SOME GHOST LEMMAS

APOSTOLOS BELIGIANNIS

How Ghosts Emerge

Let $\mathscr{A}$ be an additive category and let $\mathcal{X} \subseteq \mathscr{A}$ be a full subcategory. It is an important problem if $\mathscr{A}$ can be build from $\mathcal{X}$ in some specific way. A first step in this problem is to study the induced restricted Yoneda functor

$$\mathsf{H}_{\mathcal{X}} : \mathscr{C} \to \mathsf{Mod}\text{-}\mathcal{X}, \quad \mathsf{H}_{\mathcal{X}}(A) = \mathscr{A}(-, A)|_{\mathcal{X}}$$

where $\mathsf{Mod}\text{-}\mathcal{X}$ denotes the category of contravariant additive functors $\mathcal{X}^{\mathsf{op}} \to \mathscr{A}b$. The maps in $\mathscr{A}$ invisible by $\mathsf{H}_{\mathcal{X}}$, i.e. the maps $f : A \to B$ in $\mathscr{A}$ such that $\mathsf{H}_{\mathcal{X}}(f) = 0$, i.e. $\mathscr{A}(X, f) = 0$, $\forall X \in \mathcal{X}$, are generally called $\mathcal{X}$-*phantom maps* and they form an ideal in $\mathscr{A}$. If $\mathcal{X} = \{T\}$ consists of a single object $T \in \mathscr{A}$, then the T-phantom maps are called T-*ghost maps*. In this case the functor above takes the form $\mathsf{H}_T : \mathscr{A} \to \mathsf{Mod}\text{-}\mathsf{End}(T)$, $\mathsf{H}_T(A) = \mathscr{A}(T, A)$. More generally one may consider H-phantom maps where $\mathsf{H} : \mathscr{A} \to \mathscr{B}$ is an additive functor, i.e. maps f such that $\mathsf{H}(f) = 0$. The complexity of the ideal of $\mathcal{X}$-phantom or T-ghost maps in some sense measures the possibility to build $\mathscr{A}$ from $\mathcal{X}$ or T.

We denote by $\mathsf{add}\,\mathcal{X}$, resp. $\mathsf{Add}\,\mathcal{X}$, the full subcategory of $\mathscr{A}$ consisting of the direct summands of finite, resp. infinite set-indexed, direct sums of objects from $\mathcal{X}$.

Examples. (i) Take $\mathscr{A} = \mathbf{D}(\mathsf{Mod}\text{-}\Lambda)$ for an Artin algebra Λ and take $\mathcal{X} = \mathbf{K}^b(\mathcal{P}_\Lambda)$, the homotopy category of bounded complexes of finitely generated projective modules. Then the ideal of $\mathcal{X}$-phantom maps is zero iff $\mathbf{D}(\mathsf{Mod}\text{-}\Lambda) = \mathsf{Add}\,\mathbf{K}^b(\mathcal{P}_\Lambda)$, and this happens if and only if Λ is an iterated tilted algebra of Dynkin type.

(ii) Take $\mathscr{A}$ to be the category $\mathsf{Mod}\text{-}\Lambda$ over a ring Λ and $\mathcal{X} = \mathsf{mod}\text{-}\Lambda$ to be the category of finitely presented modules. Then any $\mathcal{X}$-phantom map is zero and this corresponds to the fact that any module is a filtered colimit of finitely presented modules.

(iii) Take $\mathscr{A} = \mathsf{mod}\text{-}\Lambda$ for an Artin algebra Λ and $\mathcal{X} = \{T_1, T_2, \cdots, T_m\}$ a finite set of modules. If the nth power of the ideal of $\mathcal{X}$-phantom maps is zero, then $\mathsf{mod}\text{-}\Lambda$ consists of all modules admitting a finite filtration of length at most n with successive factors, modules which are factors copies of the T_i.

(iv) Take $\mathscr{A} = \mathbf{D}(\mathsf{Mod}\text{-}\Lambda)$ for an Artin algebra and $\mathcal{X} = \{\Sigma^n\Lambda \,|\, n \in \mathbb{Z}\}$ to be the set of all suspensions of Λ in the derived category. If the nth power of $\mathcal{X}$-ghost maps is zero, then any complex of $\mathbf{D}(\mathsf{Mod}\text{-}\Lambda)$ is an n-fold extension of complexes of projective modules with zero differential, i.e. of complexes in $\mathsf{Add}\{\Sigma^n\Lambda \,|\, n \in \mathbb{Z}\}$.

(v) Take $\mathscr{A}$ to be the stable homotopy catregory of spectra and $\mathcal{X}$ the category of finite spectra. Then the $\mathcal{X}$-phantom ideal is square zero and any spectrum is an extension of coproducts of finite spectra.

1. A Ghost Lemma for Abelian Categories

Let $\mathscr{A}$ be an abelian category.

Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathscr{A}$ which are closed under isomorphisms and direct summands. In the sequel we use the following notations:

(i) $\mathsf{Fac}(\mathcal{U})$ is the full subcategory of $\mathscr{A}$ consisting of all factors of objects from $\mathcal{U}$.

(ii) $\mathcal{U} \diamond \mathcal{V} = \mathsf{add}\{A \in \mathscr{A} \mid \exists \text{ an exact sequence} : \ U \rightarrowtail A \twoheadrightarrow V, \ \text{where } U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Inductively we define $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$, $\forall n \geqslant 1$, for subcategories $\mathcal{U}_i$ of $\mathscr{A}$.

