes 3}).$$

In case $\mu_3 \subset F^*$ the invariant can be lifted to $K_3^M F/3$ and one can use this to show that h is injective when restricted to the subset of symbols in $K_3^M F/3$.

In this note we just describe briefly the principle of the construction of g; a more detailed paper is in preparation.

The existence of the invariant g has been suggested by J.-P. Serre [3]. I am indebted to J.-P. Serre for pointing out the relation between exceptional Jordan algebras and 3-symbols mod 3 and to T. A. Springer for introducing me to the theory of Jordan algebras.

2. Tits' first construction

As general reference for Jordan algebras within this Note *see* [1], [4]. For an associative algebra A one denotes by A^+ the Jordan algebra with the same underlying space with Jordan product

$$a\,.\,b = \frac{1}{2}(ab + ba).$$

Let F be a field ($\operatorname{Char} F \neq 2, 3$), let A be a central simple algebra over F with $\dim_F A = 9$ and let $\mu \in F^*$. In Tits' first construction one defines an exceptional central simple Jordan algebra (A, μ) with underlying space $A_0 \oplus A_1 \oplus A_2$ by an explicit multiplication table, where the A_i are three copies of A (*see* [1], Ch. IX, Sect. 12). This multiplication provides A_0 with the Jordan product of A^+, so that

A^+ is a canonical Jordan subalgebra of (A,μ). The trace T_J and the norm N_J on $J=(A,\mu)$ are given by

$$T_J(a_0,a_1,a_2)=T_A(a_0),$$
$$N_J(a_0,a_1,a_2)=N_A(a_0)+\mu N_A(a_1)+\mu^{-1}N_A(a_2)-T_A(a_0a_1a_2),$$

where T_A and N_A are the (reduced) trace and norm of A respectively ([4], 5.10).

The algebra (A,μ) is an exceptional Jordan algebra over F. It has zero divisors if and only if its normform has a nontrivial zero and if and only if $\mu\in N_A(A^*)$ ([1], p. 227 and p. 416).

We consider pairs (J,B) where J is an exceptional Jordan algebra over F and $B\subset J$ is a Jordan subalgebra which is isomorphic to $M_3(F)^+$ over the algebraic closure of F. Such subalgebras always exist ([1], Ch. IX, Lemma 2). One knows that over a separably closed field all such pairs are isomorphic ([1], Ch. IX, in particular Theorem 3). In the following we give an explicit recipe how to identify any (J,B) with some $\big((A,\mu),A^+\big)$ at least after replacing the base field by a quadratic extension. Let

$$B^\perp=\{\,x\in J\mid T_J(b\,.\,x)=0 \text{ for } b\in B\,\}.$$

Then $B\,.\,B^\perp=B^\perp$ and $J=B\oplus B^\perp$ ([1], p. 227 and p. 240). We define $C\subset \mathrm{End}_F(B^\perp)$ as the subalgebra generated by all endomorphisms

$$B^\perp\to B^\perp,\quad x\mapsto b\,.\,x$$

with $b\in B$.

Proposition 1. *The centre K of C is a separable quadratic extension of F and C is an Azumaya algebra over K of rank 9.*

We omit a proof here. However note that it suffices to consider a separably closed field F in which case (J,B) is isomorphic to the explicit model

$$\big((M_3(F),1),M_3(F)^+\big).$$

A similar remark holds for proposition 2 below.

Let $p_K\in K\otimes K$ be the canonical idempotent, that is $(1-p_K)K\otimes K$ is the kernel of the multiplication map $K\otimes K\to K$. Then $p_K\in K\otimes K\subset \mathrm{End}_K(B^\perp\otimes K)$ and there is a decomposition

$$J\otimes K=B\otimes K\oplus p_K(B^\perp\otimes K)\oplus(1-p_K)(B^\perp\otimes K).$$

This can be identified with the decomposition occuring in Tits' first construction as follows.

Proposition 2. *There exist $Z\in p_K(B^\perp\otimes K)$ such that $\lambda=N_{J\otimes K}(Z)\in K$ is a unit. For any such Z one has*

(i) $p_K(B^\perp\otimes K)=C\,.\,Z$ *and* $N_{J\otimes K}\big(p_K(B^\perp\otimes K)\big)=N_C(C^*)\,.\,\lambda$.

(ii) *There is a unique isomorphism*

$$\psi\colon J\otimes K\to(C,\lambda)$$

of Jordan algebras over K with $\psi(B\otimes K)=C^+$ such that $\psi(Z)=(0,1,0)$ and $\psi(b)=\varphi(b)$ for $b\in B$, where

$$\varphi\colon B\to C\subset End_F(B^\perp)$$

is given by $\varphi(b)(x)=\big(T_J(b)-2b\big)\,.\,x.$ □

3. The invariant $g(J)$

As usual let

$$(c) \in H^1(F,\mu_3) = F^*/(F^*)^3 \quad \text{and} \quad [A] \in H^2(F,\mu_3) \subset Br(F)$$

be the classes of $c \in F^*$ and a 9-dimensional algebra A respectively. Note that $\mu_3 \otimes \mu_3 = \mathbf{Z}/3$ as Galois module.

