ON A CUBIC IDENTITY FOR INVOLUTIVE HOPF ALGEBRAS

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Contents

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Pre-Preface

This is work in progress. The first version of this text (July 24) ended with section 5. Section 6 was added later (July 26); it is complete and essentially self-contained, up to standard notations. In Section 7 we started to give details on some presentations. It may change anytime.

Preface

The main purpose of this text is to present relation ($*$) in Lemma (1.1) with a detailed proof, in order to ask: Is there a reference for it?

I haven't been dealing with combinatorial group theory for decades and I am not an expert for Hopf algebras at all.

At some point I bumped into Lemma (1.1) which led me to the topic. Meanwhile I have looked into tons of papers, but I certainly may have missed important ones (and sure enough I haven't digested yet much of the developments since the 1980's).

Specific questions are

- Anybody seen Lemma (1.1)? Anybody seen Proposition (6.1)? Is there a geometric explanation of the B_4-operation?
- Any reference for the homomorphisms K_n in (2.1)?
- Any reference for the universal property of the functor K (see section 3.4)?
- Any reference for the functor $\overline{K}$ (4.1)? What about the universal relations which hold in commutative Hopf algebras and in cocommutative Hopf algebras? Any reference for the group homomorphisms $\overline{K}_n$ (see section 4.4)?
- Any comment on the remarks in section 4.3?

§1. The cubic identity

By a Hopf algebra we understand for simplicity a Hopf algebra over a field R. However our arguments are completely formal and work for very general types of Hopf algebras.

Let $H = (H, \mu, u, \Delta, c, S)$ be a Hopf algebra with the basic structure morphisms

$$\begin{aligned} \mu &\colon H^{\otimes 2} \to H \\ u &\colon R \to H \\ \Delta &\colon H \to H^{\otimes 2} \\ c &\colon H \to R \\ S &\colon H \to H \end{aligned}$$

Let further

$$\tau \colon H^{\otimes 2} \to H^{\otimes 2}$$

be the switch involution.

A Hopf algebra is called involutive if its antipode S is of order 2:

$$S^2 = 1$$

(for morphisms we usually write 1 for the identity).

We freely use Sweedler's sigma notation. We use it sumless (no sigma here) and without parenthesis.

We consider a bunch of morphisms $H^{\otimes 2} \to H^{\otimes 2}$:

$$\begin{aligned} \widetilde{\tau} &= (S \otimes 1) \circ \tau = \tau \circ (1 \otimes S) & x \otimes y &\mapsto S(y) \otimes x \\ \widetilde{\tau}' &= (1 \otimes S) \circ \tau = \tau \circ (S \otimes 1) & x \otimes y &\mapsto y \otimes S(x) \\ \phi &= (1 \otimes \mu) \circ (\Delta \otimes 1) & x \otimes y &\mapsto x_1 \otimes x_2 y \\ \phi' &= (1 \otimes \mu) \circ (1 \otimes S \otimes 1) \circ (\Delta \otimes 1) & x \otimes y &\mapsto x_1 \otimes S(x_2) y \end{aligned}$$

(1.1) Lemma. *Assume* $S^2 = 1$. *Then*

(∗) $$(\widetilde{\tau}\phi)^3 = 1$$

in $\mathrm{End}\big(H^{\otimes 2}\big)$.

Proof: We first show $\phi\phi' = 1$:

$$x \otimes y \mapsto x_1 \otimes S(x_2) y \mapsto x_{11} \otimes x_{12} S(x_2) y = x \otimes y$$

Here we used that S is a right inverse and understand $\Delta_2(x) = x_{11} \otimes x_{12} \otimes x_2$ where

$$\Delta_2 = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$$

Since $S^2 = 1$ one has $\widetilde{\tau}\widetilde{\tau}' = 1$. Hence $(\widetilde{\tau}\phi)^3 = 1$ reads as

$$\phi\widetilde{\tau}\phi = \widetilde{\tau}'\phi'\widetilde{\tau}'$$

The right side computes as

$$x \otimes y \mapsto y \otimes S(x) \mapsto y_1 \otimes S(y_2) S(x) \mapsto S(y_2) S(x) \otimes S(y_1)$$

and the left side as

$$\begin{aligned} x\otimes y \mapsto x_1\otimes x_2y &\mapsto S(x_2y)\otimes x_1 \\ &\mapsto S(x_2y)_1\otimes S(x_2y)_2x_1 \\ &= S\big((x_2y)_2\big)\otimes S\big((x_2y)_1\big)x_1 \\ &= S(x_{22}y_2)\otimes S(x_{21}y_1)x_1 \\ &= S(y_2)S(x_{22})\otimes S(y_1)S(x_{21})x_1 \\ &= S(y_2)S(x_{22})\otimes S(y_1)S^{-1}(x_{21})x_1 \\ &= S(y_2)S(x_{22})\otimes S(y_1)S^{-1}\big(S(x_1)x_{21}\big) \\ &= S(y_2)S(x)\otimes S(y_1) \end{aligned}$$

Here we used $S(uv)=S(v)S(u)$, again $S^2=1$ and $(xy)_i=x_iy_i$ (which is the main bialgebra axiom). □

A crucial point is that at some spot one gets $S(x_{21})x_1$ and not say $S(x_{22})x_1$. The latter would collapse as well for cocommutative H, but not in general. For example, let G be a finite group and $H=R^G$ (the commutative ring of functions on G). Then the following maps correspond:

$$\begin{aligned} H &\to H^{\otimes 2} & G^2 &\to G \\ x &\mapsto x_2\otimes S(x_3)x_1 & (g,h) &\mapsto hgh^{-1} \end{aligned}$$

§2. The categories $\mathcal{H}$ and $\mathcal{F}$

We consider two PROPs (symmetric monoidal categories generated by a single object).