For any $\mathcal{U} \subseteq \mathscr{A}$, we set: $\langle\mathcal{U}\rangle_0 = 0$, $\langle\mathcal{U}\rangle_1 = \mathcal{U}$, for $n \geqslant 2$: $\langle\mathcal{U}\rangle_n := \mathcal{U} \diamond \mathcal{U} \diamond \cdots \diamond \mathcal{U}$ (n-factors) and

$$\langle\mathcal{U}\rangle_\infty = \bigcup_{n \geqslant 0} \langle\mathcal{U}\rangle_n$$

Remark 1.1. (i) Clearly the operation $\diamond$ is associative.

(ii) Let $\mathcal{X}_i$, $1 \leqslant i \leqslant n$, be full subcategories of $\mathscr{A}$. Then clearly $\mathcal{X}_1 \diamond \mathcal{X}_2 \diamond \cdots \diamond \mathcal{X}_n$ coincides with the full subcategory $\mathsf{Filt}(\mathcal{X}_1, \cdots, \mathcal{X}_n)$ of $\mathscr{A}$ consisting of direct summands of objects A which admit a filtration

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$$

such that $A_k/A_{k-1} \in \mathcal{X}_k$, $1 \leqslant k \leqslant n$. Hence: $\mathsf{Filt}(\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_n) = \mathcal{X}_1 \diamond \mathcal{X}_2 \diamond \cdots \diamond \mathcal{X}_n$.

Date: April 14, 2008.

Definition 1.2. Let $\mathcal{X}$ be a full subcategory of $\mathscr{A}$. A map $f : A \to B$ in $\mathscr{A}$ is called **$\mathcal{X}$-phantom** if the induced map $\mathscr{A}(X, f)$ is zero, i.e. $\mathscr{A}(X, f) = 0$,. $\forall X \in \mathcal{X}$. If $\mathcal{X}$ consists of a single object T: $\mathcal{X} = \{T\}$, then an $\mathcal{X}$-phantom map is called a **T-ghost**. The set of all $\mathcal{X}$-phantom maps $A \to B$ is denoted by $\mathsf{Ph}_{\mathcal{X}}(A, B)$ and the set of T-ghost maps is denoted by $\mathsf{Gh}_T(A, B)$.

Note that $\mathsf{Ph}_{\mathcal{X}}(A, B) = \bigcap_{T\in\mathcal{X}} \mathsf{Gh}_T(A, B)$.

An **ideal** $\mathcal{I}$ of an additive category $\mathscr{A}$ is an additive subfunctor of $\mathscr{A}(-,-)$. An ideal of $\mathscr{A}$ can be described as a collection $\mathcal{I}(A, B)$ of maps in $\mathscr{A}$, $\forall A, B \in \mathscr{A}$, such that for any $f, g : A \to B$ in $\mathcal{I}$, the map $\alpha\circ(f+g)\circ\beta : X \to Y$ lies in $\mathcal{I}$ for all maps $\alpha : X \to A$ and $\beta : B \to Y$ in $\mathscr{A}$. For $n \geqslant 1$, the nth-power $\mathcal{I}^n$ of an ideal $\mathcal{I}$ consists of the collection of all maps $\mathcal{I}^n(A, B)$ in $\mathscr{A}$ which can be written as a composition of n maps in $\mathcal{I}$. Clearly $\mathcal{I}^n$ is an ideal of $\mathscr{A}$. An important example of an ideal in $\mathscr{A}$ is the Jacobson radical $\mathsf{Rad}(\mathscr{A})$: for any objects $A, B \in \mathscr{A}$, the subgroup $\mathsf{Rad}(A, B)$ of $\mathscr{A}(A, B)$ consists of all maps $f : A \to B$ such that $1_A - f \circ g : A \to A$ is invertible, for any map $g : B \to A$.

Now let $\mathscr{A}$ be abelian and $T \in \mathscr{A}$. Setting $\mathsf{Gh}_T(\mathscr{A}) = \bigcup_{A,B\in\mathscr{A}} \mathsf{Gh}_T(A, B)$ we obtain an ideal of $\mathscr{A}$. Inductively, $\forall n \geqslant 1$, we obtain an ideal $\mathsf{Gh}^n_T(\mathscr{A})$ and in particular for any object $A \in \mathscr{A}$ and any $n \geqslant 1$, we have the left ideal $\mathsf{Gh}^n_T(A,-)$ and the right ideal $\mathsf{Gh}^n_T(-, A)$.

Lemma 1.3 (**Abelian Ghost Lemma**)**.** *Let A be an abelian category and T, X are objects of $\mathscr{A}$.*

(i) *If $X \in \langle \mathsf{Fac}\, T\rangle_n$, then $\mathsf{Gh}^n_T(X,-) = 0$.*
(ii) *If $\mathsf{add}\, T$ is contravariantly finite in $\mathscr{A}$, then the following are equivalent:*
  (a) $\mathsf{Gh}^n_T(X,-) = 0$.
  (b) $X \in \langle \mathsf{Fac}\, T\rangle_n$.