Theorem. *There is an invariant up to isomorphism*

$$g(J) \in H^3(F,\mathbf{Z}/3)$$

which is natural with respect to base change and such that

$$g\big((A,\mu)\big) = [A] \cup (\mu).$$

By a result of A. S. Merkuriev and A. A. Suslin [2] one has $[A] \cup (\mu) = 0$ if and only if $\mu \in N_A(A^*)$. Moreover a cubic form has a nontrivial zero if this holds over a quadratic extension. From this and the remarks above it follows that $g(J) = 0$ if and only if J has zero divisors.

A proof of the theorem can be briefly sketched as follows. One first defines a relative invariant for the pairs (J,B) by the formula

$$g(J,B) = -\operatorname{cor}_{K/F}\big([C] \cup (\lambda)\big) \in H^3(F,\mathbf{Z}/3)$$

where C, K, λ are as in proposition 1 and proposition 2. This invariant is welldefined by proposition 2 (i). One can show that $[C]\cup(\lambda)$ is invariant under $\operatorname{Aut}(K/F)$ and therefore

$$[C] \cup (\lambda) = g(J,B)_K.$$

For a pair obtained from Tits' first construction it turns out that

$$g\big((A,\mu),A^+\big) = [A] \cup (\mu).$$

The invariant $g(J,B)$ does not depend on the choice of B. To prove this one uses the following two lemmas.

Lemma 1. *For A, μ as above let Σ be the norm variety over F with coordinates $x \in A$ and equation $N_A(x) = \mu$. Then the restriction map*

$$r\colon H^2(F,\mu_3) \to H^2(F(\Sigma),\mu_3)$$

is injective.

Proof (*see* also [5]). We may assume

$$\mu_3 \subset F^* \quad \text{and} \quad [A] = (\lambda) \cup (\rho)$$

in $H^2(F,\mu_3) \simeq H^2(F,\mathbf{Z}/3)$ for some λ, $\rho \in F^*$. Let $\nu = \lambda^k\rho^\ell$ with $(k,\ell) \notin 3\mathbf{Z} \times 3\mathbf{Z}$ and let Ω be the Brauer-Severi variety for the algebra with invariant $(\nu)\cup(\mu)$. Then Σ has a $F(\Omega)$-rational point. Hence the kernel of r is contained in the kernel of

$$H^2(F,\mu_3) \to H^2\big(F(\Omega),\mu_3\big)$$

which is generated by $(\nu) \cup (\mu)$. However the intersection of these groups over all ν is trivial. □

Lemma 2. *Let H/F be an extension of fields and let $v\colon H^* \to \mathbf{Z}$ be a discrete valuation trivial on F. Then for J over F and a pair $(J \otimes H, B)$ over H the element $g(J \otimes H, B)$ vanishes under the boundary map*

$$\partial_v\colon H^3(H, \mathbf{Z}/3) \to H^2\big(\kappa(v), \mu_3\big).$$

Proof. We may assume that $J = (A, \mu)$ after replacing F by some quadratic extension, see proposition 2. By lemma 1 we may replace F by $F(\Sigma)$. But then J has zero divisors and $g(J \otimes H, B) = 0$. □

Now let X, Y be two independent generic elements of J; their field of definition H is rational in a doubled set of coordinate functions on J. Let $B_0 \subset J \otimes H$ be the Jordan subalgebra generated by X and Y. Then $(J \otimes H, B_0)$ is a pair over H as above ([1], Ch. IX, Lemma 2). Since H is rational over F it follows from lemma 2 that $g(J \otimes H, B_0)$ is in the image of the injective restriction map

$$H^3(F, \mathbf{Z}/3) \to H^3(H, \mathbf{Z}/3).$$

Now take for $g(J)$ the preimage of $g(J \otimes H, B_0)$. By construction $g(J)$ is an invariant of J up to isomorphism. Using an appropriate specialization one finds $g(J) = g(J, B)$ for any choice of B.

References

[1] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications XXXIX, Amer. Math. Soc., Providence, R. I., 1968.

[2] A. S. Merkurjev and A. A. Suslin, K-Cohomology of Severi-Brauer varieties and the norm residue homomorphism, *Izv. Akad. Nauk S.S.S.R.*, 46, 1982, pp. 1011–1046 (=*Math. U.S.S.R. Izvestiya*, 21, 1983, pp. 307-340).

[3] J.-P. Serre, *Résumé des cours de l'année* 1990–1991, Collège de France.

[4] T. A. Springer, *Jordan algebras and algebraic groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 75, Springer-Verlag, 1973.

[5] A. A. Suslin, K-Theory and K-cohomology of certain group varieties, *Algebraic K-theory*, Adv. Soviet Math. 4, Amer. Math. Soc., Providence, R. I., 1991, pp. 53–74

NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: `markus.rost@mathematik.uni-regensburg.de`
URL: `http://www.physik.uni-regensburg.de/~rom03516`