2.1. **The category $\mathcal{H}$.** Let $\mathcal{H}$ be the "universal Hopf algebra category" which models Hopf algebras with invertible antipode. Its objects are of the form $\mathbf{H}^{\square n}$ $(n\geq 0)$. The morphisms are generated by the basic morphisms μ,u,Δ,c,S (and S^{-1}) of a Hopf algebra subject to the axioms of Hopf algebras with invertible antipode S. To mention the main bialgebra axiom: The morphisms

$$\begin{aligned} \mu\colon \mathbf{H}\,\square\,\mathbf{H} &\to \mathbf{H} \\ \Delta\colon \mathbf{H} &\to \mathbf{H}\,\square\,\mathbf{H} \end{aligned}$$

are subject to

$$\Delta\circ\mu=(\mu\,\square\,\mu)\circ(1\,\square\,\tau\,\square\,1)\circ(\Delta\,\square\,\Delta)$$

where $\tau\colon \mathbf{H}^{\square 2}\to\mathbf{H}^{\square 2}$ is the involution.

2.2. **The category $\mathcal{F}$.** Let further $\mathcal{F}$ be the category with objects the free groups

$$\mathbf{F}_n=\langle e_1,\dots,e_n|\rangle$$

with $\mathbf{F}_m\,\square\,\mathbf{F}_n=\mathbf{F}_m*\mathbf{F}_n=\mathbf{F}_{m+n}$ (free product of groups, the coproduct in the category $\mathcal{G}$ of groups) and

$$\operatorname{Hom}(\mathbf{F}_m,\mathbf{F}_n)=\operatorname{Hom}_{\mathcal{G}}(\mathbf{F}_m,\mathbf{F}_n)$$

2.3. **The functor K.** Let K be the (obvious) monoidal functor

$$K\colon \mathcal{H} \to \mathcal{F}$$

with

$$K(\mathbf{H}^{\Box n}) = \mathbf{F}_n$$

and

$$\begin{aligned} K(\mu)&\colon \mathbf{F}_2 \to \mathbf{F}_1 && e_1, e_2 \mapsto e_1 \\ K(u)&\colon \mathbf{F}_0 \to \mathbf{F}_1 && \\ K(\Delta)&\colon \mathbf{F}_1 \to \mathbf{F}_2 && e_1 \mapsto e_1 e_2 \\ K(c)&\colon \mathbf{F}_1 \to \mathbf{F}_0 && \\ K(S)&\colon \mathbf{F}_1 \to \mathbf{F}_1 && e_1 \mapsto e_1^{-1} \end{aligned}$$

It is easily checked (and basic) that the axioms of a Hopf algebra hold correspondingly in $\mathcal{F}$, so we have indeed a functor.

On morphisms this functor induces maps

$$\mathrm{Hom}_{\mathcal{H}}(\mathbf{H}^{\Box n}, \mathbf{H}^{\Box m}) \to \mathrm{Hom}(\mathbf{F}_n, \mathbf{F}_m)$$

We write

$$\Theta_n = \mathrm{Aut}_{\mathcal{H}}(\mathbf{H}^{\Box n})$$

for the automorphism groups in $\mathcal{H}$ and

$$\Phi_n = \mathrm{Aut}(\mathbf{F}_n)$$

for the automorphism group of a free group of rank n (with given basis) and denote by

(2.1) $$K_n\colon \Theta_n \to \Phi_n$$

the group homomorphisms induced by K. It is easy to lift the standard generators of Φ_n, hence the K_n are surjective.

2.4. **Interpretation by a relation for Φ_2.** If one maps relation ($*$) under

$$K_2\colon \Theta_2 \to \Phi_2$$

one gets the cubic relation $(UPO)^3 = 1$ in a classical presentation[1] of the automorphism group of a free group of with 2 generators.

§3. Further discussion of $K\colon \mathcal{H} \to \mathcal{F}$

3.1. **Universal property of $\mathcal{H}$.** The essential property of the category $\mathcal{H}$ is that for each actual Hopf algebra H in an appropriate category $\mathcal{C}$ (let us assume $\mathcal{C}$ is the category of vector spaces over a field R with the tensor product as monoidal operation), there is the monoidal functor

$$\begin{aligned} \underline{H}&\colon \mathcal{H} \to \mathcal{C} \\ \underline{H}(\mathbf{H}) &= H \\ \underline{H}(\Delta) &= \Delta_H \quad \text{etc.} \end{aligned}$$

[1] Neumann 1933 [8, §1, p. 367; §4, p. 374] (Neumann uses the transposed product of Nielsen transformations). See also Magnus-Karras-Solitar 1976 [6, Problem 3.5.2, p. 169; pp. 163]).

3.2. **Universal property of $\mathcal{F}^{\mathrm{op}}$.** A group G is nothing else than a monoidal functor

$$g\colon \mathcal{F}^{\mathrm{op}} \to \mathcal{S}$$

to the category $\mathcal{S}$ of sets: Given G, one takes for g the Hom-functor

$$g(\mathbf{F}_n) = \mathrm{Hom}_{\mathcal{G}}(\mathbf{F}_n, G) = G^n$$

and given g, the set $G = g(\mathbf{F}_1)$ comes with a group structure where the multiplication $G \times G \to G$ is given by

$$\begin{aligned} \mathbf{F}_1 &\to \mathbf{F}_2 \\ e_1 &\mapsto e_1 e_2 \end{aligned}$$

3.3. **Universal property of $\mathcal{F}$.** Let G be a finite set and let R^G be the commutative ring of functions $G \to R$. To endow G with a group-structure means the same thing as to extend R^G to a commutative Hopf algebra over R.

This remarks extends to affine algebraic groups G: If

$$G = \operatorname{Spec} R_G$$

then a group structure on G corresponds to a commutative Hopf algebra structure on R_G.

It follows that at least for the category $\mathcal{C}$ of (associative and unital) R-algebras, a monoidal functor

$$\mathcal{F} \to \mathcal{C}$$

is nothing else than a commutative Hopf algebra over R.

3.4. **Universal property of K.** In fact, this remark extends to arbitrary Hopf algebras: A Hopf algebra H is commutative if and only if its functor

$$\underline{H}\colon \mathcal{H} \to \mathcal{C}$$

admits a factorization

$$\underline{H}\colon \mathcal{H} \xrightarrow{K} \mathcal{F} \to \mathcal{C}$$

This means that $\mathcal{F}$ is the quotient category of $\mathcal{H}$ by the commutativity relation

$$\mu \circ \tau = \mu$$

Likewise $\mathcal{F}^{\mathrm{op}}$ is the quotient category of $\mathcal{H}$ by the cocommutativity relation

$$\tau \circ \Delta = \Delta$$

Details can be worked out using Proposition 1 and Exercise 3 in Mac Lane 1998 (1971) [5, III.6. Groups in Categories, p. 75–76]).

