THE HOLOMORPHIC EXTENSION OF TRIANGLE FUNCTIONS

MARKUS ROST

Introduction

Examples of triangle functions are given by the orthocenter, the circumcenter, the incenter, the excenters, the pedal points, the nine-point center, etc. Each of these functions has a "holomorphic extension" which is a complex function in four variables.

1. Triangle functions

Let E be the Euclidean plane.

Let further G be the group of Euclidean similarities of E. If we identify E with the field of complex numbers $\mathbf{C}$, then G consists of the affine transformations $x \mapsto ax + b$ with $a \neq 0$.

We let G act on E^r diagonally.

Definition. By a *triangle function* we understand a G-equivariant real analytic function

$$f\colon U \to E$$

defined on an open G-stable subset $U \subset E^3$.

Remark. One could require additionally that a triangle function is also equivariant with respect to reflections at a line. This is indeed the case for most of our examples.

A basic example of such a function is given by the orthocenter[1] $H(x, y, z)$ of a triangle x, y, z. The function $H(x, y, z)$ is defined (at least) on the subset U_H of triples x, y, $z \in E$ not lying on a line and with no perpendicular sides. It is invariant under permutations of the coordinates and one has

$$H\big(x, y, H(x, y, z)\big) = z$$

Let

$$\Phi\colon U_H \to E^4$$
$$\Phi(x, y, z) = \big(x, y, z, H(x, y, z)\big)$$

Let X be the image of Φ. X is the set of (non-degenerate) orthocentric quadrangles. It is G-stable and invariant under permutations of the coordinates of E^4. It is a smooth 6-dimensional real analytic subvariety of E^4.

Date: August 6, 2004.

[1]The orthocenter is the intersection of the altitudes; see [3, Section 1] for orthocenters, orthocentric quadrangles, pedal triangles, etc.

Lemma 1. *Let*

$$f\colon U \to E$$

be a triangle function with $U \subset U_H$. Then there exists an open G-stable subset $V \subset E^4$ with $\Phi(U) \subset V$ and a G-equivariant complex analytic function

$$\hat{f}\colon V \to E$$

such that

$$f = \hat{f} \circ \Phi$$

The function $\hat{f}$ is uniquely determined by f on every component of V containing points of $\Phi(U)$.

The function $\hat{f}$ is called the *holomorphic extension* of f. In all our examples the function $\hat{f}(x, y, z, t)$ is algebraic over $\mathbf{C}(x, y, z, t)$ (in fact, algebraic over $\mathbf{Q}(x, y, z, t)$).

The basic idea is that the subvariety $X \subset E^4 = \mathbf{C}^4$ has 2 complex coordinates, given by the G-action, and 2 further real coordinates. The complexification of the real coordinates will lead to 4 complex coordinates in total.

Conversely, let $\hat{f}\colon V \to E$ be a G-equivariant complex analytic function. Then $f|(X \cap V)$ or rather $f = \hat{f} \circ \Phi$ is a triangle function. When we consider $\hat{f}(x, y, z, t)$ on points $(x, y, z, t) \in X$, we speak of the "Euclidean case".

The following proof of the Lemma is also a device to construct $\hat{f}$.

Proof. We make an identification $E = \mathbf{C}$. Because of the G-equivariance, it suffices to consider the restrictions to the slices defined by $x = 0$, $y = 1$. Let $g(z) = f(0, 1, z)$ and $\phi(z) = \big(z, H(0, 1, z)\big)$. It suffices to find a complex analytic function $\hat{g}(z, t)$ with $g = \hat{g} \circ \phi$.

Let $G(z, s)$ be a complex analytic function with $g(z) = G(z, \bar{z})$ where $\bar{z}$ is the complex conjugate.

The orthocenter H of the triangle 0, 1, z is determined by $H \perp (1 - z)$ and $(1 - H) \perp z$. One finds

$$H(0, 1, z) = \frac{(1 - z)(z + \bar{z})}{\bar{z} - z}$$

The function

$$t = \frac{(1 - z)(z + s)}{s - z}$$

is a linear fractional function in s. Its inverse is

$$s = \frac{zt + (1 - z)z}{t + z - 1}$$

Thus

$$\hat{g}(z, t) = G\big(z, \frac{zt + (1 - z)z}{t + z - 1}\big)$$

does the job. □

2. Examples

Example 1. For the orthocenter $H(x, y, z)$ one has obviously

$$\hat{H}(x, y, z, t) = t$$

Example 2. For the centroid (center of gravity) $G(x, y, z)$ one has

$$\hat{G}(x, y, z, t) = \frac{x + y + z}{3}$$

Example 3. For the circumcenter (center of circumcircle) $O(x, y, z)$ one has

$$3G = 2O + H$$

(the line through these three points is the Euler line). Hence

$$\hat{O}(x, y, z, t) = \frac{x + y + z - t}{2}$$

Example 4. Let x_0, x_1, x_2, x_3 be an orthocentric quadruple. The nine-point circle (also called Euler circle or Feuerbach circle) is the circumcircle of the pedal triangle (which is formed by the bases of the altitudes). It contains all the midpoints m_{ij} of the sides $x_i x_j$ and $m_{ij} m_{k\ell}$ are diameters of it ($ijk\ell$ is any permutation of 1234).

The nine-point center $F(x, y, z)$ is the center of the Euler circle. It lies one the Euler line and one has

$$2F = O + H$$

Hence

$$\hat{F}(x, y, z, t) = \frac{x + y + z + t}{4}$$

Example 5 (Pedal points)**.** For a triangle x, y, z let $P(x, y, z)$ be the base of the altitude through z. Note that

$$P(x, y, z) = P(y, x, z) = P\big(x, y, H(x, y, z)\big)$$

Therefore one has for the holomorphic extension $\hat{P}$ of f the relations

$$\hat{P}(x, y, z, t) = \hat{P}(y, x, z, t) = \hat{P}(x, y, t, z) = \hat{P}(y, x, t, z)$$

It follows that the triple

$$R(x_0, x_1, x_2, x_3) = \big(\hat{P}(x_0, x_1, x_2, x_3), \hat{P}(x_0, x_2, x_3, x_1), \hat{P}(x_0, x_3, x_1, x_2)\big)$$

is equivariant with respect to the natural operations of S_4 and S_3 and the corresponding homomorphism $S_4 \to S_3$.

As for an explicit computation: One first notes that

$$P(0, 1, z) = \frac{z + \bar{z}}{2}$$

Then one can use the proof of the Lemma to compute $\hat{P}$. One finds

$$\hat{P}(x, y, z, t) = \frac{xy - zt}{x + y - z - t}$$

Remark 1. The map R is a triple of rational functions in 4 variables defined over any field F, equivariant with respect to $S_4 \to S_3$ and with respect to the action of the affine group $\mathrm{Aff}(1, F)$. It is my favorite candidate for the construction of a cubic resolvent of a quartic equation. After the previous remarks, it is justified to say that R is the holomorphic extension of the construction of the pedal triangle of an orthocentric quadrangle.