Proof. (i) The assertion is clear if $X \in \langle \mathsf{Fac}\, T\rangle$. Assume that $X \in \langle \mathsf{Fac}\, T\rangle_2$ and let $0 \to X_0 \xrightarrow{\alpha} X \xrightarrow{\beta} X_1 \to 0$ be exact, where the X_i lie in $\mathsf{Fac}\, T$, i.e. there exists epics $e_0 : T_0 \twoheadrightarrow X_0$ and $e_1 : T_1 \twoheadrightarrow X_1$, where the T_i lie in $\mathsf{add}\, T$. Let $f_0 : X \to A$ and $\beta : A \to B$ be T-ghosts. Since the composition $e_0 \circ \alpha \circ f_0 = 0$, we have $\alpha \circ f_0 = 0$ and therefore there exists a map $\rho : T_1 \to A$ such that $\beta \circ \rho = f_0$. Then $e_1 \circ f_0 \circ f_1 = e_1 \circ \beta \circ \rho \circ f_1$. However $e_1 \circ \rho \circ f_1 = 0$ since $e_1 \circ \rho \circ f_1$ is T-ghost (because f_1 is T-ghost) and T_1 lies in $\mathsf{add}\, T$. Hence $\rho \circ f_1 = 0$ and therefore $f_0 \circ f_1 = 0$, i.e. $\mathsf{Gh}^2_T(X,-) = 0$. Then the assertion follows by induction.

(ii) Assume now that $\mathsf{add}\, T$ is contravariantly finite. It suffices to show that (a) implies (b). If $\mathsf{Gh}_T(X,-) = 0$, then let $T_X \xrightarrow{f_X} X \xrightarrow{g} A \to 0$ be exact, where f_X is a right $\mathsf{add}\, T$-approximation of X and $g = \mathsf{coker}\, f_X$. Then clearly g is T-ghost, hence $g = 0$ and therefore f_X is epic, i.e. $X \in \mathsf{Fac}\, T$. Now let $\mathsf{Gh}^2_T(X,-) = 0$, and let as above $T_X \xrightarrow{f_X} X \xrightarrow{g} A \to 0$ be exact, where f_X is a right $\mathsf{add}\, T$-approximation of X and $g = \mathsf{coker}\, f_X$. Consider an exact sequence $T_A \xrightarrow{f_A} A \xrightarrow{h} B \to 0$, where f_A is a right $\mathsf{add}\, T$-approximation of A and $h = \mathsf{coker}\, f_A$. Then the composition $g \circ h$ is T-ghost out of X and therefore $g \circ h = 0$. Since g is epic, we have $h = 0$ and therefore f_A is epic, i.e. $A \in \mathsf{Fac}\, T$. If $C = \mathsf{Im}\, f_X$, then $C \in \mathsf{Fac}\, T$ and the short exact sequence $0 \to C \to X \to A \to 0$ shows that $X \in \langle \mathsf{Fac}\, T\rangle_2$. Assume now that $\mathsf{Gh}^3_T(X,-) = 0$, and let as above $T_X \xrightarrow{f_X} X \xrightarrow{g_0} A \to 0$ be exact, where f_X is a right $\mathsf{add}\, T$-approximation of X and $g_0 = \mathsf{coker}\, f_X$. Consider an exact sequence $T_A \xrightarrow{f_A} A \xrightarrow{g_1} B \to 0$, where f_A is a right $\mathsf{add}\, T$-approximation of A and $g_1 = \mathsf{coker}\, f_A$. Finally consider an exact sequence $T_B \xrightarrow{f_B} A \xrightarrow{g_2} C \to 0$, where f_B is a right $\mathsf{add}\, T$-approximation of B and $g_2 = \mathsf{coker}\, f_B$. Then the composition $g_0 \circ g_1 \circ g_2$ is T-ghost out of X and therefore $g_0 \circ g_1 \circ g_2 = 0$. Since $g_0 \circ g_1$ is epic, we have $g_3 = 0$ and therefore f_B is epic, i.e. $B \in \mathsf{Fac}\, T$. If $D = \mathsf{Im}\, f_A$, then $D \in \mathsf{Fac}\, T$ and the short exact sequence $0 \to D \to A \to B \to 0$ shows that $A \in \langle \mathsf{Fac}\, T\rangle_2$. If $C = \mathsf{Im}\, f_X$, then $C\ in\, \mathsf{Fac}\, T$ and the short exact sequence $0 \to C \to X \to A \to 0$ shows that $X \in \langle \mathsf{Fac}\, T\rangle \diamond \langle \mathsf{Fac}\, T\rangle_2 = \langle \mathsf{Fac}\, T\rangle_3$. Continuing in this way by induction we have the assertion. □

Remark 1.4. If $\mathscr{A}$ has all set-indexed coproducts, then we denote by $\mathsf{Add}\, T$ the full subcategory of $\mathscr{A}$ consisting of all direct summands of set-indexed coproducts of copies of T. The category $\mathsf{Add}\, T$ is always contravariantly finite in $\mathscr{A}$. In this case we always have:

$$X \in \langle \mathsf{Fac}\,\mathsf{Add}\, T\rangle_n \quad \text{if and only if} \quad \mathsf{Gh}^n_T(X,-) = 0$$

The above observations suggests the following notion which possibly is of some use.

Definition 1.5. The (**extension**) **dimension** $\dim \mathscr{A}$ of an abelian category $\mathscr{A}$ is defined as follows:

$$\dim \mathscr{A} := \min\{n \geqslant 0 \mid \exists\, T \in \mathscr{A} : \mathscr{A} = \langle \mathsf{add}\, T\rangle_{n+1}\}$$

Example 1.6. Let Λ be an Artin algebra. The Loewy length of Λ is denoted by $\ell\ell\Lambda$.

(i) Λ is representation finite $\Leftrightarrow$ $\dim \mathsf{mod}\text{-}\Lambda = 0$.
(ii) $\dim \mathsf{mod}\text{-}\Lambda \leqslant \ell\ell\Lambda - 1$.
  Indeed we have $\mathsf{mod}\text{-}\Lambda = \langle \Lambda/\mathbf{r}\rangle_{\ell\ell\Lambda}$.