Surprisingly, so far I found only one related reference: Conant and Kassabov (2016) [1, 4. Hopf algebras and groups].

§4. The functor $\mathcal{H} \to \mathcal{F} \times \mathcal{F}^{\mathrm{op}}$

The category $\mathcal{H}$ has the "duality" functor

$$D\colon \mathcal{H} \to \mathcal{H}^{\mathrm{op}}$$
$$D \circ D = \mathrm{id}$$
$$D(\mu) = \Delta$$
$$D(u) = c$$
$$D(S) = S$$

In diagrammatic pictures like in Kuperberg 1991 [4], the functor D is given by the flip of diagrams.

Consider the functor

$$\overline{K} = (K, K \circ D)\colon \mathcal{H} \to \mathcal{F} \times \mathcal{F}^{\mathrm{op}} \tag{4.1}$$

Since $\mathcal{F}$ controls commutative Hopf algebras and $\mathcal{F}^{\mathrm{op}}$ controls cocommutative Hopf algebras, the "kernel" of $\overline{K}$ on morphisms consists of the universal relations which hold in commutative Hopf algebras and in cocommutative Hopf algebras.

The relation $S^2 = 1$ is the basic example of such a relation (every commutative or cocommutative Hopf algebra is involutive). Therefore we call such relations "strongly involutive" relations. (And a Hopf algebra which obeys them might be called a "strongly involutive Hopf algebra".)

4.1. **Examples of strongly involutive relations.** Let $H = (H, \mu, u, \Delta, c, S)$ be a Hopf algebra. We assume that the antipode S is invertible.

Consider the relations (in Sweedler notation)

$$x_1 y S(x_2) = x_2 y S(x_1) \tag{4.2}$$

$$x_2 \otimes x_1 S(x_3) y = x_2 \otimes y x_1 S(x_3) \tag{4.3}$$

These relations are understood in $\mathrm{Hom}(H^{\otimes 2}, H)$ resp. $\mathrm{End}(H^{\otimes 2})$.

Formally, relation (4.2) means

$$\mu_2 \circ (1 \otimes 1 \otimes S) \circ (1 \otimes \tau) \circ (\Delta \otimes 1) =$$
$$\mu_2 \circ (1 \otimes 1 \otimes S) \circ (1 \otimes \tau) \circ (\tau \otimes 1) \circ (\Delta \otimes 1)$$

with $\mu_2 = \mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$ and $\tau \in \mathrm{End}(H^{\otimes 2})$ the involution.

If H is cocommutative, relation (4.2) holds since $x_1 \otimes x_2 = x_2 \otimes x_1$. If H is commutative, the relation follows from the antipode properties $x_1 S(x_2) = 1$ and $S(x_1)x_2 = 1$. Relation (4.3) is obvious in the commutative case. In the cocommutative case one has $x_1 S(x_3) = 1$.

Another consequence of (4.2) is

$$S^2 = 1$$

($y = 1$ yields $x_2 S(x_1) = 1$). Thus, if H obeys (4.2), then H is involutive.

There are also the dual relations.

It should be possible to find a generating set for all strongly involutive relations.

4.2. **Abelianization.** Let $\mathcal{Z}$ be the PROP like $\mathcal{F}$ but with objects the free abelian groups $\mathbf{Z}^n$ (with basis). There is the duality functor

$$D\colon \mathcal{Z} \to \mathcal{Z}^{\mathrm{op}}$$
$$D(X) = \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$$

The induced map on morphisms

$$\mathrm{Hom}_{\mathcal{Z}}(\mathbf{Z}^n, \mathbf{Z}^m) = \mathrm{Mat}(\mathbf{Z}, m \times n)$$

is the transpose of integral matrices: $D(M) = M^t$.

Let

$$A\colon \mathcal{F} \to \mathcal{Z}$$
$$A(\mathbf{F}_n) = \mathbf{Z}^n$$

denote the natural functor given by abelianization. Then the composition

$$\mathcal{H} \xrightarrow{K} \mathcal{F} \xrightarrow{A} \mathcal{Z}$$

commutes with D on $\mathcal{H}$, $\mathcal{Z}$ (there is no duality on $\mathcal{F}$) and we have a commutative diagram

(4.4)
$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{K} & \mathcal{F} \\ {\scriptstyle K\circ D}\big\downarrow & & \big\downarrow{\scriptstyle A} \\ \mathcal{F}^{\mathrm{op}} & \xrightarrow{D\circ A} & \mathcal{Z} \end{array}$$

4.3. **Universal property of $\mathcal{Z}$.** By the way, the category $\mathcal{Z}$ models bicommutative (commutative and cocommutative) Hopf algebras: A monoidal functor

$$\mathcal{Z} \to \mathcal{C}$$

is the same thing as a bicommutative Hopf algebra in $\mathcal{C}$.

I noticed that some years ago, but still haven't seen a reference. This is strange, since the symmetric and exterior algebras of a vector space are so prominent examples of bicommutative Hopf algebras. Not to speak of affine commutative group schemes.

4.4. **The homomorphisms $\overline{K}_n$.** On automorphisms the functor $\overline{K}$ yields group homomorphisms

$$\overline{K}_n\colon \Theta_n \to \Phi_n \times \Phi_n^{\mathrm{op}}$$

It is not difficult[2] to compute the image of $\overline{K}_n$ as

$$\mathrm{im}\, \overline{K}_n = \Psi_n := \{\, (f, g) \in \Phi_n \times \Phi_n^{\mathrm{op}} \mid \bar{f} = \bar{g}^t \,\}$$

where

$$\bar{f} = A(f) \in \mathrm{GL}_n(\mathbf{Z})$$

denotes the abelianization of $f \in \Phi_n$ and M^t is the transpose of a matrix M.

An element in the kernel of $\overline{K}_n$ is a strongly involutive relation. So an obvious question is:

What is the kernel of the group homomorphism $\overline{K}_n$?