Example 6. Let $S(x,y,z)$ denote the reflection of z along the side xy. Obviously

$$S(x,y,z) = z + 2\big(P(x,y,z) - z\big)$$

One finds

$$\hat{S}(x,y,z,t) = z + \frac{(z-x)(z-y)}{\hat{O}(x,y,z,t) - z}$$

Remark 2. Let us consider the denominators of the pedal points. Let $ijk\ell$ be a permutation of 1234 and let

$$\rho_{ij}(x_0,x_1,x_2,x_3) = x_i + x_j - x_k - x_\ell$$

In the Euclidean case (that is, x_0, x_1, x_2, x_3 is an orthocentric quadrangle) one has

$$(x_i - x_j) \perp (x_k - x_\ell)$$

It follows that in this case

$$r = |\rho_{ij}(x_0,x_1,x_2,x_3)|$$

is independent of ij ($|\cdot|$ is the Euclidean absolute value). In fact,

$$r = 2r_O = 4r_F$$

where r_O the radius of the circumcircle of any subtriangle and r_F is the radius of the Euler circle. It follows also that the quotient

$$\frac{\rho_{ij}(x_0,x_1,x_2,x_3)}{\rho_{ik}(x_0,x_1,x_2,x_3)}$$

is a complex number of norm 1. Its angle is twice the angle of the triangle x_j, x_k, x_ℓ at x_ℓ.

Example 7. [Incenters and Excenters] For a triangle x, y, z let $I(x,y,z)$ be its incenter (intersection of the angle bisectors). The holomorphic extension $\hat{I}$ has globally 4-branches with covering group $\mathbf{Z}/2 \times \mathbf{Z}/2$. There is of course the distinguished branch which extends I in an open neighborhood of $\Phi(U)$. The other three branches, obtained by analytic continuation, give by restriction via Φ the excenters of a triangle. Explicit formulas can be found in [6, End of Section 4].

Remark 3. Let I be the incenter, let J_1, J_2, J_3 be the excenters, and let O be the circumcenter of a triangle. One has

$$I + J_1 + J_2 + J_3 = 4O$$

I don't know an explicit reference for this basic fact, however see [3, Section 1.7, Exercise 5].

Example 8. Let $M(x,y,z)$ be the intersection of the angle trisectors at x, y adjacent to the side xy. In this case one finds that $\hat{M}$ has globally 9-branches with covering group $\mathbf{Z}/3 \times \mathbf{Z}/3$. Explicit formulas can be found in [6].

Remark 4. Connes' article [2] was the starting point for the considerations of this text. We get the following interpretation: Morley's theorem states that $M(x,y,z)$ and its permuted versions form an equilateral triangle, that is

$$M(x,y,z) + \zeta M(y,z,x) + \zeta^2 M(z,x,y) = 0$$

where ζ is an appropriate cube root of unity. Now Connes' Lemma means essentially the corresponding identity

$$\hat{M}(x, y, z, t) + \zeta \hat{M}(y, z, x, t) + \zeta^2 \hat{M}(z, x, y, t) = 0$$

for the holomorphic extensions. Connes' Lemma provides a geometric interpretation of $\hat{M}(x, y, z, t)$ in terms of fixed points of certain affine transformations. Namely, $\hat{M}(x_1, x_2, x_3, t)$ is the fixed point of $g_1 g_2$ where

$$g_1(u) = \sqrt[3]{\frac{x_1 + x_2 - x_3 - t}{x_3 + x_1 - x_2 - t}}(u - x_1) + x_1$$
$$g_2(u) = \sqrt[3]{\frac{x_2 + x_3 - x_1 - t}{x_1 + x_2 - x_3 - t}}(u - x_2) + x_2$$

Perhaps it is worthwhile to consider the holomorphic extensions of other so-called triangle centers and to find geometric interpretations.

Example 9. Consider the rational function

$$A(x, y, z, t) = \frac{(x - y)(y - z)(z - x)}{x + y - z - t}$$

In the Euclidean case one has

$$|A(x, y, z, t)| = 2\Delta(x, y, z)$$

where Δ denotes the area of a triangle. This is easy to deduce from

$$A(0, 1, z) = \frac{\bar{z} - z}{2}$$

Remark 5. Here is another point of view. For non-degenerate triangles x_0, x_1, x_2 in some Euclidean plane E one can use the Euler line together with its points O and H as a coordinate system. So let us identify E with $\mathbf{C}$ by the conditions $O = 0$ and $H = 1$. This way the x_0, x_1, x_2 become complex numbers subject to the relation $x_0 + x_1 + x_2 = 1$ and with circumcenter the origin. (One is tempted to call them the "Euler coordinates" of the triangle.)

For the holomorphic extensions $\hat{f}(x_0, x_1, x_2, t)$ this simply means to restrict to $t = 1$ and $x_0 + x_1 + x_2 = 1$. The function $\hat{f}$ is uniquely determined by this restriction, because of the G-equivariance. In other words:

Triangle functions are nothing else than *complex* analytic functions on the 2-simplex

$$\Delta^2 = \{\, (x_0, x_1, x_2) \in \mathbf{A}^3 \mid x_0 + x_1 + x_2 = 1 \,\}$$

One may also just restrict to $t = x_0 + x_1 + x_2$. This way we see: Triangle functions are essentially complex analytic functions on $\mathbf{A}^3$ which are homogeneous of degree 1; in other words: complex analytic sections in the bundle $\mathcal{O}(1)$ on $\mathbf{P}^2$.

Remark 6. Continuing this discussion, we may also take the permutation action of S_3 into account. Given a triangle function f, we consider also its conjugates $f \circ \sigma$, $\sigma \in S_3$.

In the following let us assume that $\hat{f}|\{t = x_0 + x_1 + x_2\}$ is rational and can be defined over some field F.

We start with the following setting: a *triangle over* F is an element x in a cubic extension K of F. K need not be a field, but let's say it is an etale ring extension of F of rank 3 (having in mind arbitrary flat ring extensions of F of rank 3). Let D

be the discriminant algebra of K and $H = K \otimes D$. Then, given a rational function $\hat{f}(x_0, x_1, x_2, t)$, the 6-tuple

$$(\hat{f} \circ \sigma)_{\sigma \in S_3}$$

yields a rational function

$$\widetilde{f}\colon K \to H$$

such that

$$\widetilde{f}(x_0, x_1, x_2) = \big(\hat{f}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}, x_0 + x_1 + x_2)\big)_{\sigma \in S_3}$$

in the case $K = F \times F \times F$.

Example 10. Suppose $\operatorname{char} F \neq 2$. For the pedal point P one finds

$$\widetilde{P}(x) = \frac{1}{2}\left(T_{K/F}(x) - \frac{N_{K/F}(x)}{x^2}\right)$$

where $T_{K/F}$, $N_{K/F}\colon K \to F$ is the trace resp. norm.

Example 11. For the point S (the reflection of x at the opposite side of the triangle, see Example 6) one finds

$$\widetilde{S}(x) = x + 2\big(\widetilde{P}(x) - x\big) = x - \frac{1}{x}\frac{d}{dt}\Phi_x(t)|_{t=x}$$

where $\Phi_x(t)$ is the characteristic polynomial of x.

3. Points on a circle

Here is a variant of “holomorphic extension”, which is actually much simpler.

Let $C_n = \mathbf{G}_{\mathrm{m}}^n$ where $\mathbf{G}_{\mathrm{m}}$ is the multiplicative group. Let further

$$D_n = \{\, (z_1, \ldots, z_n) \in C_n(\mathbf{C}) \mid |z_1| = \cdots = |z_n| \,\}$$

be the real subvariety of n-tuples of complex numbers lying on some circle around the origin. Let $G = \mathbf{G}_{\mathrm{m}}$ act on C_n diagonally. Then D_n is G-stable.

