ON THE ORTHOCENTER

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For variables x_0, x_1, x_2, x_3 there is the following general identity

$$0 = \sum_{i=0}^{3} (-1)^i x_i (x_{i-1} - x_{i+1})(x_{i-1} - x_{i+2})(x_{i+1} - x_{i+2}) \tag{1}$$

Here the indices are taken mod 4.

Formula (1) is an identity in $\mathbf{Z}[x_0, x_1, x_2, x_3]$ and may be checked by a direct computation. Alternatively, one can deduce it from the residue theorem as follows.

Consider the polynomial

$$P(t) = (t - x_0)(t - x_1)(t - x_2)(t - x_3)$$

Then

$$P'(x_i) = \prod_{j \neq i} (x_i - x_j)$$

where $P'(t)$ is the derivative. After division by

$$\delta = \prod_{i<j} (x_i - x_j)$$

identity (1) reads as

$$0 = \sum_{i=0}^{3} \frac{x_i}{P'(x_i)} \tag{2}$$

This equation follows from the residue theorem applied to the differential

$$\frac{t}{P(t)} dt$$

(It suffices to assume that the x_i are nonzero pairwise distinct complex numbers. The residue at ∞ vanishes since $\deg P > 2$.)

Since the $P'(x_i)$ are invariant under the translation $x_j \mapsto x_j + 1$ one gets also

$$0 = \sum_{i=0}^{3} \frac{1}{P'(x_i)} \tag{3}$$

(This follows as well by looking at the residues of $dt/P(t)$.)

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In the following we write $h = x_0$ and $x_{i\pm 1}$, $x_{i\pm 2}$ stand for one of x_1, x_2, x_3 with the index taken mod 3. Then there is the reformulation

$$h = -\sum_{i=1}^{3} x_i \frac{P'(h)}{P'(x_i)} = \sum_{i=1}^{3} x_i \frac{(x_{i+1}-h)}{(x_{i+2}-x_i)} \frac{(x_{i+2}-h)}{(x_{i+1}-x_i)} \tag{4}$$

of our identity. Note that $(h - x_i)$ appears with different signs in $P'(h)$ and $P'(x_i)$ which cancels the factor -1.

Assume that $x_1, x_2, x_3 \in \mathbf{C}$ are complex numbers not on a real straight line and that h is the orthocenter of the triangle x_1, x_2, x_3. (The orthocenter is the intersection of the altitudes.) This means that the opposite sides of the quadrangle h, x_1, x_2, x_3 are orthogonal:

$$x_i - h \perp x_{i+1} - x_{i+2}$$

(One speaks of an orthocentric quadrangle.) In other words, the corresponding quotients

$$\frac{x_i - h}{x_{i+1} - x_{i+2}} \in I \cdot \mathbf{R}$$

are pure imaginary complex numbers (I denotes the imaginary unit, $I^2 = -1$). Put

$$r_i = I \frac{x_i - h}{x_{i+1} - x_{i+2}} \in \mathbf{R}$$

(One has $r_i > 0$ for a positively oriented acute triangle x_1, x_2, x_3.) Then (4) yields

$$h = \sum_{i=1}^{3} r_{i+1} r_{i+2} x_i \tag{5}$$

As for (3), the invariance of the r_i under translations of the quadrangle implies

$$1 = \sum_{i=1}^{3} r_{i+1} r_{i+2}$$

We have obtained the barycentric coordinates of the orthocenter of a Euclidean triangle.

Remarkably, the functions r_i are complex algebraic (holomorphic) functions in the 4 points of the quadrangle (with values in $\mathbf{R}$ in the orthocentric case).

A similar phenomenon happens for *any* triangle function with an invariance under $\mathrm{Aff}(1, \mathbf{C})$. For more details see my text “The holomorphic extension of triangle functions”.