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[2] One uses (4.4) and the description of the kernel of $\Phi_n \to \mathrm{GL}_n(\mathbf{Z})$ in Magnus-Karras-Solitar 1976 [6, Theorem N4, p. 168].

This is perhaps too much to ask for, since it seems difficult to list generators for Θ_n efficiently. To get them one has to cover in principle all sequences of compositions of elementary morphisms in various $\mathrm{Hom}(\mathbf{H}^{\square h}, \mathbf{H}^{\square k})$ resulting in elements of Θ_n (just look at the case $n = 0$).

Even having a complete set of strongly involutive relations doesn't mean to get a hand on generators for Θ_n.

A more tractable question is

> Which elements $R_i \in \Theta_n$ are needed to have a section
> $$\Psi_n \to \Theta_n/\langle R_i \rangle$$
> to $\overline{K}_n$?

This can probably worked through using known presentations of Φ_n to get a presentation of Ψ_n, then take natural lifts of the generators and see what the relations give for Θ_n. (I have essentially done this for $n = 2$.)

This way relation (4.2) showed up.

The very first experiment was to look at a lift of the relation $(UPO)^3 = 1$ for Φ_2. The idea was to find an interesting expression in Θ_2. Surprisingly this utterly failed: For relation ($*$) one just needs $S^2 = 1$. This was the starting point of this text.

Later relation (4.3) appeared in a first systematic attempt to compute all strongly involutive relations (not completed).

§5. Extending $\overline{K}_n$

It seems to be helpful to extend both sides of $\overline{K}_n$ by certain natural operations. This results essentially in extensions by $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

We first discuss the $\mathcal{H}$-side.

5.1. **The functors T_μ, T_Δ.** The category $\mathcal{H}$ comes with a bunch of (anti-)automorphisms of order 2.

One is the duality functor D which yields automorphisms

$$\sigma \colon \Theta_n \to \Theta_n$$
$$\sigma(f) = D(f^{-1})$$

of the automorphism groups.

The formation of the opposite Hopf algebra H^{op} (cf. Montgomery 1993 [7, 1.5.11 Lemma, p. 9]) is modeled by the functor

$$T_\mu \colon \mathcal{H} \to \mathcal{H}$$
$$T_\mu(\mu) = \mu \circ \tau$$
$$T_\mu(S) = S^{-1}$$

and leaving u, Δ, c fixed.

Similarly there is the functor

$$T_\Delta \colon \mathcal{H} \to \mathcal{H}$$
$$T_\Delta(\Delta) = \tau \circ \Delta$$
$$T_\Delta(S) = S^{-1}$$

and with $D \circ T_\Delta = T_\mu \circ D$.

Consider the quotient categories

$$\mathcal{H}_\mu = \mathcal{H}/(T_\mu - 1), \qquad \mathcal{H}_\Delta = \mathcal{H}/(T_\Delta - 1)$$

The projection $\mathcal{H} \to \mathcal{H}_\mu$ is the identity on objects and is universal for the relation $T_\mu(f) = f$ for morphisms in $\mathcal{H}$. Likewise for $\mathcal{H}_\Delta$. Hence these categories model commutative resp. cocommutative Hopf algebras and one has

$$\mathcal{H}_\mu = \mathcal{F}, \qquad \mathcal{H}_\Delta = \mathcal{F}^{\mathrm{op}}$$

Note that $T_\mu \circ T_\Delta = T_\Delta \circ T_\mu$ is conjugation with S_n, where

$$S_n = S^{\square n} \in \Theta_n$$

Moreover, $S_n^2 \in \Theta_n$ is central.

Let us pass to the quotient category

$$\mathcal{H}' = \mathcal{H}/(S^2 - 1)$$

which models involutive Hopf algebras and put

$$\Theta'_n = \mathrm{Aut}_{\mathcal{H}'}(\mathbf{H}^{\square n})$$

The group Θ'_n is the quotient of Θ_n by the subgroup generated by the elements

$$1 \square \cdots \square 1 \square S^2 \square 1 \square \cdots \square 1$$

Clearly the functor $\overline{K}$ factors through $\mathcal{H}'$ and the homomorphism $\overline{K}_n$ factors through Θ'_n. (We could have passed to $\mathcal{H}'$ earlier.)

The group Θ'_n has a canonical extension by $\mathbf{Z}/2\mathbf{Z}$ given by additional elements

$$X_\mu, X_\Delta$$

subject to the relations

$$\begin{aligned} X_\mu f X_\mu^{-1} &= T_\mu(f) && (f \in \Theta_n) \\ X_\Delta f X_\Delta^{-1} &= T_\Delta(f) && (f \in \Theta_n) \\ X_\mu^2 = X_\Delta^2 &= 1 \\ X_\mu X_\Delta &= S_n \end{aligned}$$

(This extension can be formed also for Θ_n, but not in a straightforward way. For instance, one has a choice for $X_\mu X_\Delta = S_n^{\pm 1}$.)

5.2. **Extending Ψ_n.** On the $(\mathcal{F} \times \mathcal{F}^{\mathrm{op}})$-side one extends by the automorphism

$$\sigma\colon \Phi_n \times \Phi_n^{\mathrm{op}} \to \Phi_n \times \Phi_n^{\mathrm{op}}$$
$$\sigma(f, g) = (g^{-1}, f^{-1})$$

which leaves Ψ_n invariant and adds to Ψ_n the element

$$(\varepsilon, 1) \in \Phi_n \times \Phi_n^{\mathrm{op}}$$
$$\varepsilon(e_i) = e_i^{-1}$$

($(\varepsilon, \varepsilon)$ is the image of S_n and $(\varepsilon, 1)$ corresponds to X_Δ).

This results in the group

$$\overline{\Psi}_n = \{ \big((f, g), \sigma^k\big) \in (\Phi_n \times \Phi_n^{\mathrm{op}}) \rtimes \mathbf{Z}/2\mathbf{Z} \mid \bar{f} = \pm \bar{g}^t \}$$

The first incentive to consider this extension was to simplify presentations in terms of generators and relations, but clearly the corresponding extension on the $\mathcal{H}'$-side is noteworthy as well.