It is easy to see that G-equivariant real analytic functions $f\colon D_n \to \mathbf{C}$ extend uniquely to G-equivariant complex analytic functions $\hat{f}\colon C_n \to \mathbf{C}$.

Example 12. Let $n = 4$ and for $(z_1, z_2, z_3, z_4) \in D_4$ let $f(z_1, z_2, z_3, z_4)$ be the intersection of the lines $z_1 z_2$ and $z_3 z_4$. Then (cf. [4])

$$\hat{f}(z_1, z_2, z_3, z_4) = \frac{z_1^{-1} + z_2^{-1} - z_3^{-1} - z_4^{-1}}{z_1^{-1} z_2^{-1} - z_3^{-1} z_4^{-1}}$$

`The remaining sections have been appended later and could perhaps be better merged with the previous text.`

4. An Involution

4.1. **The map τ.** The group $\mathbf{G}_{\mathrm{a}}$ acts on $\mathbf{A}^4$ via

$$(x, a) \mapsto x + a = (x_0 + a, x_1 + a, x_2 + a, x_3 + a)$$

with $x = (x_0, x_1, x_2, x_3) \in \mathbf{A}^4$ and $a \in \mathbf{G}_{\mathrm{a}}$.

The group S_4 acts on $\mathbf{A}^4$ by permutation of the coordinates.

In the following $ijk\ell$ is any permutation of 1234.

Let

$$\rho_{ij}(x) = x_i + x_j - x_k - x_\ell$$

The 6 polynomials $\rho_{ij}(x)$ are $\mathbf{G}_\mathrm{a}$-invariant. One has $\rho_{ij} = -\rho_{k\ell}$. If 2 is invertible, the functions ρ_{01}, ρ_{02}, ρ_{03} form a coordinate system for $\mathbf{A}^4/\mathbf{G}_\mathrm{a}$.

Let further

$$R(x) = \rho_{ij}(x)\rho_{ik}(x)\rho_{i\ell}(x)$$

The polynomial R is independent of the choice of $ijk\ell$, it is S_4-invariant and $\mathbf{G}_\mathrm{a}$-invariant.

Let further

$$\begin{aligned} D_i(x) &= x_j^2 + x_k^2 + x_\ell^2 - x_jx_k - x_kx_\ell - x_\ell x_j \\ &= (x_j + \zeta x_k + \zeta^{-1}x_\ell)(x_j + \zeta^{-1}x_k + \zeta x_\ell) \end{aligned}$$

with $1 + \zeta + \zeta^{-1} = 0$. The polynomial D_i is $\mathbf{G}_\mathrm{a}$-invariant and invariant under permutations of $jk\ell$.

Finally consider the rational map

$$\tau\colon \mathbf{A}^4/\mathbf{G}_\mathrm{a} \to \mathbf{A}^4$$

$$\tau(x) = -\frac{1}{R(x)}\big(D_0(x), D_1(x), D_2(x), D_3(x)\big)$$

It is of degree -1.

One finds

$$\rho_{ij}\big(\tau(x)\big) = \frac{1}{\rho_{ij}(x)}$$

Thinking of the $\rho_{ij}(x)/\rho_{ik}(x)$ as angles, maybe $\tau(x)$ can be thought of as the "algebraic inversion" of the quadrangle x.

The previous relation shows

$$(\pi\circ\tau)^2 = \mathrm{id}_{\mathbf{A}^4/\mathbf{G}_\mathrm{a}}$$

where $\pi\colon \mathbf{A}^4 \to \mathbf{A}^4/\mathbf{G}_\mathrm{a}$ is the projection. In particular, $\pi\circ\tau$ is a birational isomorphism of $\mathbf{A}^4/\mathbf{G}_\mathrm{a}$.

4.2. **Coordinate free setups.** One can set up τ a little bit more coordinate free as a map

$$\tau\colon E^4/V \to (V^*)^4$$

where $E = e + V$ is a 1-dimensional affine space with underlying vector space V.

Or, let H be a quartic extension of the ground ring F. Define

$$\widetilde{D}\colon H/F \to H$$

$$\widetilde{D}(x) = -2x^2 + xT_{H/F}(x) + T_{H/F}(x^2) - Q_{H/F}(x)$$

where Q denotes the second coefficient of the characteristic polynomial. If $H = F^4$, then

$$\widetilde{D}(x) = \big(D_0(x), D_1(x), D_2(x), D_3(x)\big)$$

One has

$$\widetilde{D}^2(x) = R(x)x \mod F$$

for a certain polynomial R in the coefficients of the characteristic polynomial. Hence one can generalize τ as

$$\tau\colon H/F \to H$$

$$\tau(x) = -\frac{D(x)}{R(x)}$$

as long as R is not identically 0.

4.3. **Sections to $\mathbf{A}^n \to \mathbf{A}^n/\mathbf{G}_\mathrm{a}$.** It follows also that

$$\tau \circ \pi \circ \tau \colon \mathbf{A}^4/\mathbf{G}_\mathrm{a} \to \mathbf{A}^4$$

is a (rational) section to π which is S_4-equivariant.

Note that if 2 is not invertible, then π does not have a S_4-equivariant *linear* section.

What about S_n-equivariant rational sections to the projection $\mathbf{A}^n \to \mathbf{A}^n/\mathbf{G}_\mathrm{a}$ (with $\mathbf{G}_\mathrm{a}$ acting diagonally)?

For $n = 2$ it is easy to see that such a section does not exist if $2 = 0$.

For $n > 2$ such a section does exist over any field, basically because S_n acts generically free on $\mathbf{A}^n/\mathbf{G}_\mathrm{a}$.

For $n = 3$ such a section is given by

$$\mathbf{A}^3/\mathbf{G}_\mathrm{a} \to \mathbf{A}^3$$
$$[x_0, x_1, x_2] \mapsto$$
$$\frac{\big((2x_0 - x_1 - x_2)(x_1 - x_2)^2, (2x_1 - x_2 - x_0)(x_2 - x_0)^2, (2x_2 - x_0 - x_1)(x_0 - x_1)^2\big)}{x_0^2 + x_1^2 + x_2^2 - x_0x_1 - x_1x_2 - x_2x_0}$$

This map is closely related to the "basic line" in [5].

For $n = 4$ we have the section $\tau \circ \pi \circ \tau$, which is remarkably more or less the square of the less complicated function τ.

The cases $n = 3$, 4 can be brought in a common form as follows. Let L/F be a separable extension of degree n. For $x \in L$ let

$$P_x(t) = N_{L/F}(t - x) = t^n - T_{L/F}(x) + \cdots + (-1)^n N_{L/F}(x)$$

be the characteristic polynomial of x. Consider the function

$$g(x) = \frac{N_{L/F}\big(T_{L/F}(x) - nx\big)}{T_{L/F}\left(\dfrac{dP_x}{dt}\Big|_{t=x}\right)}$$

One has

$$g(ax + b) = ag(x)$$

for a, $b \in F$.

One finds for $n = 3$, 4 that

$$g(x) = T_{L/F}(x) \mod n$$

(This does not hold for $n = 5$.) Let further

$$f(x) = \frac{T_{L/F}(x) - g(x)}{n}$$

This has sense: We may divide by n for the universal extension $L = \mathbf{Z}[x_1, \ldots, x_n]$ over $F = L^{S_n} = \mathbf{Z}[\sigma_1, \ldots, \sigma_n]$ and then specialize.