Corollary 1.7. *Let $\mathscr{A}$ be an abelian category and T an object of $\mathscr{A}$.*

(i) *If $\mathsf{add}\, T$ is contravariantly finite in $\mathscr{A}$, then: $\mathscr{A} = \langle \mathsf{Fac}\, T\rangle_n$ if and only $\mathsf{Gh}^n_T(A,-) = 0$, $\forall A \in \mathscr{A}$.*
(ii) *If there exist objects X, A in $\mathscr{A}$ such that $\mathsf{Gh}^n_T(X,A) \neq 0$, then $X \notin \langle \mathsf{Fac}\, T\rangle_n$. In particular $X \notin \langle T\rangle_n$.*
(iii) *If $\dim \mathscr{A} = d$ and let $T \in \mathscr{A}$ be such that $\mathscr{A} = \langle T\rangle_{d+1}$. Then $\mathsf{Gh}^{d+1}_T(\mathscr{A}) = 0$.*

The Abelian Ghost Lemma 1.3 can be generalized as follows.

Proposition 1.8. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

(i) *Let*

$$H_1 \xrightarrow{\alpha_1} H_2 \xrightarrow{\alpha_2} H_3 \longrightarrow \cdots \longrightarrow H_{n-1} \xrightarrow{\alpha_{n-1}} H_n$$

be a chain of natural maps between left exact contravariant functors $H_i : \mathscr{A}^{\mathsf{op}} \longrightarrow \mathscr{B}$.
(ii) *Let $F_i : \mathscr{C}_i \to \mathscr{A}$ be covariant functors, where $\mathscr{C}_i$ are additive categories, $1 \leqslant i \leqslant n-1$.*

Assume that $\alpha_i F_i = 0$, $\forall i$, i.e. $\alpha_i F_i(X_i) = 0$, $\forall i = 1, 2, \cdots, n-1$, $\forall X_i \in \mathscr{C}_i$.

Then the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{n-1}$ vanishes on $\mathsf{Filt}\big(\mathsf{Fac}(\mathsf{Im}\, F_1), \mathsf{Fac}(\mathsf{Im}\, F_2), \cdots, \mathsf{Fac}(\mathsf{Im}\, F_{n-1})\big) = \mathsf{Fac}(\mathsf{Im}\, F_1)\diamond\mathsf{Fac}(\mathsf{Im}\, F_2)\diamond\cdots\diamond\mathsf{Fac}(\mathsf{Im}\, F_{n-1})$. In particular $\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_{n-1}$ vanishes on $\mathsf{Im}\, F_1\diamond\mathsf{Im}\, F_2\diamond\cdots\diamond\mathsf{Im}\, F_{n-1}$.

For instance in the above proposition we may choose $\mathscr{B} = \mathscr{A}b$ and $H_i = \mathscr{A}(-, A_i)$, for some objects $A_i \in \mathscr{A}$, and also $F_i : \mathcal{X}_i \hookrightarrow \mathscr{A}$ to be the inclusions of full subcategories $\mathcal{X}_i$ of $\mathscr{A}$.

Corollary 1.9. *Let $\mathscr{A}$ be an abelian category and let*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \cdots \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n$$

be a chain of maps between objects of $\mathscr{A}$. Let $\mathcal{X}_i$ be full subcategories of $\mathscr{A}$, $i = 1, \cdots, n-1$, such that $\mathscr{A}(\mathcal{X}_i, f_i) = 0$, $\forall i$. If $A \in \mathscr{A}$ is such that $\mathscr{A}(A, f_1 \circ f_2 \circ \cdots \circ f_{n-1}) \neq 0$, then $A \notin \mathsf{Filt}(\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_n)$.

In particular let $\mathcal{X}$ be a full subcategory of $\mathscr{A}$ such that $\mathscr{A}(\mathcal{X}, f_i) = 0$, $\forall i$. If $A \in \mathscr{A}$ is such that $\mathscr{A}(A, \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{n-1}) \neq 0$, then $A \notin \langle \mathcal{X}\rangle_{n-1}$.

2. A Ghost Lemma for Triangulated Categories

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ.
For any collections $\mathcal{U}$ and $\mathcal{V}$ of objects of $\mathcal{T}$, we use the following notations:

(i) $\langle \mathcal{U}\rangle := \mathsf{add}\,\{\Sigma^n U \mid n \in \mathbb{Z},\ U \in \mathcal{U}\}$.
(ii) $\mathcal{U} \star \mathcal{V} := \mathsf{add}\,\{A \in T \mid \exists \text{ triangle}: \ U \to A \to V \to \Sigma U, \text{ where } U \in \langle \mathcal{U}\rangle \text{ and } V \in \langle \mathcal{V}\rangle\}$.
(iii) Inductively we define $\mathcal{U}_1 \star \mathcal{U}_2 \star \cdots \star \mathcal{U}_n$, $\forall n \geqslant 1$, for subcategories $\mathcal{U}_i$ of $\mathcal{T}$.
(iv) For any $\mathcal{U} \subseteq \mathscr{A}$, we set: $\langle \mathcal{U}\rangle_0 = 0$, $\langle \mathcal{U}\rangle_1 = \mathcal{U}$, for $n \geqslant 2$: $\langle \mathcal{U}\rangle_n := \mathcal{U} \diamond \mathcal{U} \diamond \cdots \diamond \mathcal{U}$ (n-factors) and

$$\langle \mathcal{U}\rangle_\infty = \bigcup_{n\geqslant 0} \langle \mathcal{U}\rangle_n$$

i.e. $\langle \mathcal{U}\rangle_2 := \big\langle\langle \mathcal{U}\rangle \star \langle \mathcal{U}\rangle\big\rangle$ and $\langle \mathcal{U}\rangle_n := \big\langle\langle \mathcal{U}\rangle_{n-1} \star \langle \mathcal{U}\rangle\big\rangle$, $\forall n \geqslant 3$.

