NOTES ON RESIDUES

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The purpose of this text is to define the residue map for differentials on a smooth curve. The method described here was inspired by the definition of the Grothendieck residues in [2]. For other approaches see [3, 4, 1, 5].

Remark. In the current version of this text, we consider just smooth rational points on a curve. It is not so difficult to reduce the invariance of the residue at arbitrary smooth closed points on a curve over k to that of k-rational points, by some ad hoc arguments. However it would be desirable to extend the material of this section to arbitrary smooth closed point from the very beginning. Moreover one should give a proof of (7) (see the next Lemma) in a less ad hoc fashion.

Let k be a field and let R be the local ring of a curve over k at a smooth k-rational point. The maximal ideal of R is denoted by $\mathfrak{m}$.

Fix $n \geq 0$ and let $I = \mathfrak{m}^n$.

Let

$$\Delta\colon R \to R/I \otimes_k R$$
$$\Delta(x) = x \otimes 1 - 1 \otimes x$$

One has

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y)$$

for x, $y \in R$.

Let

$$\tilde{\mu}\colon R/I \otimes_k R \to R/I$$

be the multiplication map. Obviously $\Delta(x) \in \ker\tilde{\mu}$ for $x \in R$. If t is a prime element of R (i. e., $\mathfrak{m} = tR$), then the ideal $\ker\tilde{\mu}$ is generated by $\Delta(t)$. Since $\Delta(t)$ is a not a zero divisor, the quotient

$$\frac{\Delta(x)}{\Delta(t)} \in R/I \otimes_k R$$

is defined.

We define

$$\Delta_t(x) = \frac{\Delta(x)}{\Delta(t)} + R/I \otimes_k I \in R/I \otimes_k R/I$$

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These "difference quotients", defined for $x \in R$ and prime elements t, have the following basic properties. Let t, t' be prime elements of R and let x, $y \in R$. Then

$$\Delta_{t'}(x) = \Delta_t(x)\Delta_{t'}(t) \tag{1}$$

$$\Delta_t(t^r) = \sum_{i=0}^{r-1}(t^i \otimes t^{r-1-i}) \tag{2}$$

$$\Delta_t(xy) = (x \otimes 1)\Delta_t(y) + (1 \otimes y)\Delta_t(x) \tag{3}$$

Further, let z, $z' \in I$. From (3) we get

$$\Delta_t(xz) = (x \otimes 1)\Delta_t(z) = (1 \otimes x)\Delta_t(z) \tag{4}$$

$$\Delta_t(zz') = 0 \tag{5}$$

Let

$$\mu\colon R/I \otimes_k R/I \to R/I$$

be the multiplication map. The ideal $\ker\mu$ is generated by the elements $x \otimes 1 - 1 \otimes x$ with $x \in R$. From (4) we get for $z \in I$:

$$(\ker\mu)\Delta_t(z) = 0 \tag{6}$$

The annihilator of the ideal $\ker\mu$ consists exactly of the elements

$$\Delta_t(z) \qquad (z \in I)$$

(we don't need this in the following).

Let Ω_R be the module of differentials of R/k with derivation

$$\begin{aligned} \mathrm{d}\colon R &\to \Omega_R \\ x &\mapsto \mathrm{d}x \end{aligned}$$

The R-module Ω_R is free with basis $\mathrm{d}t$ for any prime element t of R. We denote by $\mathrm{d}_I x$ the image of $\mathrm{d}x$ in $\Omega_R/I\Omega_R$.

Lemma. *For $x \in R$ and a prime element t of R one has*

$$\mathrm{d}_I x = \mu\big(\Delta_t(x)\big)\mathrm{d}_I t \tag{7}$$

Proof. This holds for $x = t^r$ since $\mathrm{d}t^r = rt^{r-1}\mathrm{d}t$ and by (2). Moreover it holds for $x \in I^2$ since $\mathrm{d}_I I^2 = 0$ and by (5). □

Proposition. *There exist a $(R/I \otimes_k R/I)$-linear homomorphism*

$$\varphi\colon I/I^2 \to R/I \otimes_k \Omega_R/I\Omega_R$$

with

$$\varphi(z + I^2) = \Delta_t(z)(1 \otimes \mathrm{d}_I t)$$

for $z \in I$ and every prime element t of R. Here we consider I/I^2 as an $(R/I \otimes_k R/I)$-module via μ.