It is clear that

$$f(ax + b) = af(x) + b$$

for a, $b \in F$. It follows that

$$\sigma \colon L/F \to L$$
$$\sigma(x + F) = x - f(x)$$

is a section to the projection $L \to L/F$. In the split case $L = F^n$ we get exactly the sections $\mathbf{A}^n/\mathbf{G}_\mathrm{a} \to \mathbf{A}^n$ considered before.

I did not look at $n > 4$, but I think that there is a substantial difference to the cases $n \le 4$.

4.4. **Relation with triangle functions.** The automorphism $\pi \circ \tau$ has the following interpretation as a sort of algebraic version of the complex conjugation for triangle functions.

Over the field $\mathbf{C}$ of complex numbers let $f_i\colon \mathbf{C}^4 \to \mathbf{C}$, $i = 0, 1, 2$ be rational functions which are $\mathrm{Aff}(1, \mathbf{C})$ equivariant.

Denote by $\bar{f}$ the complex conjugate of f, that is $\bar{f}(x) = \overline{f(\bar{x})}$.

Let $X \subset \mathbf{C}^4$ be the (real) subvariety of orthocentric quadrangles.

Lemma 2. *The following statements are equivalent:*

(1) *For all $x \in X$ the points $f_0(x)$, $f_1(x)$, $f_2(x)$ lie on a real line.*
(2) *For all $x \in \mathbf{C}^4$ one has*

$$\frac{f_1 - f_0}{f_2 - f_0}(x) = \frac{\bar{f}_1 - \bar{f}_0}{\bar{f}_2 - \bar{f}_0}\big(\tau(x)\big)$$

□

Suppose the triangle functions f_i have real coefficients. Then the last condition amounts to the $\pi \circ \tau$-invariance of the cross ratio of the f_i.

4.5. **More remarks on the pedal point.** For an orthocentric quadrangle $x = (x_0, x_1, x_2, x_3)$ the intersection of the lines x_0x_1, x_2x_3 is the pedal point

$$P(x_0, x_1, x_2, x_3) = \frac{x_0x_1 - x_2x_3}{x_0 + x_1 - x_2 - x_3}$$

see Example 5.

For a quadrangle $x = (x_0, x_1, x_2, x_3)$ on a circle around 0, the intersection of the lines x_0x_1, x_2x_3 is given by

$$f(x_0, x_1, x_2, x_3) = \frac{x_1x_2x_3 + x_0x_2x_3 - x_0x_1x_2 - x_0x_1x_3}{x_0x_1 - x_2x_3}$$

see Example 12.

In other words,

$$f(x_0, x_1, x_2, x_3) = P(x_0^{-1}, x_1^{-1}, x_2^{-1}, x_3^{-1})^{-1}$$

Maybe the map τ is closely related with this coincidence (or maybe not).

The formula

$$P(x_0, x_1, x_2, x_3) = \frac{x_0x_1 - x_2x_3}{x_0 + x_1 - x_2 - x_3}$$

appears also in [1].

Here are further interpretations. Let a, b, c, d be four (general) points in an affine line.

Let f be the affine transformation with $f(a) = c$ and $f(d) = b$. Let g be the affine transformation with $g(a) = d$ and $g(c) = b$. Consider the commutator $h = f^{-1}g^{-1}fg$. As any commutator in the group of affine transformations, h is a translation. Moreover $h(a) = a$, and therefore h is the identity. Thus f and g commute. Their common fixed point is

$$P(a,b,c,d) = \frac{ab - cd}{a + b - c - d}$$

This way $P(a,b,c,d)$ appears as the fixed point of an action of $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{A}^1$ by affine transformations.

Lemma 3. *Let h be a transformation of a projective line. Suppose there are distinct points a, b with $h(a) = b$, $h(b) = a$. Then h^2 is the identity.*

Proof. h^2 has at least 3 distinct fixed points, namely a, b and a fixed point of h. □

Now let h be the projective transformation with $h(a) = b$, $h(b) = a$, $h(c) = d$. By the Lemma one has $h(d) = c$. Then one has $h^2 = \mathrm{id}$ and one finds

$$h(\infty) = P(a,b,c,d) = \frac{ab - cd}{a + b - c - d}$$

One can see directly that the fixed point of f above coincides with $h(\infty)$, without an explicit calculation. Namely consider $k = hf$. Then $k(a) = d$, $k(d) = a$. By the Lemma, $k^2 = \mathrm{id}$. Thus

$$hfh^{-1} = f^{-1}$$

Therefore h permutes the fixed points of f, which are ∞ and $P(a,b,c,d)$.

Further, let f, g be the projective transformations which fix c, d and which send ∞ to a resp. b. Then $fg = gf$ and one finds $fg(\infty) = P(a,b,c,d)$.

5. Exterior Algebra of a cubic extension

Let K be a cubic extension of a field F.

For $x \in K$ one denotes by

$$N_{K/F}(t - x) = t^3 - T_{K/F}(x)t^2 + Q_{K/F}(x)t - N_{K/F}(x)$$

its characteristic polynomial and by

$$x^{\#} = x^2 - T_{K/F}(x)x + Q_{K/F}(x) = \frac{N_{K/F}(x)}{x}$$

its adjoint.

The highest exterior power $\Lambda^3 K$ is a 1-dimensional F-vector space. Thus for ω, $\omega' \in \Lambda^3 K$ we have $\omega/\omega' \in F$, provided $\omega' \neq 0$.

In the following we assume (sometimes) that K/F is separable and use the canonical identifications

$$\Lambda^2 K = \mathrm{Hom}(K, \Lambda^3 K) = K \otimes \Lambda^3 K$$
$$(\Lambda^3 K)^{\otimes 2} = F$$

via the trace form. Thus for ω, $\omega' \in \Lambda^2 K$ we have $\omega/\omega' \in K$, provided ω' is nondegenerate.

Here is a basic formula which connects the $\wedge$-product with the multiplication in K:

$$ux \wedge y \wedge z + x \wedge uy \wedge z + x \wedge y \wedge uz = T_{K/F}(u) x \wedge y \wedge z$$

with u, x, y, $z \in K$.

Here are a few more formulas:

$$x(y \wedge z) + y(z \wedge x) + z(x \wedge y) = x \wedge y \wedge z$$
$$(1 \wedge x \wedge y)(1 \wedge z) + (1 \wedge y \wedge z)(1 \wedge x) + (1 \wedge z \wedge x)(1 \wedge y) = 0$$
$$x(1 \wedge y) + y(1 \wedge x) = 1 \wedge [xT(y) + yT(x) - 2xy]$$
$$\big(x(1 \wedge x \wedge y) - (1 \wedge x \wedge xy)\big)\big(y(1 \wedge y \wedge x) - (1 \wedge y \wedge xy)\big) = (1 \wedge x \wedge x^2)(1 \wedge y \wedge y^2)$$

Consider the binary cubic form

$$\psi_{K/F} \colon K/F \to \Lambda^3 K$$
$$\psi_{K/F}(x + F) = 1 \wedge x \wedge x^2$$

This form is fundamental in understanding cubic extensions, since one can recover from it the algebra K. This will be considered elsewhere.

6. Expressions for the circumcenter, the orthocenter and the Euler center

Now let $F = \mathbf{C}$, $K = \mathbf{C}^3$ and let $x = (x_1, x_2, x_3) \in K$ be a triangle.

As usual, $\bar{x}$ denotes complex conjugation.

