THE MOTIVE OF A PFISTER FORM

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Contents

Introduction

This is a draft which deserves some reorganization and supplements. However it contains all details concerning the construction and properties of the motive.

Unfortunately only very late I noticed that Proposition 1 and Proposition 9 with Corollaries should have been formulated for motives instead only for varieties. But this may be complemented without much difficulty.

1. Correspondences

In this section we recall some basic facts about correspondences and Grothendieck motives. The basic reference is [1], in particular [1, Example 16.1.12].

Let k be a fixed ground field.

1.1. Algebra of correspondences. By a variety X we understand a separated scheme of finite type over k. We denote by $\mathrm{CH}_p(X)$ the Chow group of p-dimensional cycles on X. If X is smooth, we denote by $\mathrm{CH}^p(X)$ the Chow group of p-codimensional cycles on X. In case X is irreducible, one has $\mathrm{CH}^p(X)=\mathrm{CH}_{d-p}(X)$ where $d=\dim X$.

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Let $\overline{\mathcal{V}}$ be the following additive category: The objects of $\overline{\mathcal{V}}$ are the smooth proper varieties over k. The morphism groups in $\overline{\mathcal{V}}$ are

$$\mathrm{Hom}_{\overline{\mathcal{V}}}(X,Y)=\mathrm{CH}_*(X\times Y)=\oplus_p\,\mathrm{CH}_p(X\times Y).$$

The composition law in $\overline{\mathcal{V}}$ is given by the composition of correspondences:

$$\mathrm{CH}_*(X\times Y)\times\mathrm{CH}_*(Y\times Z)\to\mathrm{CH}_*(X\times Z),$$
$$(f,g)\mapsto g\circ f=\pi_*\big(\Delta^*(f\times g)\big),$$

with

$$\Delta\colon X\times Y\times Z\to X\times Y\times Y\times Z,\qquad \Delta(x,y,z)=(x,y,y,z),$$
$$\pi\colon X\times Y\times Z\to X\times Z,\qquad \pi(x,y,z)=(x,z).$$

The *transpose* of $f\in\mathrm{Hom}_{\overline{\mathcal{V}}}(X,Y)$ is $f^t=\tau_*(f)\in\mathrm{Hom}_{\overline{\mathcal{V}}}(Y,X)$ where $\tau\colon X\times Y\to Y\times X$, $\tau(x,y)=(y,x)$. The category $\overline{\mathcal{V}}$ is endowed with the duality ${}^t\colon\overline{\mathcal{V}}\to\overline{\mathcal{V}}^{\mathrm{op}}$ given by $X^t=X$ on objects and by the transpose on morphisms.

One has $\mathrm{Hom}_{\overline{\mathcal{V}}}(\mathrm{Spec}\,k,X)=\mathrm{CH}_*(X)$. This way $X\mapsto\mathrm{CH}_*(X)$ defines a functor on $\overline{\mathcal{V}}$ (by composition of morphisms).

Let $\mathcal{V}=\mathcal{V}(k)$ be the following subcategory of $\overline{\mathcal{V}}$: The objects of $\mathcal{V}$ are the objects of $\overline{\mathcal{V}}$. The morphism groups in $\mathcal{V}$ are

$$\mathrm{Hom}(X,Y)=\oplus_i\,\mathrm{CH}_{\dim X_i}(X_i\times Y)=\oplus_j\,\mathrm{CH}^{\dim Y_j}(X\times Y_j)\subset\mathrm{Hom}_{\overline{\mathcal{V}}}(X,Y).$$

Here X_i resp. Y_j are the connected components of X resp. Y. The composition law in $\mathcal{V}$ is induced from the composition law in $\overline{\mathcal{V}}$.

We write

$$\mathrm{End}(X)=\mathrm{Hom}(X,X)$$

for the endomorphism ring of X. If X is irreducible, then

$$\mathrm{End}(X)=\mathrm{CH}_d(X\times X)=\mathrm{CH}^d(X\times X),\quad d=\dim X.$$

The assignments $X\mapsto\mathrm{CH}_p(X)$ resp. $X\mapsto\mathrm{CH}^p(X)$ are covariant resp. contravariant functors on $\mathcal{V}$. For $f\in\mathrm{Hom}(X,Y)$ we denote the associated maps by $f_*\colon\mathrm{CH}_p(X)\to\mathrm{CH}_p(Y)$ resp. $f^*\colon\mathrm{CH}^p(Y)\to\mathrm{CH}^p(X)$.

For a morphism $f\colon X\to Y$ of varietes over k we denote by the same letter the class of its graph:

$$f=[\mathrm{Graph}(f)]\in\mathrm{Hom}(X,Y).$$

In this case the maps f_* resp. f^* are the standard push forward resp. pull back maps.

1.2. Motives. By a *motive* we understand a pair (X,p) with X an object in $\mathcal{V}$ and $p\in\mathrm{End}(X)$ a projector: $p\circ p=p$. The Chow groups of (X,p) are defined by

$$\mathrm{CH}_r\big((X,p)\big)=p_*\big(\mathrm{CH}_r(X)\big),$$
$$\mathrm{CH}^r\big((X,p)\big)=p^*\big(\mathrm{CH}^r(X)\big).$$

The category $\mathcal{M}$ is defined as follows: Its objects are the pairs (X,p). Its morphism groups are

$$\mathrm{Hom}\big((X,p),(Y,q)\big)=q\circ\mathrm{Hom}(X,Y)\circ p\subset\mathrm{Hom}(X,Y).$$

The composition of morphisms in $\mathcal{M}$ is induced from the composition law in $\mathcal{V}$.