The case $n = 2$ is helped by fact that the automorphism

$$\begin{aligned} \mathrm{GL}_2(\mathbf{Z}) &\to \mathrm{GL}_2(\mathbf{Z}) \\ A &\mapsto \det(A)A^{-t} \end{aligned}$$

is inner (conjugation with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$). It follows that

$$\overline{\Psi}_2 \simeq \{\, \big((f,g),\sigma^k\big) \in (\Phi_2 \times \Phi_2) \rtimes \mathbf{Z}/2\mathbf{Z} \mid \bar{f} = \pm\bar{g} \,\}$$

with $\sigma(f,g) = (g,f)$.

`Note added August 2:` Section 7 contains a presentation for $\overline{\Psi}_2$. Lifting the relations to the extended Θ_2' seems to need just (4.2) (for now the reader is invited to try himself).

§6. The homomorphism $B_4 \to \Theta_2$

We generalize Lemma (1.1). We assume that the antipode S is invertible and do not assume $S^2 = 1$.

We consider a bunch of morphisms $H^{\otimes 2} \to H^{\otimes 2}$: First let

$$\begin{aligned} \rho &= (1 \otimes S) \circ \tau = \tau \circ (S \otimes 1) & x \otimes y &\mapsto y \otimes S(x) \\ \alpha &= (1 \otimes \mu) \circ (\Delta \otimes 1) & x \otimes y &\mapsto x_1 \otimes x_2 y \end{aligned}$$

The morphism ρ is obviously invertible, an inverse of α is given below. We put

$$\beta = \rho^{-1}\alpha\rho, \qquad \gamma = \rho^{-2}\alpha\rho^2$$

(6.1) Proposition. *One has*

(6.2) $$\alpha\beta\alpha = \beta\alpha\beta$$

(6.3) $$\beta\gamma\beta = \gamma\beta\gamma$$

(6.4) $$\alpha\gamma = \gamma\alpha$$

Hence for any Hopf algebra H we get an operation of the braid group B_4 on $H^{\otimes 2}$. (For commutative H this operation is given by the isomorphism[3] $B_4/\text{center} \to \mathrm{S}\Phi_2$.)

The proof of Proposition (6.1) takes up the rest of this section.

First some basic remarks on the antipode (cf. Montgomery 1993 [7, §1.5, p. 7, p. 9]). The antipode S is an anti-automorphism (with respect to the product and coproduct) and S^2 is an automorphism of the Hopf algebra H. It is characterized by

$$S(x_1)x_2 = 1 = x_1 S(x_2)$$

or

$$S^{-1}(x_2)x_1 = 1 = x_2 S^{-1}(x_1)$$

[3] Dyer-Formanek-Grossman 1982 [2, p. 406], Karrass-Pietrowski-Solitar 1984 [3]

The indices in Sweedler's notation[4] can be from any ordered set per variable. For instance

(6.5) $$x_1 \otimes x_{21} \otimes x_{22} = x_1 \otimes x_2 \otimes x_3 = x_{11} \otimes x_{12} \otimes x_2$$

reflects

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$$

Here is an extended and more detailed list of the morphisms:

$$\begin{aligned}
\rho &= (1 \otimes S) \circ \tau = \tau \circ (S \otimes 1) && x \otimes y \mapsto y \otimes S(x) \\
\rho^{-1} &= (S^{-1} \otimes 1) \circ \tau = \tau \circ (1 \otimes S^{-1}) && x \otimes y \mapsto S^{-1}(y) \otimes x \\
\rho^2 &= S \otimes S && x \otimes y \mapsto S(x) \otimes S(y) \\
\rho^4 &= S^2 \otimes S^2 && x \otimes y \mapsto S^2(x) \otimes S^2(y)
\end{aligned}$$

and

$$\begin{aligned}
\alpha &= (1 \otimes \mu) \circ (\Delta \otimes 1) && x \otimes y \mapsto x_1 \otimes x_2 y \\
\alpha' &= (1 \otimes \mu) \circ (1 \otimes S \otimes 1) \circ (\Delta \otimes 1) && x \otimes y \mapsto x_1 \otimes S(x_2) y \\
\beta &= \rho^{-1} \alpha \rho && x \otimes y \mapsto x S^{-1}(y_2) \otimes y_1 \\
\gamma &= \rho^{-2} \alpha \rho^2 && x \otimes y \mapsto x_2 \otimes y x_1
\end{aligned}$$

One has $\alpha' = \alpha^{-1}$ (and so α, β, γ are invertible):

$$\alpha'\alpha \colon x \otimes y \mapsto x_1 \otimes x_2 y \mapsto x_{11} \otimes x_{12} S(x_2) y = x \otimes y$$

Let us also verify the given computations of β and γ:

$$\begin{aligned}
\beta \colon x \otimes y &\mapsto y \otimes S(x) \mapsto y_1 \otimes y_2 S(x) \\
&\mapsto S^{-1}\big(y_2 S(x)\big) \otimes y_1 = x S^{-1}(y_2) \otimes y_1 \\
\gamma \colon x \otimes y &\mapsto S(x) \otimes S(y) \mapsto S(x)_1 \otimes S(x)_2 S(y) \\
&= S(x_2) \otimes S(x_1) S(y) \\
&\mapsto S^{-1}\big(S(x_2)\big) \otimes S^{-1}\big(S(x_1) S(y)\big) = x_2 \otimes y x_1
\end{aligned}$$

Further computations are

(6.6) $$\begin{aligned}
\alpha\beta \colon x \otimes y &\mapsto x S^{-1}(y_2) \otimes y_1 \mapsto x_1 S^{-1}(y_2)_1 \otimes x_2 S^{-1}(y_2)_2 y_1 \\
&= x_1 S^{-1}(y_{22}) \otimes x_2 S^{-1}(y_{21}) y_1 \\
&= x_1 S^{-1}(y) \otimes x_2
\end{aligned}$$

(6.7) $$\alpha\gamma \colon x \otimes y \mapsto x_2 \otimes y x_1 \mapsto x_{21} \otimes x_{22} y x_1$$

(6.8) $$\gamma\alpha \colon x \otimes y \mapsto x_1 \otimes x_2 y \mapsto x_{12} \otimes x_2 y x_{11}$$

Claim (6.4) is clear from (6.7) and (6.8) (cf. (6.5)).