The objects of $\langle \mathcal{U}\rangle_n$ are the objects of $\mathcal{T}$ with $\mathcal{U}$-length at least n. Note that $\langle \mathcal{U}\rangle_\infty$ coincides with the thick subcategory of $\mathcal{T}$ generated by $\mathcal{U}$.

Definition 2.1. Let $T \in \mathcal{T}$. A map $f : A \to B$ in $\mathcal{T}$ is called T-**ghost** if the induced map

$$\mathsf{Hom}_{\mathcal{T}}(T, \Sigma^n f) : \mathsf{Hom}_{\mathcal{T}}(T, \Sigma^n A) \to \mathsf{Hom}_{\mathcal{T}}(T, \Sigma^n B)$$

is zero, $\forall n \in \mathbb{Z}$.

We denote by $\mathsf{Gh}_T(A, B)$ the collection of all T-ghost maps between A and B and

$$\mathsf{Gh}_T(\mathcal{T}) := \bigcup_{A,B\in\mathcal{T}} \mathsf{Gh}_T(A, B)$$

Clearly $\mathsf{Gh}_T(\mathcal{T})$ is an ideal of $\mathcal{T}$, called the T-**ghost ideal** of $\mathcal{T}$. Therefore we may define:

(i) For any object $A \in \mathcal{T}$, the left ideal $\mathsf{Gh}_T(A, -)$ which is the additive subfunctor

$$B \longmapsto \mathsf{Gh}_T(,-)(B) = \mathsf{Gh}_T(A, B)$$

of $\mathsf{Hom}_{\mathcal{T}}(A, -)$.
(ii) The power $\mathsf{Gh}^n_T(A, -)$, $\forall n \geqslant 1$, which, for any object $B \in \mathcal{T}$, consists all maps $A \to B$ which can be written as compositions of n T-ghost maps.

From now on we fix an object $T \in \mathcal{T}$.

Lemma 2.2 (**Triangulated Ghost Lemma).** *Let $\mathcal{T}$ be a triangulated category and let T, X be objects of $\mathcal{T}$.*

(i) *If $X \in \langle T\rangle_n$, then $\mathsf{Gh}^n_T(X, -) = 0$.*

(ii) *If $\langle T\rangle$ is contravariantly finite in $\mathcal{T}$, then the following are equivalent:*
   (a) $X \in \langle T\rangle_n$.
   (b) $\mathsf{Gh}^n_T(X,-) = 0$.

Proof. (i) We use induction on the T-length of X. If $X \in \langle T\rangle$, then clearly for any object $A \in \mathcal{T}$ any T-ghost map $X \to A$ is zero. Assume that X lies in $\langle T\rangle_2$ and let

$$T_0 \xrightarrow{\alpha} X \xrightarrow{\beta} T_1 \xrightarrow{\gamma} \Sigma T_0$$

be a triangle in $\mathcal{T}$ where the T_i lie in $\langle T\rangle$. Let $f_1 : X \to A$ and $f_2 : A \to B$ be T-ghost maps. Then the composition $\alpha \circ f_1 : T_0 \to A$ is T-ghost and therefore $\alpha \circ f_1 = 0$. Hence there exists a map $\rho : T_1 \to A$ such that $f_1 = \beta \circ \rho$. Then the composition $f_1 \circ f_2 = \beta \circ \rho \circ f_2$ is T-ghost, since f_2 is T-ghost, and therefore $f_1 \circ f_2 = 0$. This clearly implies that if X lies in $\langle T\rangle_2$, then $\mathsf{Gh}^n_T(X,-) = 0$. Now the assertion follows directly by induction.

(ii) Assume now that $\langle T\rangle$ is contravariantly finite in $\mathcal{T}$. It suffices to show that (b) implies (a). Let $X \in \mathsf{Gh}^n_T(X,-) = 0$. If $n = 1$, the assertion is trivial. Assume that $n = 2$, i.e. $\mathsf{Gh}^2_T(X,-) = 0$. Since $\langle T\rangle$ is contravariantly finite in $\mathcal{T}$, there are triangles

$$\Omega^2_T X \xrightarrow{g_1} T_1 \xrightarrow{f_1} \Omega_T X \xrightarrow{h_1} \Sigma\Omega^2_T A \qquad \text{and} \qquad \Omega_T X \xrightarrow{g_0} T_0 \xrightarrow{f_0} X \xrightarrow{h_0} \Sigma\Omega_T X$$

Then the maps h_0 and h_1 are T-ghosts and then so is Σh_1. It follows that the composition $h_0 \circ \Sigma h_1 : X \to \Sigma^2\Omega^2_T X$ is zero. Consider the octahedral axiom for the composition $0 = h_0 \circ \Sigma h_1$. Then the cone A of $0 = h_0 \circ \Sigma h_1$ is a direct sum of $\Sigma^2\Omega^2_T X$ and ΣX, and there exists a triangle $\Sigma T_0 \to A \to \Sigma^2 T_1 \to \Sigma^2 T_0$. It follows that ΣX, and therefore the object X, is an extension of ΣT_0 and $\Sigma^2 T_1$. Hence X lies in $\langle T\rangle_2$. Then the assertion follows by induction. □