Associating to X the motive (X, id_X) identifies $\mathcal{V}$ as a full subcategory of $\mathcal{M}$. We simply write $X = (X, \mathrm{id}_X)$. The functors $M \mapsto \mathrm{CH}_p(M)$ resp. $M \mapsto \mathrm{CH}^p(M)$ extend the corresponding functors on $\mathcal{V}$.

The sum and the product in $\mathcal{M}$ (and in $\mathcal{V}$ and $\overline{\mathcal{V}}$) are defined by disjoint union and product:

$$(X,p) \oplus (Y,q) = (X \cup Y, p+q),$$
$$(X,p) \otimes (Y,q) = (X \times Y, p \times q).$$

1.3. The Tate motive. The endomorphism ring of $\mathbb{P}^1$ is

$$\mathrm{End}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z},$$

generated by the cycle classes $p_0 = [\mathbb{P}^1 \times P]$ and $p_1 = [P \times \mathbb{P}^1]$ with P a k-point on $\mathbb{P}^1$. The endomorphisms p_0, p_1 are orthogonal projectors. One has $(\mathbb{P}^1, p_0) = \operatorname{Spec} k$.

The *Tate motive* is the motive $L = (\mathbb{P}^1, p_1)$. For $i \geq 0$ we denote by $L^{\otimes i}$ the i-fold product $L \otimes \cdots \otimes L$ with $L^{\otimes 0} = \operatorname{Spec} k$. One has

$$\mathrm{CH}_p\big(M \otimes L^{\otimes i}\big) = \mathrm{CH}_{p-i}(M)$$

If X is irreducible, then

$$\mathrm{Hom}(X \otimes L^{\otimes i}, Y \otimes L^{\otimes j}) = \mathrm{CH}_{(\dim X)+i-j}(X \times Y)$$

For projective spaces there is the following decomposition in $\mathcal{M}$:

$$\mathbb{P}^n = \oplus_{i=0}^n L^{\otimes i}.$$

1.4. Splitting off a point. Let X be equidimensional of dimension d. We denote by $[X] \in \mathrm{CH}_d(X)$ the fundamental class of X.

Let

$$P \in \coprod_{x \in X_{(0)}} \mathbb{Z}$$

be a zero cycle. The classes

$$[P] \in \mathrm{CH}_0(X) = \mathrm{Hom}(L^{\otimes 0}, X) = \mathrm{Hom}(X, L^{\otimes d}),$$
$$[X] \in \mathrm{CH}_d(X) = \mathrm{Hom}(L^{\otimes d}, X) = \mathrm{Hom}(X, L^{\otimes 0}),$$

define morphisms

$$L^{\otimes 0} \underset{g_X}{\overset{f_P}{\rightleftarrows}} X, \qquad L^{\otimes d} \underset{g_P}{\overset{f_X}{\rightleftarrows}} X.$$

If P has degree 1, one has $g_X \circ f_P = \mathrm{id}_{L^{\otimes 0}}$ and $g_P \circ f_X = \mathrm{id}_{L^{\otimes d}}$. Then $f_P \circ g_X$, $f_X \circ g_P \in \mathrm{End}(X)$ are projectors. If additionally $d > 0$, these projectors are orthogonal. They identify $L^{\otimes 0} \oplus L^{\otimes d}$ as a direct summand of X and we have a decomposition

$$X = L^{\otimes 0} \oplus (X, \pi_P) \oplus L^{\otimes d} \tag{1}$$

with $\pi_P = \mathrm{id}_X - f_P \circ g_X - f_X \circ g_P$.

Suppose that the degree map

$$\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$$

is bijective. Then the correspondences f_P, g_P, and π_P do not depend on the choice of P. Moreover we have a canonical decomposition

$$\mathrm{End}(X) = \mathrm{End}(L^{\otimes 0}) \oplus \mathrm{End}\big((X, \pi_P)\big) \oplus \mathrm{End}(L^{\otimes d}).$$

1.5. Chow groups of fibrations. Let X, B be smooth and proper varieties over k and let $\pi\colon B \times X \to B$ be the projection. For $b \in B$ we write $X_b = \operatorname{Spec} \kappa(b) \times X$ for the fibre over b. Moreover, for $f \in \operatorname{End}(X)$ we denote by $f_b \in \operatorname{End}_{\mathcal{V}(\kappa(b))}(X_b)$ the element obtained by base change $k \to \kappa(b)$.

We will need the following observation:

Proposition 1. *Let $f \in \operatorname{End}(X)$ and suppose that*

$$(f_b)_*\big(\mathrm{CH}_p(X_b)\big) = 0$$

for all $b \in B$ and $0 \le p \le \dim B$. Then

$$f^{(1+\dim B)} \circ \operatorname{Hom}(B, X) = 0.$$

(Here the power of f is taken in the ring $\operatorname{End}(X)$.)

Proof. We use the setting of [3]. For simplicity we assume that X and B are irreducible of dimensions $d = \dim X$ and $e = \dim B$.