For a triangle we call from now on the nine-point center of a triangle x the *Euler center* and denote it by $E(x_1, x_2, x_3)$ (the letter F will be reserved for the Fermat points). Recall also other basic triangle points: the center of gravity G (or barycenter, center of mass, centroid), the circumcenter O, and the orthocenter H. (I am tempted to rename O to C and H to O, but that might be too confusing for now.)

For a triangle x consider the triangle $\Phi(x)$ geometrically obtained from x by drawing through each vertex the line parallel to the opposite side. The construction leads to the expressions

$$\Phi(x) = T_{K/F}(x) - 2x, \quad \Phi^{-1}(x) = \frac{T_{K/F}(x) - x}{2}$$

One has

$$G(x) = G\big(\Phi(x)\big)$$
$$H(x) = O\big(\Phi(x)\big)$$
$$O(x) = E\big(\Phi(x)\big)$$

The observation $H(x) = O\big(\Phi(x)\big)$ can be used to show that the altitudes are concurrent.

If we use the normalization $G(x) = 0$, then $x \mapsto \Phi(x)$ is just multiplication by -2. This implies immediately the first two of the following relations:

$$3G = 2O + H$$
$$3G = O + 2E$$
$$2E = O + H$$
$$4E = 3G + H$$

Clearly one has

$$3G(x) = T_{K/F}(x)$$

Lemma 4. *One has*

$$E(x) = \frac{1}{2} \cdot \frac{1 \wedge x^2 \wedge \bar{x}}{1 \wedge x \wedge \bar{x}}$$
$$O(x) = \frac{1 \wedge x \wedge x\bar{x}}{1 \wedge x \wedge \bar{x}}$$
$$O(x) = -\frac{1 \wedge x^{\#} \wedge \bar{x}}{1 \wedge x \wedge \bar{x}}$$
$$H(x) = \frac{1 \wedge (x^2 + x^{\#}) \wedge \bar{x}}{1 \wedge x \wedge \bar{x}}$$

Proof. Left to the reader. □

Note further that the denominator $1 \wedge x \wedge \bar{x}$ vanishes exactly when the triangle x lies on a real line.

Here is another way to look at the orthocenter. Let H be a quartic extension of the ground ring F and let

$$\theta \colon H \to H$$
$$\theta(x) = 2x^2 - xT_{H/F}(x)$$

Note that θ factors to a map $H/F \to H/F$. One has:

Lemma 5. *Let $F = \mathbf{C}$, $H = F^4$ and let $x \in H$ be a generic element.*

(1) *The quadrangles x and $\theta^2(x)$ are similar.*
(2) *The quadrangle x is orthocentric if and only if $\overline{x}$ is similar to $\theta(x)$.*

Proof. Left to the reader. But see subsection 4.2. The map θ is very close to $\widetilde{D}$. □

7. The Fermat point

See [3, Section 1.8].

Given a triangle $x = (x_1, x_2, x_3) \in K = \mathbf{C}^3$, draw at each side the outside equilateral triangle. Let y_1, y_2, y_3 be the corresponding new vertices, numbered so that x_i, x_j, y_k are the equilateral triangles. Then the lines $x_i y_i$ meet in one point, the first *Fermat point* F_1. The second Fermat point F_2 is obtained analogously by using equilateral triangles pointing inwards.

In the following we describe the holomorphic extensions of F_1 and F_2. It turns out that it is better to use the variables (x_1, x_2, x_3, O) instead of (x_1, x_2, x_3, H).

Let K be a cubic extension of a field F (of characteristic different form 3) and let $u \in K$ be a Kummer element, that is, its characteristic polynomial is of the form $t^3 - a$ with $a \in F \setminus \{0\}$. For $x \in K$ and $t \in F$ we define the rational function

$$F_u(x,t) = \frac{(1 \wedge x \wedge ux)t - (1 \wedge x \wedge ux^2)}{(1 \wedge x \wedge u)t - (1 \wedge x \wedge ux)}$$

This function has some remarkable properties.

First note that F_u depends on u only up to scalar multiplication. Now there are only two Kummer elements up to scalar multiplication, given in the split case $K = F^3$ by

$$u = (1, \zeta, \zeta^{-1})$$

with $\zeta \in \mu_3 \setminus \{1\}$.

Lemma 6. *For a triangle* $x = (x_1, x_2, x_3) \in K = \mathbf{C}^3$ *the two Fermat points are given by*

$$F_u\big(x_1, x_2, x_3, O(x_1, x_2, x_3)\big), \qquad F_{u^{-1}}\big(x_1, x_2, x_3, O(x_1, x_2, x_3)\big)$$

Proof. Left to the reader. □

Consider the two functions

$$f_u(x) = F_u(x, \infty) = \frac{1 \wedge x \wedge ux}{1 \wedge x \wedge u}$$

In the Euclidean case the interpretation is clear: The $f_u(x)$ are the Fermat points for three points on a real line (since $O = \infty$ if and only if the triangle x lies on a line).

There is also an algebraic interpretation: For $x = (x_1, x_2, x_3) \in K = F^3$ the $f_u(x)$ are the fixed points of the projective transformation $\Phi \in \mathrm{PGL}(2, F)$ with

$$\Phi \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} x_{i+1} \\ 1 \end{bmatrix} \qquad (i \bmod 3)$$

For the mid point of the $f_u(x)$ one has

$$\begin{aligned} 2k(x) &= f_u(x) + f_{u^{-1}}(x) \\ &= \frac{x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 - 6x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1} \end{aligned}$$

One finds

$$T_{K/F}(x) = 2k(x) + f(x) = f_u(x) + f_{u^{-1}}(x) + f(x)$$

where f is the function from Section 4.3 (for $n = 3$) which is also the base point of the "basic line" in [5].

It is perhaps worthwhile to consider the denominator of F_u more closely. One has for

$$\begin{aligned} K &= e_1F + e_2F + e_3F \\ x &= (x_1, x_2, x_3) \in K, \quad x_4 \in F \\ u &= (u_1, u_2, u_3) \in K \end{aligned}$$

the computation

$$\frac{(1 \wedge x \wedge ux) - x_4(1 \wedge x \wedge u)}{e_1 \wedge e_2 \wedge e_3} =$$
$$u_1(x_1 - x_4)(x_2 - x_3) + u_2(x_2 - x_4)(x_3 - x_1) + u_3(x_3 - x_4)(x_1 - x_2)$$

This function is invariant under S_4 acting on (x_1, x_2, x_3, x_4) in the standard way and on (u_1, u_2, u_3) via $S_4 \to S_3$. I have no interpretation for this.

Further, let us consider $F_u(x, t)$ as a linear fractional function in t. One has

$$F_u(x)(t) = \begin{bmatrix} \beta(x) & -\alpha(x) \\ \gamma(x) & -\beta(x) \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

where

$$\begin{aligned} \alpha(x) &= 1 \wedge x \wedge ux^2 \\ \beta(x) &= 1 \wedge x \wedge ux \\ \gamma(x) &= 1 \wedge x \wedge u \end{aligned}$$

Since the matrix has trace 0, it follows that $t \mapsto F_u(x,t)$ is of order 2 (I have no interpretation of this).

Note further that

$$\alpha(x+a) = \alpha(x) + 2a\beta(x) + a^2\gamma(x)$$

with $a \in F$. Therefore F_u is basically determined by α, and so every mystery about the Fermat points should be encoded in $\alpha(x) = \alpha_u(x)$.