Example 2.3. Let Λ be a ring. Typical examples of ghost maps in the derived category $\mathbf{D}(\mathsf{Mod}\text{-}\Lambda)$ arise from extensions of modules: elements of $\mathsf{Ext}^n(Y,X)$ give rise to maps in $\mathsf{Gh}^n_\Lambda(Y,\Sigma^n X)$. Indeed Let if $X \rightarrowtail A \twoheadrightarrow Y$ is an element of $\mathsf{Ext}^1_\Lambda(Y,X)$. Then the map $Y \to \Sigma X$ in the derived category is Λ-ghost. In fact we have $\mathsf{Ext}^1_\Lambda(Y,X) \cong \mathsf{Gh}_\Lambda(Y,\Sigma X)$. If $X \rightarrowtail A \to B \twoheadrightarrow Y$ is an element of $\mathsf{Ext}^2_\Lambda(Y,X)$ and $Z = \mathsf{Im}(A \to B)$, then in the derived category we have Λ-ghost maps $Y \to \Sigma Z$ and $Z \to \Sigma X$. Hence we have a ghost map $Y \to \Sigma^2 X = Y \to \Sigma Z \to \Sigma^2 X$ which lies in $\mathsf{Gh}^2_\Lambda(Y,\Sigma^2 X)$.

Example 2.4. Let $\mathscr{A}$ be an abelian category with enough projectives. For simplicity we assume that $\mathscr{A}$ admits a projective generator P. Then for any object $A \in \mathscr{A}$ the following are equivalent:

(i) $\mathsf{Gh}^{n+1}_P(A,-) = 0$.
(ii) $\mathsf{pd}\, A \leqslant n$.
(iii) $A \in \langle P\rangle_{n+1}$.

In this example we denote the suspension in $\mathbf{D}^b(\mathscr{A})$ be $[1]$. Let $\cdots \to P^2 \to P^1 \to P^0 \to A \to 0$ be a projective resolution of A. It is build from extensions:

$$\Omega(A) \rightarrowtail P^0 \twoheadrightarrow A, \quad \Omega^2 A \rightarrowtail P^1 \twoheadrightarrow \Omega(A), \quad \Omega^3 A \rightarrowtail P^2 \twoheadrightarrow \Omega^2(A), \quad \cdots$$

The above extensions give rise to triangles in $\mathbf{D}^b(\mathscr{A})$:

$$\Omega(A) \to P^0 \to A \to \Omega(A)[1], \quad \Omega^2 A \to P^1 \to \Omega(A) \to \Omega^2(A)[1], \quad \Omega^3 A \to P^2 \to \Omega^2(A) \to \Omega^3(A)[1], \quad \cdots \tag{2.1}$$

Clearly the maps $\Omega^n(A) \to \Omega^{n+1}(A)[1]$ are P-ghosts and therefore we have a sequence of P-ghost maps:

$$A \to \Omega(A)[1] \to \Omega^2(A)[2] \to \Omega^3(A)[3] \to \cdots$$

(i) $\Rightarrow$ (ii) Assume that $\mathsf{Gh}^{n+1}_P(A,-) = 0$. For $n = 0$, the assertion is trivial, since then the P-ghost map $A \to \Omega(A)[1]$ is zero hence the triangle $\Omega(A) \to P^0 \to A \to \Omega(A)[1]$ splits and therefore A is projective. If $n = 1$, then the composition $A \to \Omega(A)[1] \to \Omega^2(A)[2]$ of P-ghost maps is zero. This implies, from the first triangle in (2.1), that the map $\Omega(A)[1] \to \Omega^2(A)[2]$ factors through the map $\Omega(A)[1] \to P^0[1]$ say via a map $P^0[1] \to \Omega^2(A)[2]$. However this map corresponds to an extension in $\mathsf{Ext}^1(P^0,\Omega^2(A)) = 0$. Hence the map $\Omega(A)[1] \to \Omega^2(A)[2]$, or equivalently the map $\Omega(A) \to \Omega^2(A)[1]$, is zero. It follows that the second triangle in (2.1) splits and therefore $\Omega(A)$ is projective as a direct summand of P^1, i.e. $\mathsf{pd}\, A \leqslant 1$. Continuing in this way, we deduce that if $\mathsf{Gh}^{n+1}_P(A,-) = 0$, then $\mathsf{pd}\, A \leqslant n$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) Assume that $\mathsf{pd}\, A \leqslant n$. If $n = 0$, then the assertion is clear. If $n = 1$, then the projective resolution $P^1 \rightarrowtail P^0 \twoheadrightarrow A$ induces a triangle $P^1 \to P^0 \to A \to P^1[1]$ in $\mathbf{D}^b(\mathscr{A})$. Hence $A \in \langle P\rangle_2$. By induction it follows that if $\mathsf{pd}\, A \leqslant n$, then $A \in \langle P\rangle_{n+1}$. The implication (iii) $\Rightarrow$ (i) follows from the Ghost Lemma.

It follows that for any $A \in \mathscr{A}$:

$$\mathsf{pd}\, A \;=\; \min\big\{n \geqslant 0 \,|\, \mathsf{Gh}^{n+1}_P(A,-) = 0\big\} \;=\; \min\big\{n \geqslant 0 \,|\, A \in \langle P\rangle_{n+1}\big\}$$

Note that the ghost ideal can be very large, even for familiar abelian categories. For instance since $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{R}$, it follows that $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = \mathsf{Gh}_{\mathbb{Z}}(\mathbb{Q},\Sigma\mathbb{Z}) = \mathbb{R}$. However in this case $\mathsf{Gh}^2_{\mathbb{Z}}(\mathbf{D}(\mathsf{Mod}\text{-}\mathbb{Z})) = 0$.