First note that the map $\operatorname{Hom}(B, X) \to \operatorname{Hom}(B, X)$, $g \mapsto f \circ g$ is given by the composite of the maps

$$\mathrm{CH}_p(B \times X) \xrightarrow{\times f} \mathrm{CH}_{p+d}(B \times X \times X \times X), \tag{2}$$

$$\mathrm{CH}_{p+d}(B \times X \times X \times X) \xrightarrow{\Delta^*} \mathrm{CH}_p(B \times X \times X), \tag{3}$$

$$\mathrm{CH}_p(B \times X \times X) \xrightarrow{g_*} \mathrm{CH}_p(B \times X), \tag{4}$$

with $\Delta(b, x, y) = (b, x, x, y)$ and $g(b, x, y) = (b, y)$.

Similarly, after replacing B by $\operatorname{Spec} \kappa(b)$ and f by f_b, the composition of these maps yield the action of f_b on $\mathrm{CH}_p\big(X_b\big)$.

Let $Z = X$, $X \times X$, or $X \times X \times X$. The projections $\pi\colon B \times Z \to B$ induce spectral sequences

$$E^2_{p,q} = A_p\big(B, A_q[\pi, M]\big) \Longrightarrow A_{p+q}(B \times Z, M), \tag{5}$$

see [3, Sect. 8]. Here M can be any cycle module in the sense of [3]; for our purpose we may restrict to the case $M = K^{\mathrm{M}}_*$ given by Milnor's K-theory.

The maps (2), (3), and (4) extend to the Chow groups $A_p(B \times Z, M)$ and, similarly, to the groups $A_p(Z_b, M)$. Moreover they are compatibel with the spectral sequences (5). For the product map $\times f$ this is fairly obvious from the definitions, see [3, Sect. 14]. For the pull back map Δ^* see [3, Theorem 12.7]. For the push forward map g_* see [3, Proposition 8.5.1].

The filtration on the Chow groups of $B \times X$ corresponding to the spectral sequence (5) for $Z = X$ is

$$\mathrm{CH}_{p,0}(\pi) \subset \mathrm{CH}_{p,1}(\pi) \subset \cdots \subset \mathrm{CH}_p(B \times X)$$

where $\mathrm{CH}_{p,r}(\pi) \subset \mathrm{CH}_p(B \times X)$ is the subgroup generated by the classes of cycles $V \subset Z$ with $\dim \pi(V) \le r$.

Consider the quotients

$$\overline{\mathrm{CH}}_{p,r}(\pi) = \mathrm{CH}_{p,r}(\pi) / \mathrm{CH}_{p,r-1}(\pi)$$

The direct summands $\widetilde{E}^2_{r,p-r}$ of the groups $E^2_{r,p-r}$ corresponding to the quotients $\overline{\mathrm{CH}}_{p,r}(\pi)$ are the cokernels of the divisor maps

$$\coprod_{b \in B_{(r+1)}} A_{p-r}(X_b, K^{\mathrm{M}}_*, 1-p+r) \xrightarrow{d} \coprod_{b \in B_{(r)}} \mathrm{CH}_{p-r}(X_b).$$

The group $\overline{\mathrm{CH}}_{p,r}(\pi)$ is a quotient of $\widetilde{E}^2_{r,p-r}$. Therefore f acts trivially on $\overline{\mathrm{CH}}_{p,r}(\pi)$ as long as f_b acts trivially on $\mathrm{CH}_{p-r}(X_b)$ for all $b \in B$.

Since the filtration on $\mathrm{CH}_p(B \times X)$ has length $1+e$ and since f acts trivially on the filtration quotients, it follows that the action of f on $\mathrm{CH}_p(B \times X)$ is nilpotent of order $1+e$. □

2. Isotropic and split quadrics

For the remaining parts of this article, the characteristic of the ground field k is different from 2. For generalities on quadratic forms and in particular Pfister forms, see [2, 4].

For a quadratic form φ we denote by X_φ the associated projective quadric. If not mentioned otherwise, we assume φ to be regular so that X_φ is smooth. One has $\dim X_\varphi = \dim \varphi - 2$.

2.1. Motives of quadrics and Witt equivalence. We call a quadric X_φ *isotropic* if φ is isotropic. A quadric X is isotropic if and only if X has a k-rational point.

The following proposition shows that the motive of a quadric X_φ depends essentially (up to elementary operations involving the Tate motive) on the class of φ in the Witt ring of k.

Proposition 2. *Let $\varphi = \mathbb{H} \perp \psi$ where $\mathbb{H}$ is a hyperbolic plane and let $X = X_\varphi$ and $Y = X_\psi$. Then*

$$X = L^{\otimes 0} \oplus Y \otimes L \oplus L^{\otimes d}$$

where $d = \dim X$.

The case $d = 0$ is easily verified and therefore we may assume $d > 0$. We choose coordinates (u, v, y) such that $\varphi(u, v, y) = uv + \psi(y)$.

The proof of Proposition 2 is based on the stratification of X given by the following data:

$$\begin{aligned} &Z = \{\, u = 0 \,\} \subset X, \\ &Z' = Z \setminus P, \quad P = [0, 1, 0], \\ &r\colon Z' \to Y, \quad r\big([0, v, y]\big) = [y]. \end{aligned}$$

Then $X \setminus Z \simeq \mathbb{A}^d$ with $d = \dim X$, the point P is the singular point of Z, and r is a 1-dimensional vector bundle.

