NORM VARIETIES AND ALGEBRAIC COBORDISM

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Abstract. We outline briefly results and examples related with the bijectivity of the norm residue homomorphism. We define norm varieties and describe some constructions. Further we discuss degree formulas which form a major tool to handle norm varieties. Finally we formulate Hilbert's 90 for symbols which is the hard part of the bijectivity of the norm residue homomorphism, modulo a theorem of Voevodsky.

Introduction

This text is a brief outline of results and examples related with the bijectivity of the norm residue homomorphism—also called "Bloch-Kato conjecture" and, for the mod 2 case, "Milnor conjecture".

The starting point was a result of Voevodsky which he communicated in 1996. Voevodsky's theorem basically reduces the Bloch-Kato conjecture to the existence of norm varieties and to what I call Hilbert's 90 for symbols. Unfortunately there is no text available on Voevodsky's theorem.

In this exposition p is a prime, k is a field with $\operatorname{char} k \neq p$ and $K_n^M k$ denotes Milnor's n-th K-group of k [15], [19].

Elements in $K_n^M k/p$ of the form

$$u = \{a_1, \dots, a_n\} \bmod p$$

are called symbols (mod p, of weight n).

A field extension F of k is called a splitting field of u if $u_F = 0$ in $K_n^M F/p$.

Let

$$\begin{aligned} h_{(n,p)} \colon K_n^M k/p &\to H^n_{\text{ét}}(k, \mu_p^{\otimes n}) \\ \{a_1, \dots, a_n\} &\mapsto (a_1, \dots, a_n) \end{aligned}$$

be the norm residue homomorphism.

1. Norm Varieties

All successful approaches to the Bloch-Kato conjecture consist of an investigation of appropriate generic splitting varieties of symbols. This goes back to the work of Merkurjev and Suslin on the case $n = 2$ who studied the K-cohomology of Severi-Brauer varieties [12]. Similarly, for the case $p = 2$ (for $n = 3$ by Merkurjev, Suslin [14] and the author [18], for all n by Voevodsky [23]) one considers certain quadrics associated with Pfister forms. For a long time it was not clear which sort of varieties one should consider for arbitrary n, p. In some cases one knew candidates, but these were non-smooth varieties and desingularizations appeared to be difficult to handle.

1991 *Mathematics Subject Classification.* 12G05.

Key words and phrases. Milnor's K-ring, Galois cohomology, Cobordism.

Finally Voevodsky proposed a surprising characterization of the necessary varieties. It involves characteristic numbers and yields a beautiful relation between symbols and cobordism theory.

Definition. Let $u = \{a_1, \dots, a_n\} \bmod p$ be a symbol. Assume that $u \neq 0$. A *norm variety* for u is a smooth proper irreducible variety X over k such that

(1) The function field $k(X)$ of X splits u.
(2) $\dim X = d := p^{n-1} - 1$
(3) $\frac{s_d(X)}{p} \not\equiv 0 \mod p$

Here $s_d(X) \in \mathbf{Z}$ denotes the characteristic number of X given by the d-th Newton polynomial in the Chern classes of TX. It is known (by Milnor) that in dimensions $d = p^n - 1$ the number $s_d(X)$ is p-divisible for any X. If $k \subset \mathbf{C}$ one may rephrase condition (3) by saying that $X(\mathbf{C})$ is indecomposable in the complex cobordism ring mod p.

We will observe in section 2 that the conditions for a norm variety are birational invariant.

The name "norm variety" originates from some constructions of norm varieties, see section 3.

We conclude this section with the "classical" examples of norm varieties.

Example. The case $n = 2$. Assume that k contains a primitive p-th root ζ of unity. For a, $b \in k^*$ let $A_\zeta(a, b)$ be the central simple k-algebra with presentation

$$A_\zeta(a, b) = \langle\, u, v \mid u^p = a, v^p = b, vu = \zeta uv \,\rangle$$

The Severi-Brauer variety $X(a, b)$ of $A_\zeta(a, b)$ is a norm variety for the symbol $\{a, b\} \bmod p$.