Moreover (6.2) implies (6.3) by a conjugation with ρ.

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[4]Sweedler's notation is usually explained as an abbreviation for sums in (real) tensor products. I think one should rather set it up as a formal calculus for morphisms in $\mathcal{H}$. This wouldn't change anything in practice, but would be more satisfactory.

It remains to verify (6.2). Using (6.6) one finds

$$\begin{aligned}(\alpha\beta)\alpha\colon x\otimes y &\mapsto x_1\otimes x_2y\mapsto x_{11}S^{-1}(x_2y)\otimes x_{12}\\ &= x_{11}S^{-1}(y)S^{-1}(x_2)\otimes x_{12}\\ \beta(\alpha\beta)\colon x\otimes y &\mapsto x_1S^{-1}(y)\otimes x_2\mapsto x_1S^{-1}(y)S^{-1}(x_{22})\otimes x_{21}\end{aligned}$$

and (6.2) follows (cf. (6.5)). This completes the proof of Proposition (6.1).

§7. Presentations

`The material of §7 could be arranged better, but I think the current version readable.`

7.1. **Presentation of Φ_2.** According to Neumann 1933 [8, §1, p. 367; §4, p. 374] or Magnus-Karras-Solitar 1976 [6, Problem 3.5.2, p. 169; pp. 163], the group Φ_2 has the following presentation. This goes back to Nielsen 1924 [9].

(7.1) Theorem. *The automorphism group Φ_2 of the free group*

$$\mathbf{F}_2 = \langle e_1, e_2 | \rangle$$

is generated by the elements O, P, U given by

$$O\colon \begin{array}{l} e_1\mapsto e_1^{-1}\\ e_2\mapsto e_2\end{array} \qquad P\colon\quad e_1\leftrightarrow e_2 \qquad U\colon \begin{array}{l} e_1\mapsto e_1e_2\\ e_2\mapsto e_2\end{array}$$

A complete set of relations is

$$\begin{aligned} O^2 = P^2 = (OP)^4 &= 1\\ (UO)^2 &= (OU)^2\\ (UPOP)^2 &= 1\\ (UOP)^3 &= 1\end{aligned}$$

□

(7.2) Remark. The referenced articles [8, 6] use transposed product of Nielsen transformations, while we use the composition of homomorphisms. This matters only for the last relation.

(7.3) Remark. The subgroup generated by O, P maps isomorphically to the subgroup

$$\left\{\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}, \begin{pmatrix}0 & \pm1\\ \pm1 & 0\end{pmatrix}\right\}\subset \mathrm{GL}_2(\mathbf{Z})$$

One has

$$OP\colon \begin{array}{l} e_1\mapsto e_2\\ e_2\mapsto e_1^{-1}\end{array} \qquad (OP)^2\colon\quad e_i\mapsto e_i^{-1}\quad (i=1,2)$$

The element OP is of order 4 and $(OP)^2$ commutes with O, P.

(7.4) Remark. The group $\langle\, a, b, c \mid abc = 1\,\rangle$ has the automorphism $a\mapsto b\mapsto c\mapsto a$ of order 3. The automorphism UOP is of this form:

$$UOP\colon\quad e_1\mapsto e_2\mapsto (e_1e_2)^{-1}\mapsto e_1$$

(7.5) Remark. We will frequently use the equivalence of relations

$$(XY)^2 = (YX)^2 \iff [X, YXY] = 1 \iff (\text{if } Y^2 = 1)\ [X, YXY^{-1}] = 1$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator and of

$$(XY)^2 = 1 \iff XYX^{-1} = Y^{-1} \quad (\text{if } X^2 = 1)$$

7.2. **A variation.** Let

$$V = (OP)^{-1}U^{-1}(OP)\colon \quad \begin{array}{l} e_1 \mapsto e_1 \\ e_2 \mapsto e_1 e_2 \end{array}$$

If one replaces U by V as generator (this is suggested by the homomorphism $K \circ D$) one gets the presentation

(7.6) Corollary. *The group Φ_2 is generated by O, P, V. A complete set of relations is*

$$\begin{aligned} O^2 = P^2 = (OP)^4 &= 1 \\ (VO)^2 &= 1 \\ (VPOP)^2 &= (POPV)^2 \\ (VPO)^3 &= 1 \end{aligned}$$

Proof: This follows easily from Theorem (7.1) by noting that

$$(OP)^{-1}O(OP) = POP$$

and $O^2 = P^2 = 1$. □

7.3. **Presentation of some "small" subgroups.** For the presentation of Ψ_2 we need some preparations.

For elements A, B in a group subject to

$$A^2 = B^2 = (AB)^4 = 1$$

we use the notations

$$\begin{aligned} \overline{A} &= BAB \\ E &= A\overline{A} = \overline{A}A = (AB)^2 = (BA)^2 \\ \widehat{B} &= ABA \end{aligned}$$

Let

$$G_0 = \langle\, A, B \mid A^2 = B^2 = (AB)^4 = 1 \,\rangle$$

The group G_0 is isomorphic to the semi-direct product $\mathbf{Z}_4 \rtimes \mathbf{Z}_2$ with $\mathbf{Z}_2$ acting on $\mathbf{Z}_4$ non-trivially. The element E generates the center of G_0. The elements of order 4 are $(AB)^{\pm 1}$. The conjugacy classes of the elements of order 2 are $\{E\}$, $\{A, \overline{A}\}$ and $\{B, \widehat{B}\}$. The subgroup

$$G_{00} = \langle\, A, \overline{A} \mid A^2 = \overline{A}^2 = (A\overline{A})^2 = 1 \,\rangle \subset G_0$$

is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Let G_{10} be the group generated by A, $\overline{A}$, C subject to the relations

$$A^2 = \overline{A}^2 = (A\overline{A})^2 = 1 \tag{7.7}$$

$$(CA)^2 = (AC)^2 \tag{7.8}$$

$$(C\overline{A})^2 = (\overline{A}C)^2 \tag{7.9}$$

$$(CAC\overline{A})^2 = 1 \tag{7.10}$$

We use the notation $X_Y = Y^{-1}XY$.