To give a further example here: For the feet f_i of the three altitudes (the intersections of the lines hx_i and $x_{i-1}x_{i+1}$ in the orthocentric quadrangle h, x_1, x_2, x_3) one has the formula

$$f_i = \frac{hx_i - x_{i-1}x_{i+1}}{h + x_i - x_{i-1} - x_{i+1}}$$

(It was this computation which lead to the presentation (4).)

We conclude with a discussion of a simpler relation, namely

$$0 = \sum_{i=1}^{3} (x_i - h)(x_{i+1} - x_{i+2}) \tag{6}$$

Formula (6) is an identity in $\mathbf{Z}[h, x_1, x_2, x_3]$ and therefore valid in any ring. In particular it holds in the symmetric algebra

$$S^{\bullet}_{\mathbf{R}}\mathbf{C}$$

of the $\mathbf{R}$-vector space $\mathbf{C}$. This way (6) becomes an identity in $S^2_{\mathbf{R}}\mathbf{C}$. After applying the Euclidean metric

$$S^2_{\mathbf{R}}\mathbf{C} \to \mathbf{R}$$

$$vw \mapsto \langle v, w \rangle = \frac{v\overline{w} + \overline{v}w}{2}$$

one obtains

$$0 = \sum_{i=1}^{3} \langle x_i - h, x_{i+1} - x_{i+2} \rangle \tag{7}$$

This means that any two of the three orthogonality relations

$$x_i - h \perp x_{i+1} - x_{i+2}$$

imply the third. In other words: The intersection of two altitudes of the triangle x_1, x_2, x_3 lies on the third altitude.

A direct verification of formula (6) is rather immediate. If one wishes, one may deduce it from the residue theorem as well, this time looking at the differentials

$$\frac{1}{Q(t)}dt, \qquad \frac{t}{Q(t)}dt$$

with $Q(t) = (t - x_1)(t - x_2)(t - x_3)$.

Here is another game to prove (6). Let

$$u = \frac{x_1 + x_2 + x_3 - h}{2} \tag{8}$$

Then

$$2(x_{i\pm1} - u) = (h - x_i) \pm (x_{i+1} - x_{i-1})$$

Taking the difference of the squares gives

$$(x_{i+1} - u)^2 - (x_{i-1} - u)^2 = (h - x_i)(x_{i+1} - x_{i-1}) \tag{9}$$

Summing up over i mod 3 results in (6).

Applying $\langle\ ,\ \rangle$ to (9) shows

$$\langle h - x_i, x_{i+1} - x_{i-1} \rangle = ||x_{i+1} - u||^2 - ||x_{i-1} - u||^2$$

It follows that h is the orthocenter of the triangle x_1, x_2, x_3 if and only if u is its circumcenter (point of equal distance to the x_i, $i = 1, 2, 3$). This way (8) yields the Euler equation

$$3G = H + 2U$$

where

$$G = \frac{x_1 + x_2 + x_3}{3}$$

is the center of gravity, H is the orthocenter and U the circumcenter of a Euclidean triangle.

One more remark. The right term of (6) has a Σ_4-invariance: It is invariant under any permutation of the 4 variables x_i ($x_0 = h$), with a sign change for odd permutations. To make this evident, one may write (6) as

$$0 = \sum \operatorname{sgn}(i,j,k,\ell)(x_i - x_j)(x_k - x_\ell) \tag{10}$$

Here the sum is taken over the 3 unordered partitions

$$\{\{i,j\},\{k,\ell\}\} \qquad (\{i,j,k,\ell\} = \{0,1,2,3\})$$

of the 4-element index set. I am not aware of an argument for the evidence of (10) without breaking the Σ_4-symmetry.

Applying $\langle\ ,\ \rangle$ to (10) yields

$$0 = \sum \operatorname{sgn}(i,j,k,\ell)\langle x_i - x_j, x_k - x_\ell\rangle$$

It shows that if two of the diagonal pairs of the quadrangle x_0, x_1, x_2, x_3 meet orthogonally, so does the third.

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