ON QUADRATIC FORMS ISOTROPIC OVER THE FUNCTION FIELD OF A CONIC

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Consider a quadratic extension $L = F(\sqrt{a})$ of a field F ($\operatorname{Char} F \neq 2$). The behaviour of quadratic forms over F under base extension $\varphi \to \varphi_L$ is well understood, since any anisotropic form φ is isomorphic to $\psi \perp \langle 1, -a \rangle \varrho$ with forms ψ, ϱ over F such that ψ_L is anisotropic [S, 2. Sect. 5]. This implies that the anisotropic part $(\varphi_L)_{\mathrm{an}}$ of φ_L is isomorphic to ψ_L and therefore already defined over F and that if φ is anisotropic and φ_L is hyperbolic then φ is a multiple of $\langle 1, -a \rangle$.

Now let K be the function field of a conic, i.e. K is for some $a, b \in F^*$ isomorphic to the fraction field of $R = F[s,t]/(s^2 - at^2 - b)$. Then K is the universal splitting field of the form $\langle 1, -a, -b \rangle$ and in view of the decomposition mentioned above it is natural to ask whether a form φ which becomes isotropic over K contains a subform similar to $\langle 1, -a, -b \rangle$. This however is not true in general; see [L, Sect. 6] for further information. The purpose of this note is to prove

Proposition. *Let φ be a form over F. Then there exist a number p, forms φ_i, ψ_i $(i = 0, \dots, p)$ and elements $c_i \in F^*$ $(i = 0, \dots, p-1)$ such that $\varphi = \varphi_0$ and*

i) $\varphi_i \simeq c_i \langle 1, -a \rangle \perp \psi_i, \quad i = 0, \dots, p-1;$
ii) $\varphi_{i+1} \simeq c_i b \langle 1, -a \rangle \perp \psi_i, \quad i = 0, \dots, p-1;$
iii) $\big((\varphi_p)_K\big)_{\mathrm{an}} \cong \big((\varphi_p)_{\mathrm{an}}\big)_K.$

This proposition shows that the extension $K|F$ has similar splitting properties as the quadratic extension $L|F$:

Corollary. *Let φ be a form over F. Then there exists a form ψ over F such that $(\varphi_K)_{\mathrm{an}}$ is isomorphic to ψ_K. If φ is anisotropic and φ_K is hyperbolic then φ is a multiple of $\langle 1, -a, -b, ab \rangle$.*

The first statement of the corollary has been proved by Arason in [ELW, Appendix II]; it follows from the proposition by taking $\psi = (\varphi_p)_{\mathrm{an}}$, since all φ_i are isomorphic over K. The second statement is well known, see e.g. [A, Sect. 2] or [S, 4. Sect. 5]; it is a consequence of the proposition and Witt cancellation. The method of proof presented here is direct and constructive and might indicate a way to handle the extension $K|F$ also for other questions.

Note that $R = F[t] \oplus sF[t]$ as F-vector space. Define $d\colon R \to \mathbb{N} \cup \{-\infty\}$ by

$$d(P + sQ) = \max\{\deg P, 1 + \deg Q\} \quad \text{for } P, Q \in F[t]$$

(here $\deg 0 = -\infty$). Moreover let $R_n = \{\, r \in R \mid d(r) \leq n \,\}$. R_n is a F-vector subspace of R and one has $R_0 = F$ and $R_n \cdot R_m \subset R_{n+m}$.

Lemma. *Let $\varphi\colon V \to F$ be an anisotropic form and suppose that for some $n \geq 1$ there exist*

$$v \in (V \otimes_F R_n) \setminus (V \otimes_F R_{n-1})$$

The text appeared as: Math. Ann. 288 (1990), no. 3, 511–513.

such that $\varphi(v) = 0 \in R$.

Then there exists a subspace $L \subset V$ of dimension 2 such that

1) *$\varphi|L \simeq c\langle 1, -a\rangle$ for some $c \in F^*$,*
2) *there exists a nonzero $\tilde{v} \in V \otimes_F R_{n-1}$ such that $\tilde{\varphi}(\tilde{v}) = 0$, where $\tilde{\varphi} = b(\varphi|L) \perp (\varphi|W)$ and $W = L^{\perp}$.*

Proof of the proposition. We use induction on $\dim \varphi_{\mathrm{an}}$. It is clear that we may assume that φ is anisotropic and φ_K is isotropic.

Since K is the fraction field of R there exist $n \geq 0$ and a nonzero $v \in V \otimes_F R_n$ such that $\varphi(v) = 0$. We proceed by induction on n. If $n = 0$, then $v \in V$ and φ would be isotropic over F; hence $n \geq 1$. We may assume $v \notin V \otimes_F R_{n-1}$ and we take $\varphi_1 = \tilde{\varphi}$ where $\tilde{\varphi}$ is the form in the lemma. If $\tilde{\varphi}$ is anisotropic we apply the induction hypothesis for $n-1$ and if $\tilde{\varphi}$ is isotropic we apply the induction hypothesis for $\dim \tilde{\varphi}_{\mathrm{an}} < \dim \varphi$. In any case we find forms $\tilde{\varphi} = \tilde{\varphi}_0, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_p$ as in the proposition and $\varphi = \varphi_0$, $\varphi_i = \tilde{\varphi}_{i-1}$ $(i = 1, \ldots, p+1)$ is a sequence as required. □

In order to prove the lemma we write

$$v = v_0 + \sum_{i=1}^{n} v_i s t^{i-1} + w_i t^i; \quad v_i,\, w_i \in V.$$

Claim. $\langle v_n, w_n\rangle_\varphi = 0$ and $\varphi(w_n) = -a\varphi(v_n)$.