Suppose for a moment that we work in a category of motives which include the motives of arbitrary (possibly non smooth and non proper) varieties. Then the data above immediately give the desired decomposition of the motive of X: The decomposition $X = Z \cup \mathbb{A}^d$ indicates that $X = Z \oplus L^{\otimes d}$. Similarly one would have $Z = L^{\otimes 0} \oplus Z'$. Furthermore, since r is a 1-dimensional vector bundle, homotopy invariance gives $Z' = Y \otimes L$.

Constructions of such a category of motives have been announced by Morel and Voevodsky, and (in a different manner by extending the framework of [3] to a bivariant theory of cycles) by the author. However, for the sake of completeness, we continue here to work in the category $\mathcal{M}$. The arguments are of a completely formal nature starting from the described data.

Let

$$S \subset Y \times X$$

be the closure of the image of the morphism

$$Z' \xrightarrow{(r,i)} Y \times X$$

where $i\colon Z' \to X$ is the inclusion. The cycle class $f = [S] \in \mathrm{CH}_{d-1}(Y \times X)$ defines a morphism

$$Y \otimes L \xrightarrow{f} X.$$

Recall the decomposition (1). One has $g_X \circ f = 0$ and $g_P \circ f = 0$ by dimension reasons. Therefore f defines actually a morphism

$$Y \otimes L \xrightarrow{\hat f} (X, \pi_P).$$

By Manin's identity principle [1, Example 16.1.12], $\hat f$ is an isomorphism, if for any B in $\mathcal{V}$ and $p \geq 1$ the map

$$(\mathrm{id}_B \times \hat f)_*\colon\ \mathrm{CH}_{p-1}(B \otimes Y) \to \mathrm{CH}_p\bigl(B \otimes (X, \pi_P)\bigr)$$

is bijective. This follows from the following two Lemmata.

Lemma 3. *Let $X' = X \setminus P$ and let $i\colon Z' \to X'$, $j\colon X' \to X$ be the inclusions. The following diagram is commutative*:

$$\begin{array}{ccc}
\mathrm{CH}_{p-1}(B \otimes Y) & \xrightarrow{f_*} & \mathrm{CH}_p(B \otimes X) \\
{\scriptstyle r^*}\big\downarrow & & \big\downarrow{\scriptstyle j^*} \\
\mathrm{CH}_p(B \otimes Z') & \xrightarrow{i_*} & \mathrm{CH}_p(B \otimes X')
\end{array}$$

The pull back map r^* is bijective.

Proof. The bijectivity of r^* follows from the fact that r is a 1-dimensional vector bundle.

Let $S' = S \cap (Y \times X')$. The scheme S' is just the image of the closed immersion $Z' \to Y \times X'$.

The map $j^* \circ f_*$ is given by taking cross product with X', intersecting with $B \times S'$ and taking push forward along Y. Now $S' \subset Y \times Z'$ and $Z' \subset X'$ is a smooth hyperplane. Therefore we can also take first cross product with Z', intersect with $B \times S'$ inside $B \times Y \times Z'$, take push forward along Y (this yields the pull back map r^*, since S' is the transpose of the graph of r) and finally take the direct image with i_*. $\square$

Lemma 4. *There is an isomorphism $\rho\colon \mathrm{CH}_p\bigl(B \otimes (X, \pi_P)\bigr) \to \mathrm{CH}_p(B \otimes Z')$ making the following diagram commutative:*

$$\begin{array}{ccc}
\mathrm{CH}_p\bigl(B \otimes (X, \pi_P)\bigr) & \xrightarrow{\text{inclusion}} & \mathrm{CH}_p(B \otimes X) \\
{\scriptstyle \rho}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle j^*} \\
\mathrm{CH}_p(B \otimes Z') & \xrightarrow{i_*} & \mathrm{CH}_p(B \otimes X')
\end{array}$$

The map i_* is injective.

Proof. The map j^* fits into an exact sequence

$$\mathrm{CH}_p(B \times P) \xrightarrow{(i_P)_*} \mathrm{CH}_p(B \times X) \xrightarrow{j^*} \mathrm{CH}_p(B \times X') \to 0$$

The push forward map $(i_P)_*$ is injective, since the projection $\lambda_X : X \to \operatorname{Spec} k$ gives a left inverse.

The map i_* fits into a long exact sequence (see [3, Sect. 5])

$$\cdots \to A_{p+1}(B \times X', K_*^{\mathrm{M}}, -p) \xrightarrow{\ell^*} A_{p+1}(B \times \mathbb{A}^d, K_*^{\mathrm{M}}, -p) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathrm{CH}_p(B \times Z') \xrightarrow{i_*} \mathrm{CH}_p(B \times X') \xrightarrow{\ell^*} \mathrm{CH}_p(B \times \mathbb{A}^d) \to 0$$

Here the maps ℓ^* are induced from the inclusion $\ell \colon \mathbb{A}^d \to X'$.

Let $\lambda \colon \mathbb{A}^d \to \operatorname{Spec} k$ be the projection. The pullback maps $\lambda^* \colon A_{p-d}(B) \to A_p(B \times \mathbb{A}^d)$ are isomorphims. The maps $\lambda_X^* \circ (\lambda^*)^{-1}$ are left inverses to ∂. Hence the long exact sequence splits up into short exact sequences.