Example. The case $p = 2$. For $a_1, \dots, a_n \in k^*$ one denotes by

$$\langle\!\langle a_1, \dots, a_n \rangle\!\rangle = \bigotimes_1^n \langle 1, -a_i \rangle$$

the associated n-fold Pfister form [9], [21]. The quadratic form

$$\varphi = \langle\!\langle a_1, \dots, a_{n-1} \rangle\!\rangle \perp \langle -a_n \rangle$$

is called a Pfister neighbor. The projective quadric $Q(\varphi)$ defined by $\varphi = 0$ is a norm variety for the symbol $\{a_1, \dots, a_n\} \bmod 2$.

2. Degree formulas

The theme of "degree formulas" goes back to Voevodsky's first text on the Milnor conjecture (although he never formulated explicitly a "formula") [22]. In this section we formulate the degree formula for the characteristic numbers s_d. It shows the birational invariance of the notion of norm varieties.

The first proof of this formula relied on Voevodsky's stable homotopy theory of algebraic varieties. Later we found a rather elementary approach [11], which is in spirit very close to "elementary" approaches to the complex cobordism ring [16], [4].

For our approach to Hilbert's 90 for symbols we use also "higher degree formulas" which again were first settled using Voevodsky's stable homotopy theory [3]. These follow meanwhile also from the "general degree formula" proved by Morel and Levine [10] in characteristic 0 using factorization theorems for birational maps [1].

We fix a prime p and a number d of the form $d = p^n - 1$.

For a proper variety X over k let

$$I(X) = \deg\big(\mathrm{CH}_0(X)\big) \subset \mathbf{Z}$$

be the image of the degree map on the group of 0-cycles. One has $I(X) = i(X)\mathbf{Z}$ where $i(X)$ is the "index" of X, i. e., the gcd of the degrees $[k(x) : k]$ of the residue class field extensions of the closed points x of X. If X has a k-point (in particular if k is algebraically closed), then $I(X) = \mathbf{Z}$. The group $I(X)$ is a birational invariant of X. We put

$$J(X) = I(X) + p\mathbf{Z}$$

Let X, Y be irreducible smooth proper varieties over k with $\dim Y = \dim X = d$ and let $f : Y \to X$ be a morphism. Define $\deg f$ as follows: If $\dim f(Y) < \dim X$, then $\deg f = 0$. Otherwise $\deg f \in \mathbf{N}$ is the degree of the extension $k(Y)/k(X)$ of the function fields.

Theorem (Degree formula for s_d)**.**

$$\frac{s_d(Y)}{p} = (\deg f)\frac{s_d(X)}{p} \mod J(X)$$

Corollary. *The class*

$$\frac{s_d(X)}{p} \bmod J(X) \in \mathbf{Z}/J(X)$$

is a birational invariant.

Remark. If X has a k-rational point, then $J(X) = \mathbf{Z}$ and the degree formula is empty. The degree formula and the birational invariants $s_d(X)/p \bmod J(X)$ are phenomena which are interesting only over non-algebraically closed fields. Over the complex numbers the only characteristic numbers which are birational invariant are the Todd numbers.

We apply the degree formula to norm varieties. Let u be a nontrivial symbol mod p and let X be a norm variety for u. Since $k(X)$ splits u, so does any residue class field $k(x)$ for $x \in X$. As u is of exponent p, it follows that $J(X) = p\mathbf{Z}$.

Corollary (Voevodsky)**.** *Let u be a nontrivial symbol and let X be a norm variety of u. Let further Y be a smooth proper irreducible variety with* $\dim Y = \dim X$ *and let $f : Y \to X$ be a morphism. Then Y is a norm variety for u if and only if* $\deg f$ *is prime to p.*

It follows in particular that the notion of norm variety is birational invariant. Therefore we may call any irreducible variety U (not necessarily smooth or proper) a norm variety of a symbol u if U is birational isomorphic to a smooth and proper norm variety of u.