Proof. We first show that

$$\Delta_t(z)(1 \otimes \mathrm{d}_I t)$$

with $z \in I$ does not depend on the choice of t. Indeed, let t, t' be prime elements of R. The elements

$$\Delta_{t'}(t), \quad 1 \otimes \mu\big(\Delta_{t'}(t)\big)$$

have the same image under μ. Using (1), (6), and (7) one finds

$$\Delta_{t'}(z)(1\otimes \mathrm{d}_I t') = \Delta_t(z)\Delta_{t'}(t)(1\otimes \mathrm{d}_I t') = \Delta_t(z)\big(1\otimes \mu\big(\Delta_{t'}(t)\big)\mathrm{d}_I t'\big) = \Delta_t(z)(1\otimes \mathrm{d}_I t)$$

The proposition follows now from (4) and (5). □

Let F be the fraction field of R and let $\Omega_F = F \otimes_R \Omega_R$. Let z be an element of R with $I = zR$ and let

$$\frac{1}{z}\colon \Omega_R/I\Omega_R \to \Omega_F/\Omega_R$$
$$\omega + I\Omega_R \mapsto \frac{\omega}{z} + \Omega_R$$

We define

$$\Phi_I\colon\ \mathrm{Hom}_k(R/I, k) \to \Omega_F/\Omega_R$$
$$\Phi_I(\alpha) = (\alpha \otimes \frac{1}{z})\big(\varphi(z + I^2)\big)$$

The proposition shows that Φ_I is R-linear and does not depend on the choice of z (with $I = zR$).

Lemma. *Let t be a prime element of R. Then*

$$\Phi_I(\alpha) = \sum_{i=0}^{n-1} \alpha(t^i)\frac{\mathrm{d}t}{t^{i+1}} + \Omega_R \tag{8}$$

Proof. Choosing $z = t^n$ we get

$$\Phi_I(\alpha) = (\alpha \otimes \frac{1}{t^n})\Big(\sum_{i=0}^{n-1}(t^i \otimes t^{n-1-i})(1\otimes \mathrm{d}_I t)\Big)$$
$$= \frac{1}{t^n}\sum_{i=0}^{n-1}\alpha(t^i)t^{n-1-i}\mathrm{d}t + \Omega_R$$

and the claim follows. □

Let

$$\mathrm{Hom}_k^c(R,k) = \varinjlim_n \mathrm{Hom}_k(R/\mathbf{m}^n, k) \subset \mathrm{Hom}_k(R,k)$$

be the "continuous k-dual" of R. By (8), the maps Φ_I for various I paste together and we get (using that t^{-i}, $i > 0$ is a k-basis of F/R):

Local Residue Theorem. *There exist a R-linear bijection*

$$\Phi\colon\ \mathrm{Hom}_k^c(R,k) \to \Omega_F/\Omega_R$$

with

$$\Phi(\alpha) = \sum_{i=0}^{\infty}\alpha(t^i)\frac{\mathrm{d}t}{t^{i+1}} + \Omega_R \tag{9}$$

for every prime element t of R. □

One may call Φ the "local residue isomorphism", cf. Proposition 7.2 and Proposition 8.4 in [2, Chapter III]. The residue map will now be defined as

$$\mathrm{Res}\colon \Omega_F \to k$$
$$\mathrm{Res}(\omega) = \big(\Phi^{-1}(\omega + \Omega_R)\big)(1)$$

The map Res has indeed the characteristic properties

$$\mathrm{Res}(\Omega_R) = 0, \qquad \mathrm{Res}(\frac{\mathrm{d}t}{t^n}) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

of the residue map (the second computation follows from (9)).

References

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