Putting things together yields a decomposition

$$\mathrm{CH}_p(B \times X) = \mathrm{CH}_p(B \times P) \oplus \mathrm{CH}_p(B \times Z') \oplus \mathrm{CH}_p(B \times \mathbb{A}^d)$$

It is easy to see that this decomposition coincides with the decomposition given by (1). $\square$

The proof of Proposition 2 is now complete.

One may check that the inverse of the correspondence $\hat{f}$ is given by the cycle $T = \sigma(S) \subset X \times Y$ where

$$\sigma \colon Y \times X \to X \times Y,$$
$$\sigma\big([y'], [u, v, y]\big) = \big([v, u, y], [y']\big).$$

(T is the transpose of S reflected by $u \leftrightarrow v$.)

2.2. Zero cycles on a quadric. We recall the computation of the group of zero cycles of a quadric (see [5]).

Lemma 5. *For a quadric X the degree map* $\deg \colon \mathrm{CH}_0(X) \to \mathbb{Z}$ *is injective. Its image is*

$$\deg\big(\mathrm{CH}_0(X) \to \mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \textit{if } X \textit{ is isotropic,} \\ 2\mathbb{Z} & \textit{if } X \textit{ is anisotropic.} \end{cases}$$

2.3. The Chow ring of a split quadric. The following material is well known.

For $n \geq 2$ let τ_n be the following n-dimensional quadratic form:

$$\tau_n = \begin{cases} \sum_{i=1}^m x_i y_i + z^2 & \text{if } n = 2m+1, \\ \sum_{i=1}^m x_i y_i & \text{if } n = 2m. \end{cases}$$

We call a quadratic form φ *split* if φ is similar to τ_n for some n.

Lemma 6. *A quadratic form φ $(\dim \varphi \geq 2)$ is split if and only if X_φ is a direct sum of powers of the Tate motive.*

In this case one has with $d = \dim X_\varphi$

$$X_\varphi = \begin{cases} \oplus_{i=0}^d L^{\otimes i} & \textit{if } d \textit{ is odd,} \\ \oplus_{i=0}^d L^{\otimes i} \oplus L^{\otimes (d/2)} & \textit{if } d \textit{ is even.} \end{cases}$$

Proof. Using Proposition 2 one reduces to the case when φ is anisotropic. But then X_φ has no rational point and cannot be a sum of powers of the Tate motive (cf. Lemma 5). ☐

A quadric is called *split* if it satisfies the properties of Lemma 6.

Lemma 7. *Let X be a split quadric. Then*

$$\operatorname{End}(X) = \oplus_p \operatorname{End}_{\mathbb{Z}}\big(\mathrm{CH}_p(X)\big)$$

Proof. This holds in fact for any sum of powers of the Tate motive. ☐

Lemma 8. *Let $X = X_{\tau_n}$ and $d = \dim X = n - 2$. Then*

(i) *For $p < d/2$ one has $\mathrm{CH}^p(X) = \mathbb{Z}$ generated by the class of the plane section*

$$\{\, x_1 = \cdots = x_p = 0 \,\}.$$

(ii) *If $d = 2m - 2$, then $\mathrm{CH}^{d/2}(X) = \mathbb{Z} \oplus \mathbb{Z}$ generated by the classes of the two maximal linear subspaces*

$$\{\, x_1 = \cdots = x_{m-1} = x_m = 0 \,\},$$
$$\{\, x_1 = \cdots = x_{m-1} = y_m = 0 \,\}.$$

(iii) *For $p > d/2$ one has $\mathrm{CH}^p(X) = \mathbb{Z}$ generated by the class of the linear subspace*

$$\{\, x_1 = \cdots = x_m = z = y_1 = \cdots = y_{p-m} = 0 \,\} \qquad \text{if } d = 2m - 1,$$
$$\{\, x_1 = \cdots = x_m = y_1 = \cdots = y_{p+1-m} = 0 \,\} \qquad \text{if } d = 2m - 2.$$

Proof. This well known fact follows e.g. by analyzing the proof of Proposition 2. In fact, it easy to check that the maps

$$f_*\colon\ \mathrm{CH}_{p-1}(Y) \to \mathrm{CH}_p(X)$$

sends plane sections to plane sections and linear subspaces to linear subspaces. ☐

Let $h \in \mathrm{CH}^1(X)$ and $u \in \mathrm{CH}^{(d+1)/2}(X)$, v, $w \in \mathrm{CH}^{d/2}(X)$ the generators described in Lemma 8. The Chow ring of X_{τ_n} has as a ring over $\mathbb{Z}$ the following presentation ($d > 0$):

$$\langle\, h, u \mid h^{(d+1)/2} = 2u,\ h^{d+1} = 0,\ u^2 = 0 \,\rangle \qquad \text{if } d \equiv 1 \bmod 2,$$
$$\langle\, h, u \mid h^{d/2} = v + w,\ hv = hw,\ vw = 0,\ v^2 = w^2 = h^d,\ h^{d+1} = 0 \,\rangle \quad \text{if } d \equiv 0 \bmod 4,$$
$$\langle\, h, u \mid h^{d/2} = v + w,\ hv = hw,\ vw = h^d,\ v^2 = w^2 = 0,\ h^{d+1} = 0 \,\rangle \quad \text{if } d \equiv 2 \bmod 4.$$

3. Endomorphisms of quadrics

We are now ready to prove the following proposition, which provides an important tool to construct correspondences between quadrics.