3. Existence of norm varieties

Theorem. *Norm varieties exists for every symbol $u \in K_n^M k/p$ for every p and every n.*

As we have noted, for the case $n = 2$ one can take appropriate Severi-Brauer varieties (if k contains the p-th roots of unity) and for the case $p = 2$ one can take appropriate quadrics.

In this exposition we describe a proof for the case $n = 3$ using fix-point theorems of Conner and Floyd in order to compute the non-triviality of the characteristic numbers. Our first proof for the general case used also Conner-Floyd fix-point theory. Later we found two further methods which are comparatively simpler. However the Conner-Floyd fix-point theorem is still used in our approach to Hilbert's 90 for symbols.

Let $u = \{a, b, c\} \bmod p$ with $a,\ b,\ c \in k^*$. Assume that k contains a primitive p-th root ζ of unity, let $A = A_\zeta(a, b)$ and let

$$MS(A, c) = \{\, x \in A \mid \mathrm{Nrd}(x) = c \,\}$$

We call $MS(A, c)$ the Merkurjev-Suslin variety associated with A and c. The symbol u is trivial if and only if $MS(A, c)$ has a rational point [12]. The variety $MS(A, c)$ is a twisted form of $\mathrm{SL}(p)$.

Theorem. *Suppose $u \neq 0$. Then $MS(A, c)$ is a norm variety for u.*

Let us indicate a proof for a subfield $k \subset \mathbf{C}$ (and for $p > 2$). Let $U = MS(A, c)$. It is easy to see that $k(U)$ splits u. Moreover one has $\dim U = \dim A - 1 = p^2 - 1$. It remains to show that there exists a proper smooth completion X of U with nontrivial characteristic number.

Let

$$\bar{U} = \{\, [x, t] \in \mathbf{P}(A \oplus k) \mid \mathrm{Nrd}(x) = ct^p \,\}$$

be the naive completion of U. We let the group $G = \mathbf{Z}/p \times \mathbf{Z}/p$ act on the algebra A via

$$(r, s) \cdot u = \zeta^r u, \quad (r, s) \cdot v = \zeta^s v$$

This action extends to an action on $\mathbf{P}(A \oplus k)$ (with the trivial action on k) which induces a G-action on $\bar{U}$. Let $\mathrm{Fix}(\bar{U})$ be the fixed point scheme of this action. One finds that $\mathrm{Fix}(\bar{U})$ consists just of the p isolated points $[1, \zeta^i \sqrt[p]{c}]$, $i = 1, \ldots, p$, which are all contained in U.

The variety U is smooth, but $\bar{U}$ is not. However, by equivariant resolution of singularities [2], there exists a smooth proper G-variety X together with a G-morphism $X \to \bar{U}$ which is a birational isomorphism and an isomorphism over U. It remains to show that

$$\frac{s_d(X)}{p} \not\equiv 0 \mod p$$

For this we may pass to topology and try to compute $s_d\big(X(\mathbf{C})\big)$. We note that for odd p, the Chern number s_d is also a Pontryagin number and depends only on the differentiable structure of the given variety. Note further that X has the same G-fixed points as $\bar{U}$ since the desingularization took place only outside U.

Consider the variety

$$Z = \Big\{ \Big[\textstyle\sum_{i,j=1}^p x_{ij} u^i v^j, t\Big] \in \mathbf{P}(A \oplus k) \;\Big|\; \textstyle\sum_{i,j=1}^p x_{ij}^p = ct^p \Big\}$$

This variety is a smooth hypersurface and it is easy to check

$$\frac{s_d(Z)}{p} \not\equiv 0 \mod p$$

As a G-variety, the variety Z has the same fixed points as X ("same" means that the collections of fix-points together with the G-structure on the tangent spaces are isomorphic). Let M be the differentiable manifold obtained from $X(\mathbf{C})$ and $-Z(\mathbf{C})$ by a multi-fold connected sum along corresponding fixed points. Then M

is a G-manifold without fixed points. By the theory of Conner and Floyd [5], [7] applied to $(\mathbf{Z}/p)^2$-manifolds of dimension $d = p^2 - 1$ one has

$$\frac{s_d(M)}{p} \equiv 0 \mod p$$

Thus

$$\frac{s_d(X)}{p} \equiv \frac{s_d(Z)}{p} \mod p$$

and the desired non-triviality is established.