Proof of the claim:

$$\begin{aligned} 0 &= \varphi(v) \bmod R_{2n-1} \\ &= \varphi(v_n)s^2t^{2(n-1)} + 2\langle v_n, w_n\rangle_\varphi s t^{2n-1} + \varphi(w_n)t^{2n} \bmod R_{2n-1} \\ &= (\varphi(v_n)a + \varphi(w_n))t^{2n} + 2\langle v_n, w_n\rangle_\varphi s t^{2n-1} \bmod R_{2n-1}. \end{aligned}$$

The claim follows since t^{2n} and st^{2n-1} define F-independent vectors of R/R_{2n-1}.

Note that $v_n \neq 0$ and $w_n \neq 0$ since $v \notin V \otimes_F R_{n-1}$ and φ is anisotropic. Let

$$L = F[z]/(z^2 - a)$$

and let $\alpha \in L^*$ be the class of z. We identify L with $\langle v_n, w_n\rangle_F \subset V$ by $1 \to v_n$ and $\alpha \to w_n$. Then the claim shows that $\varphi|L = cN_{L|F}$ with $c = \varphi(v_n)$ and $N_{L|F}\colon L \to F$, $e + \alpha f \to e^2 - af^2$ the norm form.

Now write $v = x + y$ with $x \in L \otimes_F R$ and $y \in W \otimes_F R$, $W = L^{\perp}$. Then $x \in (s + t\alpha)t^{n-1} + L \otimes_F R_{n-1}$ and $y \in W \otimes_F R_{n-1}$.

Put $\tilde{v} = b^{-1}(s - t\alpha)x + y$. Then $\tilde{v}$ is a zero of the form $\tilde{\varphi} = b(\varphi|L) \perp (\varphi|W)$, since

$$\begin{aligned} b\varphi(b^{-1}(s - t\alpha)x) &= bcN_{L|F}(b^{-1}(s - t\alpha)x) = bcb^{-2}(s^2 - at^2)N_{L|F}(x) \\ &= cN_{L|F}(x) = \varphi(x). \end{aligned}$$

It remains to show that $\tilde{v} \in V \otimes_F R_{n-1}$. In order to do this we have to show that $(s - t\alpha)x \in L \otimes_F R_{n-1}$.

Case I: $n \geq 2$. Then there exist μ, $\lambda \in L$ and $\tilde{x} \in L \otimes_F R_{n-2}$ such that

$$x = (s + t\alpha)t^{n-1} + (s + t\alpha)t^{n-2}\mu + t^{n-1}\lambda + \tilde{x}$$

We have with $\omega = t^{n-1} + t^{n-2}\mu$ and $\bar\lambda$ the conjugate of λ under $\alpha \to -\alpha$:

$$\begin{aligned}
0 &= \varphi(v) \bmod R_{2n-2} = \varphi(x) \bmod R_{2n-2} \\
&= cN_{L|F}((s+t\alpha)\omega + t^{n-1}\lambda) \bmod R_{2n-2} \\
&= c[N_{L|F}((s+t\alpha)\omega) + \operatorname{tr}_{L|F}((s+t\alpha)\omega t^{n-1}\bar\lambda) + N_{L|F}(t^{n-1}\lambda)] \bmod R_{2n-2} \\
&= c[b\cdot N_{L|F}(\omega) + \operatorname{tr}_{L|F}((s+t\alpha)t^{2n-2}\bar\lambda) + 0] \bmod R_{2n-2} \\
&= c[0 + st^{2n-2}\operatorname{tr}_{L|F}\bar\lambda + t^{2n-1}\operatorname{tr}_{L|F}(\alpha\bar\lambda)] \bmod R_{2n-2}
\end{aligned}$$

Hence the traces of $\bar\lambda$ and $\alpha\bar\lambda$ are zero and therefore $\lambda = 0$. Finally:

$$(s - t\alpha)x = b\omega + (s - t\alpha)\tilde{x} \in L \otimes_F R_{n-1}.$$

Case II: $n = 1$. Then $x = s + t\alpha + \lambda$ for some $\lambda \in L$ and it suffices to show $\lambda = 0$. However

$$0 = \varphi(v) = b + s\operatorname{tr}_{L|F}\bar\lambda + t\ \operatorname{tr}_{L|F}\alpha\bar\lambda + N_{L|F}(\lambda) + \varphi(y)$$

and therefore again $\lambda = 0$ since $\varphi(y) \in F$. □

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