Proposition 9. *For $d \geq 0$ there exist a number $N(d)$ with the following property*:
Let X be a smooth projective quadric of dimension d, let $f \in \operatorname{End}(X)$ and let F/k be some field extension. If $f_F = 0 \in \operatorname{End}_{\mathcal{V}(F)}(X_F)$, then $f^{N(d)} = 0$.

Proof. We argue by induction on d. Let φ be a quadratic form defining X.
Suppose first that X has a k-rational point P. Then $\varphi = \mathbb{H} \perp \psi$ as in Proposition 2 and

$$\operatorname{End}(X) = [X \times P]\mathbb{Z} \oplus \operatorname{End}(X_\psi) \oplus [P \times X]\mathbb{Z}.$$

If $f_F = 0$, then necessarily $f \in \operatorname{End}(X_\psi)$, since the other summands do not change under field extensions. By induction we have $f^{N(d-2)} = 0$.

In the general case we apply Proposition 1 with $B = X$. Since all the quadrics X_b have a rational point, we may apply the previous step and see that $f_b^{N(d-2)} = 0$ for all $b \in B$. It suffices to take $N(d) = (1+d)N(d-2)$. □

Corollary 10. *Let X be a smooth projective quadric, let $f \in \operatorname{End}(X)$ and let F/k be some field extension.*

(i) *If f_F is nilpotent, then f is nilpotent.*
(ii) *If f_F is an isomorphism, then f is an isomorphism.*

Proof. (i) is clear from Proposition 9.

Suppose that f_F is an isomorphism. We may assume that X_F is a split quadric. Then, by Lemma 7, f_F is completely determined by its action on the groups $\mathrm{CH}_p(X_F) \simeq \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$. Thus f_F satisfies an equation $(t^2 - 1)(t^2 + mt \pm 1) = 0$ with $m \in \mathbb{Z}$. It follows that the inverse of f_F is an integral polynomial in f_F and therefore defined over k.

Hence we may assume that $f_F = (\mathrm{id}_X)_F$. But then (ii) is immediate from (i): If $f = 1 + g$ and g is nilpotent, then f is invertible. □

Corollary 11. *Let X, Z be quadrics and $f \in \operatorname{Hom}(X, Z)$. If F/k is a field extension such that X_F and Z_F are split and if*

$$(f_F)_*\colon\ \mathrm{CH}_p(X_F) \to \mathrm{CH}_p(Z_F)$$

is an isomorphism for all p, then f is an isomorphism.

Proof. We must have $\dim X = \dim Z$. Let $g = f^t$. Then $(g_F)_*\colon\ \mathrm{CH}_p(Z_F) \to \mathrm{CH}_p(X_F)$ is also an isomorphism for all p. By Lemma 7, the endomorphism $(f \circ g)_F$ is an isomorphism. By Corollary 10 (ii) $f \circ g$ is an isomorphism. Similarly it follows that $g \circ f$ is an isomorphism. □

Definition 12. Let X be a quadric of dimension d. We call a projector $p \in \operatorname{End}(X)$ *special*, if there exists a quadric Z of dimension $d-2$ such that

$$(X, \mathrm{id}_X - p) \simeq Z \otimes L.$$

Example: (cf. (1) and Proposition 2) If X is isotropic, then π_P is a special projector on X.

Proposition 13. *On a quadric X there exists at most one special projector.*

Proof. First suppose that X is isotropic. Then there is an isomorphism

$$(X, p) \oplus Z \otimes L \xrightarrow{\sim} (X, \pi_P) \oplus Y \otimes L$$

with Y as in Proposition 2. Since

$$\operatorname{Hom}\big(Z \otimes L, (X, \pi_P)\big) = 0$$

by dimension reasons, the induced morphism $Z \otimes L \to Y \otimes L$ has a left inverse. Hence Z is a direct summand of Y. The morphism $Z \to Y$ must be an isomorphism by Corollary 11. It follows that $(X, \pi_P) \simeq (X, p)$. Hence

$$\operatorname{Hom}\big((X, p), Y \otimes L\big) = 0$$

similar as above and the decomposition is unique.

In the general case let p, p' be two special projectors on X with quadrics Z resp. Z' and isomorphisms

$$(X,p) \oplus Z \otimes L \xrightarrow{\sim} X \xrightarrow{\sim} (X,p') \oplus Z' \otimes L.$$

Let $F = k(Z)$. Note that X_F is isotropic. Namely Z_F is isotropic and any F-rational point on Z_F corresponds to an element $u \in \mathrm{CH}_1(X_F)$ which gives in the split case a generator. Intersecting u with a hyperplane section gives an F-rational point on X_F. By the previous step we have

$$\mathrm{Hom}\big(Z_F \otimes L, (X,p)_F\big) = 0.$$

By a filtration argument one sees that $\mathrm{Hom}\big(Z \otimes L, (X,p')\big)$ is trivial. Similarly for $\mathrm{Hom}\big((X,p), Z' \otimes L\big) = 0$. Hence the decomposition is unique. □

Lemma 14. *Let p be a special projector. Then $p^t = p$.*

Proof. If $f\colon (X, \mathrm{id}_X - p) \to Z \otimes L$ is an isomorphism, then $f^t\colon Z \otimes L \to (X, \mathrm{id}_X - p^t)$ is an isomorphism. Hence p^t is special und must be equal to p. □