The functions Φ_n. We conclude this section with examples of norm varieties for the general case.

Let $a_1, a_2, \ldots$ be a sequence of elements in k^*. We define functions $\Phi_n = \Phi_{a_1,\ldots,a_n}$ in p^n variables inductively as follows.

$$\Phi_0(t) = t^p$$

$$\Phi_n(T_0, \ldots, T_{p-1}) = \Phi_{n-1}(T_0) \prod_{i=1}^{p-1} \bigl(1 - a_n \Phi_{n-1}(T_i)\bigr)$$

Here the T_i stand for tuples of p^{n-1} variables. Let $U(a_1, \ldots, a_n)$ be the variety defined by

$$\Phi_{a_1,\ldots,a_{n-1}}(T) = a_n$$

Theorem. *Suppose that the symbol $u = \{a_1, \ldots, a_n\} \bmod p$ is nontrivial. Then $U(a_1, \ldots, a_n)$ is a norm variety of u.*

4. Hilbert's 90 for symbols

The bijectivity of the norm residue homomorphisms has always been considered as a sort of higher version of the classical Hilbert's Theorem 90 (which establishes the bijectivity for $n = 1$). In fact, there are various variants of the Bloch-Kato conjecture which are obvious generalizations of Hilbert's Theorem 90: The Hilbert's Theorem 90 for K_n^M of cyclic extensions or the vanishing of the motivic cohomology group $H^{n+1}\bigl(k, \mathbf{Z}(n)\bigr)$. In this section we describe a variant which on one hand is very elementary to formulate and on the other hand is the really hard part of the Bloch-Kato conjecture (modulo Voevodsky's theorem).

Let $u = \{a_1, \ldots, a_n\} \in K_n^M k/p$ be a symbol. Consider the norm map

$$\mathcal{N}_u = \sum_F N_{F/k} \colon \bigoplus_F K_1 F \to K_1 k$$

where F runs through the finite field extensions of k (contained in some algebraic closure of k) which split u. Hilbert's Theorem 90 for u states that $\ker \mathcal{N}_u$ is generated by the "obvious" elements.

To make this precise, we consider two types of basic relations between the norm maps $N_{F/k}$.

Let F_1, F_2 be finite field extensions of k. Then the sequence

$$(1) \qquad K_1(F_1 \otimes F_2) \xrightarrow{(N_{F_1\otimes F_2/F_1}, -N_{F_1\otimes F_2/F_2})} K_1F_1 \oplus K_1F_2 \xrightarrow{N_{F_1/k} + N_{F_2/k}} K_1 k$$

is a complex.

Further, if K/k is of transcendence degree 1, then the sequence

$$K_2K \xrightarrow{d_K} \bigoplus_v K_1\kappa(v) \xrightarrow{N} K_1k \tag{2}$$

is a complex. Here v runs through the valuations of K/k, d_K is given by the tame symbols at each v and N is the sum of the norm maps $N_{\kappa(v)/k}$. The sum formula $N\circ d_K=0$ is also known as Weil's formula.

We now restrict again to splitting fields of u. The maps in (1) yield a map

$$\mathcal{R}_u=\sum_{F_1,F_2}(N_{F_1\otimes F_2/F_1},-N_{F_1\otimes F_2/F_2})\colon \bigoplus_{F_1,F_2} K_1(F_1\otimes F_2)\to\bigoplus_F K_1F$$

with $\mathcal{N}_u\circ\mathcal{R}_u=0$. Let C be the cokernel of $\mathcal{R}_u$ and let $\mathcal{N}'_u\colon C\to K_1k$ be the map induced by $\mathcal{N}_u$. Then the maps in (2) yield a map

$$\mathcal{S}_u=\sum_K d_K\colon \bigoplus_K K_2K\to C$$

with $\mathcal{N}'_u\circ\mathcal{S}_u=0$ where K runs through the splitting fields of u of transcendence degree 1 over k (contained in some universal field). Let $H_0(u,K_1)$ be the cokernel of $\mathcal{S}_u$ and let $N_u\colon H_0(u,K_1)\to K_1k$ be the map induced by $\mathcal{N}'_u$.