Consider further the cubic map

$$A\colon K \to \operatorname{Hom}(K, \Lambda^3 K) = \Lambda^2 K$$
$$x \mapsto (y \mapsto 1 \wedge x \wedge yx^2)$$

If $u \in K$ is a Kummer element, then $K = F + uF + u^{-1}F$ and the functions $\alpha_u(x)$, $\alpha_{u^{-1}}(x)$ and $\psi(x) = 1 \wedge x \wedge x^2$ are just the corresponding components of A.

8. On the cubic form in 6 variables

First, more exterior algebra of a cubic extension. Let K be a separable cubic extension of a field F. The norm for K defines a cubic form

$$N_{K/F}\colon \Lambda^2 K \to \Lambda^3 K$$

Let

$$f\colon K \times K \to \Lambda^3(K/F)$$
$$f(x,y) = 1 \wedge (x+y) \wedge (xy)$$

see [7].

For x, $y \in K$ with y generic we define

$$Z_y(x) = \frac{y \wedge x}{y \wedge 1}$$

One finds

$$Z_{cy}(ax+b) = aZ_y(x) + b \qquad (a,\ b,\ c \in F,\ c \neq 0) \tag{1}$$

$$f\big(x, Z_y(x)\big) = 0 \tag{2}$$

$$Z_y(x) = \frac{y^{\#}(1 \wedge y \wedge x) + (y^2 \wedge y \wedge x)}{N_{K/F}(y \wedge 1)} \tag{3}$$

Equation (2) provides us with the solution $z = Z_y(x)$ of the equation $f(x,z) = 0$ in z.

We now restrict to the case $K = F \times F \times F$. Write

$$z = Z_y(x) \in F \times F \times F$$

One finds that

$$\frac{x_j - z_i}{x_k - z_i} = \frac{y_j}{y_k} \tag{4}$$

is an alternative defining equation for $Z_y(x)$.

We now restrict to the Euclidean case $F = \mathbf{C}$.

Equation (3) means that the triangle $Z_y(x)$ is similar to the triangle y^{-1}.

Let us consider equation (4). It means that the sum of the angles of the triangles $\Delta_k = (x_i, x_j, z_k)$ at z_k is 2π. Let us further assume that $|y_1| = |y_2| = |y_3|$, where $|\cdot|$ is the Euclidean norm. Then equation (4) shows the $|x_j - z_i| = |x_k - z_i|$ and we can consider the circles C_k around z_k through x_i, x_k. It is an exercise to observe that these circles meet in a common point. In the case when $y = (1, \zeta, \zeta^{-1})$, this point is the Fermat point, see [3, Section 1.8, Exercise 3].

We recall Napoleon's theorem: The barycenters of the (outside) equilateral triangles used to define the Fermat point form itself an equilateral triangle. See [3, Section 1.8, Exercise 3].

Now these barycenters are exactly the points z_k in the case when when $y = (1, \zeta, \zeta^{-1})$ is a Kummer element with ζ a suitable primitive cube root of unity. Napoleon's theorem follows now from equation (3) which shows that the triangle z is similar to y^{-1}. It follows also directly from the computation

$$Z_{(1,\zeta,\zeta^{-1})}(x_1, x_2, x_3) = \frac{(x_2 - \zeta x_3, x_3 - \zeta x_1, x_1 - \zeta x_2)}{1 - \zeta}$$

9. More on the cubic form in 6 variables

What about a geometric interpretation of the equation $f(x, y) = 0$?

Let $K = F \times F \times F$. Then canonically $\Lambda^3 K = F$ up to a sign due to a choice of the order of the idempotents of K.

One has

$$f(x_1, x_2, x_3, y_1, y_2, y_3) =$$
$$(x_1 - y_2)(x_2 - y_3)(x_3 - y_1) - (x_1 - y_3)(x_2 - y_1)(x_3 - y_2)$$

Let $u = (u_1, u_3, u_5, u_2, u_4, u_6)$. Then $f(u) = 0$ means essentially that

$$\prod_{i=1}^{6} (u_{i+1} - u_i)^{(-1)^i} = 1$$

In other words, the condition $f(u) = 0$ means that for the hexagon u_1, u_2, u_3, u_4, u_5, u_6 the alternating product of the sides is $= 1$. This interpretation however does not reflect the symmetry of f under $\mathbf{Z}/2 \times S_4 = \mathbf{Z}/2 \wr S_3$.

Lemma 7. *Suppose all $x_1, x_2, x_3, y_1, y_2, y_3$ are pairwise distinct. Then $f(x, y) = 0$ if and only if there exists $\varphi \in \mathrm{PGL}(2, F) = \mathrm{Aut}(\mathbf{P}^1)$ such that*

$$\varphi(x_i) = y_i, \quad \varphi(y_i) = x_i \qquad (i = 1,\ 2,\ 3)$$

In this case, φ is uniquely determined and one has $\varphi^2 = 1$. □

Clearly, φ is uniquely determined by $\varphi(x_i) = y_i$, $i = 1, 2$ and one has $\varphi^2 = 1$ by Lemma 3. As for an explicit description, one finds with $v = (x_1, y_1, x_2, y_2)$

$$\varphi = \begin{bmatrix} \beta(v) & -\alpha(v) \\ \gamma(v) & -\beta(v) \end{bmatrix}$$

where

$$\begin{aligned} \alpha(v) &= x_1 y_1 (x_2 + y_2) - x_2 y_2 (x_1 + y_1) \\ \beta(v) &= x_1 y_1 - x_2 y_2 \\ \gamma(v) &= x_1 + y_1 - x_2 - y_2 \end{aligned}$$

Note further that

$$\begin{aligned} \alpha(v + a) &= \alpha(v) + 2a\beta(v) + a^2 \gamma(v) \\ \alpha(v^{-1}) &= \frac{-1}{x_1 y_1 x_2 y_2} \gamma(v) \end{aligned}$$

with $a \in F$. Therefore φ can be pulled out of the noses of γ and PGL(2).

A special case is the pedal point

$$\varphi(\infty) = P(x_1, y_1, x_2, y_2)$$

This and the interpretation of $P(x_1, y_1, x_2, y_2)$ in [1] shows:

Lemma 8. *Let*

$$u = (u_{01}, u_{02}, u_{03}, u_{23}, u_{31}, u_{12}) \in (\mathbf{CP}^1)^6$$

be a generic 6*-tuple of points in the Riemann sphere. Then the following statements are equivalent:*

(1) $f(u) = 0$.
(2) *The* 4 *circumcircles* $u_{01}u_{02}u_{03}$, $u_{01}u_{12}u_{31}$, $u_{02}u_{12}u_{23}$, $u_{03}u_{23}u_{31}$ *meet in a common point.*
(3) *The* 4 *circumcircles* $u_{23}u_{31}u_{12}$, $u_{23}u_{03}u_{02}$, $u_{31}u_{03}u_{01}$, $u_{12}u_{01}u_{02}$ *meet in a common point.*

What about describing these points of intersection?

Here is another way to phrase this. Consider 4 generic circles C_i through a point P. Then the 6 points u_{ij} of intersection with $C_i \cap C_j = \{P, u_{ij}\}$ satisfy the relation $f(u_{01}, u_{02}, u_{03}, u_{23}, u_{31}, u_{12}) = 0$ and any (generic) 6-tuple u with $f(u) = 0$ is obtained this way.