Lemma 15. *Let X, Z be quadrics with $\dim Z = \dim X - 2$ and let F be a field extension of k such that X, Z are split. Suppose that there exists a morphism*

$$f\colon Z \otimes L \to X$$

such that

$$(f_F)_*\colon\ \mathrm{CH}_{(p-1)}(Z_F) \to \mathrm{CH}_p(X_F)$$

is an isomorphism for all $1 < p < d$. Then there exists a special projector p on X and f induces an isomorphism

$$Z \otimes L \simeq (X,p).$$

Proof. The morphism $f^t \circ f$ is seen to be an isomorphism, by similar arguments as above. Then $\mathrm{id}_X - f \circ (f^t \circ f)^{-1} \circ f^t$ is a special projector. □

Let $p \in \mathrm{End}(X)$ be a special projector and $M = (X,p)$. We compute the endomorphism ring of M. Let F be a splitting field of X and let ($d = \dim X$)

$$\omega\colon\ \mathrm{End}(M) \to \mathrm{End}(M_F) = \mathrm{End}(L^{\otimes 0} \oplus L^{\otimes d}) = \mathbb{Z} \oplus \mathbb{Z}$$

be the natural map. Let

$$\Gamma = 2\mathbb{Z} \oplus 2\mathbb{Z} + (1,1)\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}.$$

Lemma 16. *The map ω is injective. One has*

$$\mathrm{Im}\,\omega = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } X \text{ is isotropic,} \\ \Gamma & \text{if } X \text{ is anisotropic.} \end{cases}$$

Proof. We have $(1,1) = \omega(\mathrm{id}_M) \in \mathrm{Im}\,\omega$. Moreover $2\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathrm{Im}\,\omega$. Namely X becomes isotropic over a quadratic extension F/k and then

$$\omega \circ \mathrm{cor}_{F/k}\big(\mathrm{End}(M_F)\big) = 2\mathbb{Z} \oplus 2\mathbb{Z}.$$

If $(1,0) \in \mathrm{Im}\,\omega$, there exists a zero cycle of degree 1 on X and X is isotropic. It remains to show that ω is injective.

Let $B = X$ and let $\pi\colon B \times X \to B$ be the projection to the second factor and let

$$E^2_{s,t} = A_s\big(B, A_t[\pi, K_*^{\mathrm{M}}]\big) \Longrightarrow A_{s+t}(B \times X, K_*^{\mathrm{M}}) \tag{6}$$

be the associated spectral sequence, see [3, Sect. 8]. This spectral sequence commutes with the projector p acting on X. Hence we have a spectral sequence

$$E^2_{s,t} = A_s\big(B, p_*\big(A_t[\pi, K^{\mathrm{M}}_*]\big)\big) \Longrightarrow p_*\big(A_{s+t}(B \times X, K^{\mathrm{M}}_*)\big) \tag{7}$$

Since all fibers X_b are isotropic, we have

$$p_*\big(A_t[\pi, K^{\mathrm{M}}_*]\big) = \begin{cases} 0 & \text{if } 0 < t < d, \\ K^{\mathrm{M}}_* & \text{if } t = 0,\, d. \end{cases}$$

It follows that there is a short exact sequence

$$\mathrm{CH}_0(B) \to \mathrm{CH}_d(B \otimes M) \to \mathrm{CH}_d(B) \to 0.$$

Hence $\mathrm{CH}_d(B \otimes M)$ is free of rank ≤ 2 and the same is true for the subgroup $\mathrm{End}(M)$. □

4. Construction of the motive

Let $a_n \in k^*$, $n \geq 1$ be a sequence of elements. We denote by

$$\varphi_n = \langle\langle a_1, \ldots, a_n \rangle\rangle = \otimes_{i=1}^n \langle 1, -a_i \rangle$$

the Pfister form corresponding to $a_1, \ldots, a_n$. Moreover φ'_n denotes the prime subform of φ_n defined by $\varphi_n = \langle 1 \rangle \perp \varphi'_n$. Furthermore we put $\psi_n = \varphi_{n-1} \perp \langle -a_n \rangle$. Note that

$$\psi_n = \langle 1, -a_n \rangle \perp \varphi'_n.$$

We write $X_n = X_{\psi_n}$ and $Z_n = X_{\varphi'_n}$. Let $d_n = \dim X_n = 2^{n-1} - 1$.

Theorem 17. *On the quadric X_n there exist a special projector p_n.*

Let $M_n = (X_n, p_n)$. Then

$$(X_n, \mathrm{id}_{X_n} - p_n) \simeq Z_{n-1} \otimes L \tag{8}$$

and

$$Z_n \simeq M_n \otimes \bigoplus_{i=0}^{d_n - 1} L^{\otimes i} \tag{9}$$

Proof. For $n = 1$, 2 one takes $p_n = \mathrm{id}_{X_n}$.

Step 1: We first assume that the theorem holds for n and construct p_{n+1} with property (8).

We may assume that X_n is anisotropic.

We first choose a certain element

$$\Theta \in \mathrm{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) = \mathrm{CH}_{2d_n}(M_n \otimes X_{n+1}).$$

Let $F = k(X_n)$ and consider the sequence

$$\mathrm{CH}_{2d_n}(M_n \otimes X_{n+1}) \underset{p_n}{\overset{\text{incl.}}{\rightleftarrows}} \mathrm{CH}_{2d_n}(X_n \times X_{n+1}) \xrightarrow{j} \mathrm{CH}_{d_n}\big((X_{n+1})_F\big)$$

Here j is the projection map. j is surjective (by taking closure of cycles). Since $(X_{n+1})_F$ is split, the rightmost term is $\simeq \mathbb{Z}$ generated by the class u of a maximal linear subspace.