Hilbert's 90 for symbols. *For every symbol u the norm map*

$$N_u\colon H_0(u,K_1)\to K_1k$$

is injective.

Example. If $u=0$, then it is easy to see that N_u is injective. In fact, it is a trivial exercise to check that $\mathcal{N}'_u$ is injective.

Example. The case $n=1$. The splitting fields F of $u=\{a\} \bmod p$ are exactly the field extensions of k containing a p-th root of a. It is an easy exercise to reduce the injectivity of N_u (in fact of $\mathcal{N}'_u$) to the classical Hilbert's Theorem 90, i. e., the exactness of

$$K_1L \xrightarrow{1-\sigma} K_1L \xrightarrow{N_{L/k}} K_1k$$

for a cyclic extension L/k of degree p with σ a generator of $\mathrm{Gal}(L/k)$.

Example. The case $n=2$. Assume that k contains a primitive p-th root ζ of unity. The splitting fields F of $u=\{a,b\} \bmod p$ are exactly the splitting fields of the algebra $A_\zeta(a,b)$. One can show that

$$H_0(u,K_1)=K_1A_\zeta(a,b)$$

with N_u corresponding to the reduced norm map Nrd [13]. Hence in this case Hilbert's 90 for u reduces to the classical fact $SK_1A=0$ for central simple algebras of prime degree [6].

Example. The case $p=2$. The splitting fields F of $u=\{a_1,\dots,a_n\} \bmod 2$ are exactly the field extensions of k which split the Pfister form $\langle\!\langle a_1,\dots,a_n\rangle\!\rangle$ or, equivalently, over which the Pfister neighbor $\langle\!\langle a_1,\dots,a_{n-1}\rangle\!\rangle\perp\langle -a_n\rangle$ becomes isotropic. Hilbert's 90 for symbols mod 2 had been first established in [17]. This text considered similar norm maps associated with any quadratic form (which are not injective in general). A treatment of the special case of Pfister forms is contained in [8].

Remark. One can show that the group $H_0(u, K_1)$ as defined above is also the quotient of $\oplus_F K_1 F$ by the R-trivial elements in $\ker \mathcal{N}_u$. This is quite analogous to the description of $K_1 A$ of a central simple algebra A: The group $K_1 A$ is the quotient of A^* by the subgroup of R-trivial elements in the kernel of $\mathrm{Nrd}\colon A^* \to F^*$. Similarly for the case $p = 2$: In this case the injectivity of N_u is related with the fact that for Pfister neighbors φ the kernel of the spinor norm $\mathrm{SO}(\varphi) \to k^*/(k^*)^2$ is R-trivial.

In our approach to Hilbert's 90 for symbols one needs a parameterization of the splitting fields of symbols.

Definition. Let $u = \{a_1, \ldots, a_n\} \bmod p$ be a symbol. A *p-generic splitting variety* for u is a smooth variety X over k such that for every splitting field F of u there exists a finite extension F'/F of degree prime to p and a morphism $\operatorname{Spec} F' \to X$.

Theorem. *Suppose* $\operatorname{char} k = 0$. *Let* $m \geq 3$ *and suppose for* $n \leq m$ *and every symbol* $u = \{a_1, \ldots, a_n\} \bmod p$ *over all fields over* k *there exists a* p*-generic splitting variety for* u *of dimension* $p^{n-1} - 1$. *Then Hilbert's 90 holds for such symbols.*

The proof of this theorem is outlined in [20].

For $n = 2$ one can take here the Severi-Brauer varieties and for $n = 3$ the Merkurjev-Suslin varieties. Hence we have:

Corollary. *Suppose* $\operatorname{char} k = 0$. *Then Hilbert's 90 holds for symbols of weight* ≤ 3.

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