(7.11) Lemma. *The group G_{10} is generated by G_{00} and C with relations*

$$[C, C_A] = 1 \tag{7.12}$$

$$[C, C_{\overline{A}}] = 1 \tag{7.13}$$

$$[C, C_E] = 1 \tag{7.14}$$

$$CC_AC_{\overline{A}}C_E = 1 \tag{7.15}$$

where $E = A\overline{A}$. In other words,

$$G_{10} = \frac{\mathbf{Z}[G_{00}]}{\mathbf{Z}} \rtimes G_{00} \simeq \mathbf{Z}^3 \rtimes (\mathbf{Z}_2 \times \mathbf{Z}_2)$$

Here $\mathbf{Z}[G_{00}]$ is the group ring of G_{00} and

$$\mathbf{Z} = (1 + A + \overline{A} + E)\mathbf{Z} \subset \mathbf{Z}[G_{00}]$$

is the invariant subspace.

Proof: Relations (7.12), (7.13) reformulations of (7.8), (7.9), respectively (under presence of (7.7)). Relation (7.10) is the same as

$$CC_AC_EC_{\overline{A}} = 1$$

which after conjugation yields

$$C_ACC_{\overline{A}}C_E = 1$$

Together with (7.12) (7.13) these yield

$$[C_A, C_{\overline{A}}] = 1$$

which is (7.14) after a conjugation. Now (7.15) is immediate. □

7.4. **Presentation of Ψ_2.** Let G_1 be the group with generators

$$A, B, C$$

and relations

$$A^2 = B^2 = (AB)^4 = 1 \tag{7.16}$$

$$(CA)^2 = (AC)^2 \tag{7.17}$$

$$(C\overline{A})^2 = (\overline{A}C)^2 \tag{7.18}$$

$$(CAC\overline{A})^2 = 1 \tag{7.19}$$

$$(ABC)^3 = 1 \tag{7.20}$$

There is the obvious homomorphism $G_{10} \to G_1$ (it is injective, as can be seen later from the injectivity of $\lambda\colon G_1 \to \Phi_2 \times \Phi_2^{\mathrm{op}}$). Thus the relations of Lemma (7.11) hold in G_1.

(7.21) Lemma. *Let*

$$x_2 = (CA)^2$$
$$x_1 = Bx_2B$$

Then

$$Ax_2A = x_2 \qquad Ax_1A = x_1^{-1}$$
$$Bx_2B = x_1 \qquad Bx_1B = x_2$$
$$Cx_2C^{-1} = x_2 \qquad C^r x_1 C^{-r} = x_1 x_2^r \quad (r \in \mathbf{Z})$$

In particular, the subgroup $\langle x_1, x_2\rangle \subset G_1$ generated by x_1, x_2 is a normal subgroup.

Proof: We freely use (7.16) and its consequences without extra reference. By (7.17), the elements A, C commute with x_2. The conjugations with B are obvious. $(Ax_1)^2 = 1$ is the same as (7.15).

Put $C' = CA$. Then (7.20) reads as $(BC')^3 = 1$ and one gets

$$\begin{aligned}(x_1C')^2 &= (BC'C'BC')^2\\ &= (BC')C'(BC')^2C'(BC')\\ &= (BC')C'(BC')^{-1}C'(BC')\\ &= (BC')BC'(BC') = 1\end{aligned}$$

Hence

$$x_2^{-1} = (x_1CA)^2x_2^{-1} = x_1CAx_1(CA)^{-1} = x_1Cx_1^{-1}C^{-1}$$

Since C commutes with x_2 this yields the computation of $C^r x_1 C^{-r}$.

`[Can this be simplified, perhaps using` B_4`?]`

□

(7.22) Lemma. *One has $G_1 = \Psi_2$. More precisely, the homomorphism*

$$\lambda = (\lambda_1, \lambda_2)\colon G_1 \to \Phi_2 \times \Phi_2^{\mathrm{op}}$$
$$\lambda(A) = (O, O)$$
$$\lambda(B) = (P, P)$$
$$\lambda(C) = (U, V)$$

exists and induces an isomorphism

$$G_1 \to \Psi_2 = \{\, (f, g) \in \Phi_2 \times \Phi_2^{\mathrm{op}} \mid f^{\mathrm{ab}} = (g^{\mathrm{ab}})^t \,\}$$

Proof: The λ_i are defined on the generators of G_1. We first show that the relations of G_1 are respected.

Let $\overline{N} \lhd G_1$ be the normal subgroup generated by $(C\overline{A})^2$. For the quotient $G_1/\overline{N}$ relation (7.19) can be dropped since

$$C\overline{A}CA = (C\overline{A})^2E$$

and $E^2 = 1$. Theorem (7.1) yields $G_1/N = \Phi_2$ with respect to the assignments for λ_1 on generators.

For λ_2 one considers similarly the normal subgroup $N \lhd G_1$ generated by $x_1 = (CA)^2$ and the quotient G_1/N. Again relation (7.19) can be dropped since

$$CAC\overline{A} = (CA)^2E$$

Corollary (7.6) yields $G_1/N = \Phi_2^{\mathrm{op}}$ because of

$$\lambda_2(ABC) = \lambda_2(C)\lambda_2(B)\lambda_2(A) = VPO$$

Having now defined λ, we recall a basic fact about the abelianization of Φ_2. The following sequence is exact (Magnus-Karras-Solitar 1976 [6, Corollary N4, p. 169])

(7.23)
$$1 \to \mathbf{F}_2 \xrightarrow{\Gamma} \Phi_2 \xrightarrow{\mathrm{ab}} \mathrm{GL}_2(\mathbf{Z}) \to 1$$

where

$$\Gamma\colon \mathbf{F}_2 \to \Phi_2$$
$$x \mapsto \Gamma_x$$
$$\Gamma_x(y) = xyx^{-1}$$

identifies $\mathbf{F}_2$ with its group of inner automorphisms.