We claim that $j(\ker p_n) = 0$. To check this, one may pass to a splitting field in which case the claim is easy to verify. Let Θ be an element which maps to the generator u.

Let F be a splitting field of φ_n and let

$$\begin{aligned}\rho\colon \operatorname{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) \to \operatorname{Hom}\big((M_n)_F \otimes L^{\otimes d_n}, (X_{n+1})_F\big) &= \\ = \operatorname{Hom}\big(L^{\otimes d_n} \otimes L^{\otimes 2d_n}, (X_{n+1})_F\big) &= \mathbb{Z} \oplus \mathbb{Z}\end{aligned}$$

be the restriction map. Note that

$$\rho(\Theta) \in (1, \mathbb{Z}).$$

Lemma 18. *If X_n is anisotropic, then $2\mathbb{Z} \oplus 2\mathbb{Z} \subset \operatorname{Im}\rho \subset \Gamma$.*

Proof. The first inclusion is easily seen by passing to a quadratic splitting field and taking norms.

We replace k by $k(X_{n+1})$. Then X_{n+1} is isotropic, but X_n is still anisotropic. One finds using the induction hypothesis:

$$\begin{aligned}\operatorname{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) &= \operatorname{Hom}(M_n \otimes L^{\otimes d_n}, L^{\otimes 0} \oplus L^{\otimes d_{n+1}} \oplus M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i}) \\ &= \operatorname{Hom}(M_n \otimes L^{\otimes d_n}, M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i})\end{aligned}$$

The map ρ is then given by projection onto the summand for $i = d_n$ followed by passing to the split case. Hence it factors through the map $\omega\colon \operatorname{End}(M_n) \to \mathbb{Z} \oplus \mathbb{Z}$ and Lemma 16 yields the claim. $\square$

Since $\rho(\Theta) \notin 2\mathbb{Z} \oplus 2\mathbb{Z}$ we must have $\Gamma \subset \operatorname{Im}\rho$ and we may assume that

$$\rho(\Theta) = (1, 1).$$

Let $h \in \mathrm{CH}^1(X_{n+1})$ be a hyperplane section and put

$$f = \Theta \cdot (1 + h + h^2 + \cdots + h^{d_n - 1}) \in$$
$$\operatorname{Hom}(M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i}, X_{n+1}) = \operatorname{Hom}(Z_n \otimes L, X_{n+1}).$$

One easily checks that f satisfies the assumptions in Lemma 15. Hence Lemma 15 yields the projector p_{n+1} together with property (8).

Step 2: We now assume the existence of p_n with property (8) and show (9).

We may again assume that X_n is anisotropic.

This time we look for an element

$$\Theta \in \operatorname{Hom}(M_n \otimes L^{\otimes d_n - 1}, Z_n) = \mathrm{CH}_{2d_n - 1}(M_n \otimes Z_n).$$

Similar as above we have sequence (with $F = k(X_n)$)

$$\mathrm{CH}_{2d_n-1}(M_n \otimes Z_n) \underset{p_n}{\overset{\text{incl.}}{\rightleftarrows}} \mathrm{CH}_{2d_n-1}(X_n \times Z_n) \xrightarrow{j} \mathrm{CH}_{d_n-1}\big((Z_n)_F\big) = u\mathbb{Z}$$

and find Θ mapping to u.

By a spectral sequence argument (with base Z_n and using the fact that $M_n \simeq L^{\otimes 0} \oplus L^{\otimes d_n}$ over any residue class field of Z_n) on gets an exact sequence

$$\mathrm{CH}_{d_n-1}(Z_n) \xrightarrow{M_n \times} \mathrm{CH}_{2d_n-1}(M_n \otimes Z_n) \xrightarrow{\ell} \mathrm{CH}_{2d_n-1}(Z_n) \to 0.$$

Suppose that $\ell(\Theta) \in \mathrm{CH}_{2d_n-1}(Z_n) = [Z_n]\mathbb{Z}$ is even. Then we can arrange $\ell(\Theta) = 0$ still keeping $j(\Theta) = u$. But then Θ would come from $\mathrm{CH}_{d_n-1}(Z_n)$ and the maximal linear subspace u would be defined over k. This contradicts the fact that X_n is anisotropic. We my therefore find Θ with

$$\ell(\Theta) = [Z_n], \qquad j(\Theta) = u.$$

Let $h \in \mathrm{CH}^1(Z_n)$ be a hyperplane section and put

$$f = \Theta \cdot (1 + h + h^2 + \cdots + h^{d_n-1}) \in \mathrm{Hom}(M_n \otimes \bigoplus_{i=0}^{d_n-1} L^{\otimes i}, Z_n).$$

Since $p_n^t = p_n$, the transpose f^t is well defined.

It is easy to check that $f \circ f^t$ satisfies the assumption of Corollary 11 and hence $f \circ f^t$ is an isomorphism.

Similarly one proves that $f^t \circ f$ is an isomorphism (by extending Corollary 11 to motives of the form $M_n \otimes \mathbb{P}^m$). □

Proposition 19. *One has*

$$X_{\varphi_n} \simeq M_n \otimes \mathbb{P}^{d_n}.$$

Proof. This follows as for (9). □

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