Finally we mention a modular property of f. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}(2, F)$$

act on $u = (u_1, u_2, u_3, u_4, u_5, u_6)$ diagonally by

$$u_i \mapsto \frac{au_i + b}{cu_i + d}$$

Then

$$(f \circ g)(u) = \frac{(ad - bc)^3}{\prod_{i=1}^6 (cu_i + d)} f(u)$$

10. The Kiepert hyperbola

Lemma 9. *Fix* $t \in \mathbf{R}$ *and let*

$$b = \frac{1 + it}{2}, \quad c = \frac{1 - it}{2}$$

where $i^2 = -1$. *For a triangle* $x = (x_0, x_1, x_2) \in \mathbf{C}^3$ *define the points*

$$y_i = bx_{i+1} + cx_{i+2} \qquad (i \bmod 3)$$

Then the lines ℓ_i *through* x_i *and* y_i *meet in a point, denoted by* $P_t(x)$.

One may rephrase this as follows:

Corollary 1. *Fix an angle* α. *For a triangle erect on each side the isosceles triangle with base angles* α. *Then the lines through the vertices of the triangle and the new vertex of the opposite isosceles triangle are concurrent (=meet in a point).*

For a triangle x let $\mathcal{H}(x)$ be the real curve determined by $t \mapsto P_t(x)$. It turns out that $\mathcal{H}(x)$ is in general a hyperbola. It is called the Kiepert Hyperbola.

Here are special points on $\mathcal{H}(x)$:

- If $t = 0$ (or $\alpha = 0$) the points y_i are the midpoints of the sides of the triangle. Therefore $P_0(x)$ is the center of gravity.
- If $t = \infty$ (or $\alpha = \pm\pi/2$) the points y_i lie at infinity in direction orthogonal to the corresponding side. Therefore $P_\infty(x)$ is the orthocenter.
- If $t = \pm\sqrt{3}$ (or $\alpha = \pm\pi/3$) the points y_i are vertices of equilateral triangles erected at each side. Therefore $P_{\pm\sqrt{3}}(x)$ are the (first and second) Fermat points.
- If the angle α is the angle at x_i (with appropriate sign) then y_j lies on the line $x_i x_k$. In this case the point of concurrence is just x_i. Thus, for a general triangle x, there are values $t = t_i$ such that $P_{t_i}(x) = x_i$.

Proof of Lemma 9. The lines ℓ_i have the parameterizations

$$\ell_i \colon x_i + r_i(y_i - x_i), \qquad y_i = b x_{i+1} + \bar{b} x_{i+2} \qquad (r_i \in \mathbf{R})$$

The lines have a common point if $D = 0$ where

$$D = \det \begin{pmatrix} x_1 - x_0 & y_0 - x_0 & y_1 - x_1 & 0 \\ x_2 - x_0 & y_0 - x_0 & 0 & y_2 - x_2 \\ \overline{x}_1 - \overline{x}_0 & \overline{y}_0 - \overline{x}_0 & \overline{y}_1 - \overline{x}_1 & 0 \\ \overline{x}_2 - \overline{x}_0 & \overline{y}_0 - \overline{x}_0 & 0 & \overline{y}_2 - \overline{x}_2 \end{pmatrix}$$

Now D is a priori a cubic polynomial in t. But there are at least 4 zeros: $t = 0$ and the t_i, corresponding to the center of gravity and the points of the triangle. Thus $D = 0$. □

I haven't computed $P_t(x)$ explicitly in general, only for three points on a real line. For $t \neq 0$ and $z \in \mathbf{R}$ one finds

$$P_t(0, 1, z) = \frac{z(\bar{b}z + b)}{z^2 - z + 1}$$

Note that for $t = \pm\sqrt{3}$ this simplifies to

$$P_{\pm\sqrt{3}}(0, 1, z) = \frac{z}{bz + \bar{b}}$$

Here is the equation for the Kiepert hyperbola. One considers the functions (with, say, $x \in K = \mathbf{C}^3$)

$$\beta(x) = T(x)T(x\overline{x}^{\#})$$
$$\Phi(x) = \beta(x) - \overline{\beta(x)}$$

Then one has

$$\mathcal{H}(x) = \{\, z \in \mathbf{C} \mid \Phi(x - z) = 0 \,\}$$

The Kiepert hyperbola is the hyperbola determined by the points of the triangle x, its barycenter $T(x)/3$ and its orthocenter $H(x)$. It is relatively easy to see that $\Phi(x - x_i) = 0$ and $\Phi(x - T(x)/3) = 0$. To see that $\Phi(x - H(x)) = 0$, one considers the function

$$\alpha(x) = x(1 \wedge \overline{x}) \in \Lambda^2 K = \Lambda^3 K \otimes K$$

Then one observes the following identity:

$$T\big(\alpha(x)^2 - \overline{\alpha(x)}^2\big) = -2\Phi(x) \tag{5}$$

and uses the fact that $H(x) = 0$ if and only if $\alpha(x) + \overline{\alpha(x)} = 0$.

Remark 7. Along the way we have used a bunch of identities like (5). Maybe one can set up the whole topic in terms of an appropriate tensor category.

11. On the radius of the incircle

Let $x = (x_1, x_2, x_3) \in \mathbf{C}^3$ be a (generic) Euclidean triangle with the origin as circum center and with circum radius R, i. e., $|x_1| = |x_2| = |x_3| = R$. Let further r, r_1, r_2, r_3 be the radii of the incircle resp. excircles. The functions $x \mapsto r/R$, $x \mapsto r_i/R$ are real analytic functions on $(\mathbf{C}^\times/\mathbf{R}^\times)^3$, invariant also under the diagonal action of $\mathbf{C}^\times/\mathbf{R}^\times$. When considering their holomorphic extensions (see Section 3), it turns out that they generate a biquadratic extension of $\mathbf{C}(x_1, x_2, x_3)$. This will be described in the following.

Let F be field. For $x = (x_1, x_2, x_3) \in F^3$ we consider the functions

$$\begin{aligned} N(x) &= x_1x_2x_3 \\ M(x) &= (x_1 - x_2)^2x_3 + (x_2 - x_3)^2x_1 + (x_3 - x_1)^2x_2 \\ \Delta(x) &= (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2 \end{aligned}$$

We consider the polynomial

$$P_x(t) = t^4 - 8t^3 - 2\frac{M(x)}{N(x)}t^2 + \frac{\Delta(x)}{N(x)^2}$$

Note that the functions $M/N(x)$, $\Delta/N^2(x)$ are invariant under $x \mapsto ax$ ($a \in F^\times$) and $x \mapsto x^{-1}$.

The zeros of P_x can be described as follows: Let z_i with $z_i^2 = x_i$. Then

$$\rho = \frac{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)}{z_1z_2z_3}$$

and its 3 conjugates under $z_i \mapsto \pm z_i$ are the zeros of P_x. Therefore it is clear that the zeros lie in the biquadratic extension generated by z_i/z_j. (This is the biquadratic extension mentioned in Example 7.)