One has

$$O^{\mathrm{ab}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{\mathrm{ab}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U^{\mathrm{ab}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V^{\mathrm{ab}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Since O^{ab} and P^{ab} are symmetric and $V^{\mathrm{ab}} = (U^{\mathrm{ab}})^t$ it follows that $\lambda(G_1) \subset \Psi_2$.

The homomorphism λ_2 is surjective with kernel N. It remains to show that $\lambda_1|N$ induces an isomorphism onto $\Gamma(F_2)$.

Now N is generated as a group by x_1, x_2 (see Lemma (7.21)) and one finds

$$\lambda_1(x_2) = (UO)^2\colon \quad \begin{aligned} e_1 &\mapsto e_2^{-1}e_1e_2 \\ e_2 &\mapsto e_2 \end{aligned}$$

Thus

$$\lambda_1(x_2) = \Gamma_{e_2}^{-1}$$

and

$$\lambda_1(x_1) = P\lambda_1(x_2)P = \Gamma_{P(e_2)}^{-1} = \Gamma_{e_1}^{-1}$$

□

(7.24) Corollary (of the last proof).

(1) *N is freely generated by x_1, x_2.*
(2) $[N, \overline{N}] = 1$
(3) *The homomorphism λ_1 is given by the action of $G_1/\overline{N}$ on N with free generators x_1^{-1}, x_2^{-1}.*

Proof: This is clear, except perhaps for (2). As we have seen

$$\lambda(N) = \big(\Gamma(F_2), 1\big)$$
$$\lambda(\overline{N}) = \big(1, \Gamma(F_2)\big)$$

Since λ is injective, the claim is follows.

□

7.5. **Presentation of $\overline{\Psi}_2$.** Define automorphisms

$$\begin{aligned} \mu, \sigma\colon G_1 &\to G_1 \\ \mu(A) = \sigma(A) &= A \\ \mu(B) = \sigma(B) &= B \\ \mu(C) &= AC^{-1}A \\ \sigma(C) &= \widehat{B}C^{-1}\widehat{B} \end{aligned}$$

Let us verify that these definitions respect the relations of G_1. As for the relations (7.17), (7.18): These are preserved by μ (since $A\overline{A} = \overline{A}A$) and flipped by σ (since $\widehat{B}A\widehat{B} = \overline{A}$). Similarly, (7.19) is preserved by μ, σ. Finally,

$$\begin{aligned} \mu(ABC) &= ABAC^{-1}A = AC(ABC)^{-1}(AC)^{-1} \\ \sigma(ABC) &= AB\widehat{B}C^{-1}\widehat{B} = \widehat{B}C(ABC)^{-1}(\widehat{B}C)^{-1} \end{aligned}$$

shows that (7.20) is preserved by μ, σ as well.

Note that (for $u \in G_1$)

$$\begin{aligned} \mu^2(u) = \sigma^2(u) &= u \\ (\sigma\mu)(u) &= (AB)u(AB)^{-1} \\ (\mu\sigma)(u) &= (AB)^{-1}u(AB) \\ (\sigma\mu)^2(u) &= EuE \end{aligned}$$

We extend the group G_1 to G_2, $G_2' = G_1 \rtimes \mathbf{Z}_2$ by adding the automorphisms μ, σ

$$\begin{aligned} G_2 &= \langle\, G_1, X \mid X^2 = (XA)^2 = (XB)^2 = (XCA)^2 = 1 \,\rangle \\ G_2' &= \langle\, G_1, T \mid T^2 = (TA)^2 = (TB)^2 = (TC\widehat{B})^2 = 1 \,\rangle \end{aligned}$$

Note that one of the relations (7.17), (7.18) can be dropped from G_2' since they are flipped by conjugation with T.

The combined extension

$$G_3 = \langle\, G_2, T \mid T^2 = (TA)^2 = (TB)^2 = (TC\widehat{B})^2 = 1,\ (TX)^2 = E \,\rangle$$

is an extension of G_1 by $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Denote by $\varepsilon \in \Phi_2$ the automorphism

$$\varepsilon = (OP)^2\colon \quad e_i \mapsto e_i^{-1} \quad (i = 1, 2)$$

and let

$$\begin{aligned} \mu', \sigma'\colon \Phi_2 \times \Phi_2^{\mathrm{op}} &\to \Phi_2 \times \Phi_2^{\mathrm{op}} \\ \mu'(f, g) &= (\varepsilon f \varepsilon^{-1}, g) \\ \sigma'(f, g) &= (g^{-1}, f^{-1}) \end{aligned}$$

(7.25) Lemma. *There are the equalities of homomorphisms $G_1 \to \Phi_2 \times \Phi_2^{\mathrm{op}}$:*

$$\begin{aligned} \lambda \circ \mu &= \mu' \circ \lambda \\ \lambda \circ \sigma &= \sigma' \circ \lambda \end{aligned}$$

Proof: It suffices to check equality on the generators A, B, C of G_1. For A, B the claims are obvious. For C one finds

$$\mu(C) = AC^{-1}A = \begin{cases} ECE \bmod \overline{N} \\ C \bmod N \end{cases}$$

and

$$\begin{aligned} (\lambda \circ \sigma)(C) &= \big((OPO)U^{-1}(OPO), (OPO)V^{-1}(OPO)\big) \\ &= \big(OVO, (OPO)POUOP(OPO)\big) \\ &= \big(OVO, \overline{O}U\overline{O}\big) \\ &= \big(V^{-1}, U^{-1}\big) \end{aligned}$$

□

(7.26) Corollary. *One has $G_3 = \overline{\Psi}_2$. More precisely, the homomorphism*

$$\begin{aligned} \overline{\lambda} \colon G_3 &\to (\Phi_2 \times \Phi_2^{\mathrm{op}}) \rtimes \{1, \sigma'\} \\ \overline{\lambda}|G_1 &= (\lambda, 1) \\ \overline{\lambda}(X) &= \big((\varepsilon, 1), 1\big) \\ \overline{\lambda}(T) &= \big((1, 1), \sigma\big) \end{aligned}$$

$$\lambda =\colon G_1 \to \Phi_2 \times \Phi_2^{\mathrm{op}}$$

induces an isomorphism $G_3 \to \overline{\Psi}_2$. □

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