Remark 2.5. Let $\mathcal{T}$ be a triangulated category with all (set-indexed) coproducts. Then for any object T in $\mathcal{T}$, the full subcategory $\langle T\rangle^{\oplus} := \mathsf{Add}\{\Sigma^n T \mid n \in \mathbb{Z}\}$ is contravariantly finite in $\mathcal{T}$.

In this case $\langle T\rangle^{\oplus}_n$ consists of the direct summands of objects obtained by n-fold extensions of arbitrary direct sums of shifts of copies of T. The triangulated ghost lemma in this setting reads as follows:

Infinite Triangulated Ghost Lemma: Let T be a triangulated category with all set-indexed coproducts. Then for any object $X \in \mathcal{T}$: $\quad X \in \langle T\rangle^{\oplus}_n \;\Leftrightarrow\; \mathsf{Gh}^n_T(X,-) = 0$.

Moreover let $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_{n-1}$ be full subcategories of $\mathcal{T}$, each closed under shifts and consisting of compact objects. If A is a compact object, not lying in $\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_{n-1}$, then there exists a chain $A \xrightarrow{\alpha_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{\alpha_n} X_n$ of maps between objects in $\mathcal{T}$, such that each $\mathcal{T}(\mathcal{X}_i, \alpha_i) = 0$ and the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n \neq 0$.

In particular if X and T are compact, then: $X \in \langle T\rangle_n \;\Leftrightarrow\; \mathsf{Gh}^n_T(X,-) = 0$.

Corollary 2.6. *Let T be an object of $\mathcal{T}$ such that $\langle T\rangle$ is contravariantly finite in $\mathcal{T}$. Then the following are equivalent:*

(i) *There exists $d \geqslant 0$: $\mathcal{T} = \langle T\rangle_{d+1}$ (and d is minimal with this property).*
(ii) *There exists $d \geqslant 0$: $\mathsf{Gh}^{d+1}_T(\mathcal{T}) = 0$ (and d is minimal with this property).*

Definition 2.7. The **dimension** $\dim \mathcal{T}$ of $\mathcal{T}$ is defined as follows:

$$\dim \mathcal{T} := \min\{n \geqslant 0 \mid \exists\ T \in \mathcal{T}:\ \langle T\rangle_{n+1} = \mathcal{T}\}$$

It follows that if $\dim \mathcal{T} = d$ and $T \in \mathcal{T}$ is such that $\mathcal{T} = \langle T\rangle_n$, then $\mathsf{Gh}^{d+1}_T(\mathcal{T}) = 0$.

Corollary 2.8. *Let T be an object of $\mathcal{T}$ and let $\mathsf{Thick}(T)$ be the thick subcategory of $\mathcal{T}$ generated by T. Then for any $X \in \mathsf{Thick}(T)$, the left ideal $\mathsf{Gh}_T(X,-)$ is nilpotent. In particular if T is a classical generator of $\mathcal{T}$, then the left ideal $\mathsf{Gh}_T(X,-)$ is nilpotent for any object $X \in \mathcal{T}$.*

Recall that an object T of $\mathcal{T}$ is a *generator* of $\mathcal{T}$ if $\mathcal{T}(T,\Sigma^n A) = 0$ implies that $A = 0$. The object T is called a *classical generator* of $\mathcal{T}$ if the thick subcategory generated by T coincides with $\mathcal{T}$, i.e. $\mathcal{T} = \langle T\rangle_\infty$.

Remark 2.9. Let T be an object of $\mathcal{T}$. It is easy to see that the following are equivalent:

(i) T is a *generator* of $\mathcal{T}$.
(ii) The ideal of T-ghost maps is contained in the Jacobson radical $\mathsf{Rad}(\mathcal{T})$ of $\mathcal{T}$.

Let T be a generator of $\mathcal{T}$ and assume that $\langle T\rangle$ is contravariantly finite in $\mathcal{T}$. If any object of $\mathcal{T}$ has semiprimary endomorphism ring, then by using the ghost Lemma it is easy to see that T is a classical generator.

Example 2.10. (The original Ghost Lemma, see [J.L. Kelly: *Chain maps inducing zero homology maps*, Proc. Camb. Phil. Soc. **61** (1965), 847–854.])

Let Λ be a ring and let $T = \Lambda$ considered, as a complex concentrated in degree zero, in the homotopy category $\mathbf{K}(\mathsf{Mod}\text{-}\Lambda)$. Then the ideal of Λ-ghosts are the maps of complexes $f^\bullet : A^\bullet \to B^\bullet$ such that its cohomology $\mathsf{H}^n(f^\bullet) : \mathsf{H}^n(A^\bullet) \to \mathsf{H}^n(B^\bullet)$ is the zero map, $\forall n \in \mathbb{Z}$. Kelly's original result says that if $X^\bullet$ is a complex of projectives such that for each $k \in \mathbb{Z}$, the modules $\mathsf{B}^k(X^\bullet)$ and $\mathsf{H}^k(X^\bullet)$ have projective dimension less than n. Then any composition $X^\bullet \to A^\bullet_1 \to \cdots \to A^\bullet_n$ of maps in $\mathbf{K}(\mathsf{Mod}\text{-}\Lambda)$, each inducing the zero map in cohomology, is zero. This follows from the fact that such a complex $A^\bullet$ is, in the homotopy category, an n-fold extension of the category of complexes of projectives with zero differential. Of course the last category equals $\langle\Lambda\rangle^{\oplus} = \langle\mathsf{Add}\,\Lambda\rangle \subseteq \mathbf{K}(\mathsf{Mod}\text{-}\Lambda)$. Then apply the Infinite Triangulated Ghost Lemma above. On the other hand, by Corollaty 2.6, for any perfect complex $A^\bullet \in \mathbf{K}^b(\mathcal{P}_\Lambda)$, i.e. a bounded complex with finitely generated projective components, the ideal $\mathsf{Gh}_\Lambda(A^\bullet,-)$ is nilpotent. The above trivially hold true for any abelian category with enough projectives. This is the case originally considered by Kelly.