Lemma 10. *Let $x = (x_1, x_2, x_3) \in \mathbf{C}^3$ be a generic Euclidean triangle with $|x_1| = |x_2| = |x_3| = R$ and let r, r_1, r_2, r_3 be the radii of its incircle resp. excircles. Then*

$$-\frac{2r}{R}, \quad +\frac{2r_1}{R}, \quad +\frac{2r_2}{R}, \quad +\frac{2r_3}{R}$$

are the zeros of P_x. □

Lemma 10 includes some standard relations for the radii R, r, r_i (with a_i the side lengths of the triangle):

Corollary 2. *Then*

$$\begin{aligned} 4R &= r_1 + r_2 + r_3 - r \\ \frac{a_1^2 + a_2^2 + a_3^2}{2} &= r_1r_2 + r_2r_3 + r_3r_1 - r(r_1 + r_2 + r_3) \\ 0 &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r} \\ \frac{(a_1a_2a_3)^2}{R^2} &= rr_1r_2r_3 \end{aligned}$$

Note also that

$$4|A|R = \sqrt{|\Delta(x)|}$$

where $|A|$ denote the area of the triangle x (see Example 9). Recall also that $|A| = rs = r_i s_i$ where $s = (a_1 + a_2 + a_3)/2$ is the semi-perimeter and where $s_i = s - a_i$

Remark 8. Describe the Soddy centers. It seems to me that their holomorphic extensions lie in $K = F[z_i/z_j] \otimes F[\alpha]/(\alpha^2 + \Delta(x))$ and are conjugate under $\alpha \mapsto -\alpha$.

We remark that the tri-quadratic extension K contains also the incenters of the other subtriangles of the orthocentric quadrangle of the given triangle.

It seems that the Soddy centers, the Gergonne point and the incenter lie on a line.

Remark 9. The following observation might be helpful: Let K/F be a biquadratic extension with intermediate quadratic subextensions K_i. Let x be an element of K. If $T_{K/F}(x) = 0$, then

$$T_{K/K_1}(x) T_{K/K_2}(x) T_{K/K_3}(x) = T_{K/F}(x^{\#})$$

Proof.

$$(x + x_1)(x + x_2)(x + x_3) = x^2(x + x_1 + x_2 + x_3) + x(x_1 x_2 + x_2 x_3 + x_3 x_1) + x_1 x_2 x_3$$

□

12. The conjugate (or dual?) quadrangle

In this section E is a 2-dimensional affine space over some field F and

$$V = E - e_0$$

denotes its underlying vector space. Let $x = (x_1, x_2, x_3, x_4) \in E^4$ with $x_i \neq x_j$ for $i \neq j$. Let further ℓ_{ij} be the line through x_i and x_j and let

$$L_{ij} = L_{ji} = [x_i - x_j] \in \mathbf{P}(V)$$

be the corresponding points in the projective space of V. We suppose that the points L_{ij} are distinct, i. e., no three of the x_i lie on a line.

Lemma 11. *The 6 points L_{ij} "stand in involution". More precisely, there exist $\tau \in \mathrm{PGL}(V) = \mathrm{Aut}\big(\mathbf{P}(V)\big)$ with $\tau^2 = 1$ and such that*

$$\tau(L_{ij}) = L_{k\ell}$$

for any permutation $ijk\ell$ of 1234.

Proof. We give an explicit formula for τ. Let $v_i = x_i - x_4$, $i = 1, 2, 3$ and define

$$\Phi = \Phi_{x_1,x_2,x_3,x_4} \colon V \to V \otimes (\Lambda^2 V)^{\otimes 2}$$

$$\Phi(w) = \sum_{i=1}^{3} v_i (v_{i+1} \wedge v_{i+2})(v_{i+1} \wedge w)$$

with the indices reduced mod 3.

Note that $\dim \Lambda^2 V = 1$. One finds

$$\Phi(v_i) = (v_{i+1} - v_{i+2})(v_i \wedge v_{i+1})(v_{i+2} \wedge v_i)$$

$$\Phi\big(v_{i+1} - v_{i+2}\big) = -v_i (v_{i+1} \wedge v_{i+2})\big((v_{i+1} - v_i) \wedge (v_{i+2} - v_i)\big)$$

and therefore

$$\Phi^2 = -(v_i \wedge v_{i+1})(v_{i+1} \wedge v_{i+2})(v_{i+2} \wedge v_i)\big((v_{i+1} - v_i) \wedge (v_{i+2} - v_i)\big)\mathrm{id}_V$$

Now let $\tau = \mathbf{P}(\Phi)$ be the induced map between the projective spaces. □

Remark 10. For a permutation $\sigma \in S_4$ one finds

$$\Phi_{x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}} = (\operatorname{sgn}\sigma)\Phi_{x_1,x_2,x_3,x_4}$$

Remark 11. I don't know a geometric explanation for the existence of τ. It appeared to me first as a consequence of Lemma 7 above and of Lemma 3 in [7].

Remark 12. The converse of Lemma 11 is also true (see also Lemma 4 in [7]): Given 6 points $L_{ij} \in \mathbf{P}(V)$, $1 \le i < j \le 4$, there exist 4 points $x_i \in E$ with $L_{ij} = [x_i - x_j]$. This way the involution τ gives rise to a self map on nondegenerate quadrangles in 2-dimensional affine space up to translations and scalar multiplication:

Given $x = (x_1, x_2, x_3, x_4) \in E^4$ one may construct a new quadrupel $x' = (x'_1, x'_2, x'_3, x'_4) \in E^4$, well defined up to translations and scalar multiplication, such that x'_i lies on lines parallel to ℓ_{jk}, $\ell_{k\ell}$, $\ell_{\ell j}$.

An explicit formula for x' is given by $x'_i = \Phi_{x_1,x_2,x_3,0}(x_i)$, $i = 1, 2, 3$ and $x'_4 = 0$.

Or, for

$$x = (0, x_1, x_2, x_3) \in V^4$$

one may take

$$x' = \left(0, \frac{x_2 - x_3}{x_2 \wedge x_3}, \frac{x_3 - x_1}{x_3 \wedge x_1}, \frac{x_1 - x_2}{x_1 \wedge x_2}\right) \in (V^*)^4$$

If $F = \mathbf{R}$ and $E = \mathbf{C}$ one may take for $x = (0, 1, u, z)$ the quadrangle

$$x' = \big(0, u - z, (u - z)h(u, z), (u - z)h(z, u)\big)$$

with

$$h(u, z) = \frac{(\bar{u}z - u\bar{z})(1 - z)}{(z - \bar{z})(u - z)}$$

One observes that

$$\lim_{t\to\pm\infty} h(it, z) = \frac{(1 - z)(z + \bar{z})}{\bar{z} - z} = H(0, 1, z)$$

is the orthocenter of $(0, 1, z)$.

References

[1] A. Connes, *Symetries Galoisiennes et Renormalisation*, Seminaire Poincare Octobre 2002, arXiv:math.QA/0211199.

[2] ———, *A new proof of Morley's theorem*, Les relations entre les mathématiques et la physique théorique, Inst. Hautes Études Sci., Bures, 1998, pp. 43–46, MR 99m:51027.

[3] H. S. M. Coxeter, *Introduction to geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Reprint of the 1969 edition, MR 90a:51001.

[4] C. Lubin, *A proof of Morley's theorem*, Amer. Math. Monthly **62** (1955), 110–112, MR 16,848c.

[5] M. Rost, *Notes on cubic equations*, Notes, http://www.mathematik.uni-bielefeld.de/˜rost /binary.html, 2003.

[6] ———, *Notes on Morley's theorem*, Notes, http://www.mathematik.uni-bielefeld.de/˜rost /binary.html, 2003.

[7] ———, *The variety of angles*, Notes, http://www.mathematik.uni-bielefeld.de/˜rost /binary.html, 2004.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: rost@mathematik.uni-bielefeld.de
URL: http://www.mathematik.uni-bielefeld.de/~rost