Example 2.11. Let $\mathscr{A}$ be an abelian category with exact coproducts and enough projectives. Let P be a projective generator of $\mathscr{A}$. For any complex $X^\bullet \in \mathbf{D}(\mathscr{A})$, define its *$P$-ghost dimension* $\mathsf{gh.dim}_P X^\bullet$, resp. *$P$-extension dimension* $\mathsf{ext.dim}_P X^\bullet$, to be the nilpotency index of the left ideal of P-ghost maps out of $X^\bullet$ (or ∞ if the ideal is not nilpotent), resp. the minimum $n \geqslant 0$ such that $X^\bullet$ lies in $\langle P\rangle^{\oplus}_{n+1}$ ((or ∞) if no such n exists). Then for the associated P-ghost dimension $\mathsf{gh.dim}_P\mathscr{A}$ and P-extension dimension $\mathsf{ext.dim}_P\mathscr{A}$ of $\mathscr{A}$ we have: $\mathsf{gl.dim}\,\mathscr{A} = \mathsf{gh.dim}_P\mathscr{A} = \mathsf{ext.dim}_P\mathscr{A}$.

2.1. **The Abelian Ghost Lemma implies the Triangulated Ghost Lemma.** Let $\mathscr{A}$ be an abelian category and let $\mathcal{U}$ be a full subcategory, closed under isomorphisms and direct summands.

We denote by $\widehat{\mathcal{U}} = \mathsf{Sub}\,\mathsf{Fac}(\mathcal{U})$ the full subcategory of subquotients of $\mathcal{U}$. Note that $\widehat{\mathcal{U}} = \mathsf{Sub}\,\mathsf{Fac}(\mathcal{U}) = \mathsf{Fac}\,\mathsf{Sub}(\mathcal{U})$, and $\widehat{\mathcal{U}}$ is an exact abelian subcategory of $\mathscr{A}$.

The following is a triangulated analogue of Proposition 1.8.

Proposition 2.12. *Let $\mathcal{T}$ be a triangulated category and let*

(i)
$$H_1 \xrightarrow{\alpha_1} H_2 \xrightarrow{\alpha_2} H_3 \longrightarrow \cdots \longrightarrow H_{n-1} \xrightarrow{\alpha_{n-1}} H_n \qquad (*)$$
be a chain of natural maps between cohomological functors $H_i : \mathcal{T}^{\mathsf{op}} \longrightarrow \mathscr{B}$, where $\mathscr{B}$ is abelian.

(ii) *$F_1, F_2, \cdots, F_{n-1} : \mathscr{C} \to \mathcal{T}$ be covariant functors, where $\mathscr{C}$ is any additive category.*

Assume that $\alpha_i F_i = 0$, $\forall i$. If $\mathcal{X}_i$ denotes the closure of each full subcategory $\mathsf{Im}\,F_i$ under the suspension functor, then the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{n-1}$ vanishes on $\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_{n-1}$.

Proof. (sketch) Let $\mathscr{A}(\mathcal{T})$ be the category of coherent functors $\mathcal{T}^{\mathsf{op}} \to \mathscr{A}b$. It is well-known that $\mathscr{A}(\mathcal{T})$ is a Frobenius abelian category and the Yoneda embedding $\mathsf{Y} : \mathcal{T} \hookrightarrow \mathscr{A}(\mathcal{T})$, $A \to \mathcal{T}(-, A)$ is a homomological functor which is universal in the following sense: any cohomological functor $H : \mathcal{T} \to \mathscr{B}$ to an abelian category $\mathscr{B}$ admits a unique exact extension $H^* : \mathscr{A}(\mathcal{T}) \to \mathscr{B}$ such that $H^* \circ \mathsf{Y} = H$. It follows that the chain of cohomological functors $(*)$ induces a chain of exact functors $\mathscr{A}(\mathcal{T})^{\mathsf{op}} \to \mathscr{B}$:

$$H_1^* \xrightarrow{\alpha_1^*} H_2^* \xrightarrow{\alpha_2^*} H_3^* \longrightarrow \cdots \longrightarrow H_{n-1}^* \xrightarrow{\alpha_{n-1}^*} H_n^* \qquad (**)$$

Then the assertion follows from the Abelian Ghost Lemma stated in Proposition 1.8 and the following three observations:

1: The composition $\alpha_1^* \circ \alpha_2^* \circ \cdots \circ \alpha_{n-1}^*$ vanishes on $\widehat{\mathsf{Y}(\mathcal{X}_1)} \diamond \widehat{\mathsf{Y}(\mathcal{X}_2)} \diamond \cdots \diamond \widehat{\mathsf{Y}(\mathcal{X}_{n-1})}$.

2: $\mathsf{Y}(\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_{n-1}) \subseteq \widehat{\mathsf{Y}(\mathcal{X}_1)} \diamond \widehat{\mathsf{Y}(\mathcal{X}_2)} \diamond \cdots \diamond \widehat{\mathsf{Y}(\mathcal{X}_{n-1})}$.

3: $(\alpha_1^* \circ \alpha_2^* \circ \cdots \circ \alpha_{n-1}^*)|_{\mathsf{Y}(\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_{n-1})} = (\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{n-1})|_{\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_{n-1}}$. □

Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
E-mail address: abeligia@cc.uoi.gr