Classification problems in the representation theory of finite-dimensional algebras

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1 Introduction

1.1 Combinatorics arising in the representation theory of algebras

The aim of the representation theory of algebras is to understand the category Mod$-A$ of modules over a given associative unital k-algebra A where k is a commutative ring. We will restrict ourselves to the case that k is a field and A is finite-dimensional over k. Familiar examples for algebras of this kind are the group algebras kG for a finite group G or the factor algebras $k[X]/I$ of the polynomial ring $k[X]$ by a non-zero ideal I.

Frequently, we will assume that the field k is algebraically closed. Moreover, we will mainly deal only with the full subcategory mod$-A$ of Mod$-A$ given by all right A-modules M which are finite-dimensional as vector spaces over k.

The key opening representation theory of finite-dimensional algebras to combinatorial methods is the observation that these algebras as well as their categories of finite-dimensional modules can be described to a large extent by locally finite quivers. Recall that 'quiver' is the short notation for 'oriented graph' which is used commonly in representation theory. The quiver describing the algebra is sometimes called the 'Gabriel quiver' whereas for the quiver of the category of finite-dimensional modules meanwhile 'Auslander-Reiten quiver' is the standard name.

In the Auslander-Reiten quiver an indecomposable module shrinks to a mere vertex. A compromise between this total loss of structure and the wish to work only with combinatorial data is furnished by looking at the corresponding element of the module in the Grothendieck group (with respect to short exact sequences). Note, that this Grothendieck group is always of the shape $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

For general algebras the Auslander-Reiten quiver together with the Grothendieck group does not lead to many effective combinatorial algorithms. The situation improves drastically if one considers strongly simply connected algebras. Associated with the representation theory of these algebras another combinatorial structure namely partially ordered sets arise. Quivers, Grothendieck groups, partially ordered sets together with various actions on these structures form the combinatorial environment of the classifications and algorithms which we will present.

1.2 Representation types, classifications and CREP

In the last 30 years a lot of effort was made to study and classify finite-dimensional algebras with respect to their representation type. An algebra is said to be of finite representation type if there are only finitely many isomorphism classes of indecomposable finite-dimensional modules. A generalization is given by the tame algebras which are characterized by having for any fixed dimension only finitely many 1-parameter families of isomorphism classes of indecomposable modules. By a theorem of Drozd algebras which are not tame have to be wild which means that the classification of their indecomposable modules is as difficult as the classification of pairs of square matrices under simultaneous conjugation.

In the study of finite representation type it turned out that it is better to pass to the universal covering of a suitable degeneration called the 'standard form' of a given algebra. Thus one arrives at a strongly simply connected algebra. For the study of tame algebras a similar approach seems to work although up to now there are no complete results in this case.

In order to understand finite representation type for strongly simply connected algebras, the aim was to classify the minimal representation infinite algebras and those algebras which have a faithful indecomposable module. This led to the classifications of the tame concealed algebras and the sincere representation directed algebras. At this place computers had to enter the business because it turned out that there were so many exceptional structures yielding huge amounts of combinatorial data that otherwise the explicit classifications would have been impossible. Our main aim is to review these classifications introducing the necessary material to understand the algorithms which were used.

The original classifications were performed by stand-alone programs. Meanwhile we have developed the program system CREP (Combinatorial REPresentation theory) which allows to deal with algorithms, examples and classifications in a more conceptual and systematic way (see [DN2]). Another aim of these notes is to illustrate this progress and to indicate new developments. In particular, we will complete a classification of certain tame incidence algebras associated with partially ordered sets using our algorithms on thin start modules and rather recent results of Skowroński on tame strongly simply connected algebras of polynomial growth.

1.3 Outline of contents

Presently there exist three books offering an introduction to the representation theory of finite-dimensional algebras, namely [ARS], [GR] and [Ri2]. Although, [Ri2] is the oldest one, its main topic are advanced examples of algebras of tame representation type, and therefore the introduction to the general theory is short and several proofs (e.g. existence of almost split sequences) are skipped. In our opinion [ARS] gives the most comprehensive introduction to the theory. Concerning computer applications the disadvantage of this book is that it deals with Artin algebras and their module categories whereas currently the most powerful computer algorithms work for finite-dimensional algebras over an algebraically closed field. However, the representation theory of algebras over an algebraically closed field is the main topic in [GR]. Using the language of functors and categories even for the basic concepts, this book seems to be meant for more advanced readers. Nevertheless, we feel that this approach is really also the best-suited for the computer applications which we want to present since it leads immediately to concepts like quivers and partially ordered sets.

Therefore, in sections 2 to 4 we introduce the basic material about representation theory of finite-dimensional algebras following [GR] in notation and referring to this monograph for proofs. In section 5 we give the precise definitions of the representation types and provide a brief account about covering theory and degenerations. Section 6 is concentrated on the classifications of the tame concealed, sincere representation directed and minimal wild concealed algebras. We explain how the completely separating algebras i.e. the schurian strongly simply connected algebras are made accessible for a computer. Moreover, we present some few results from tilting theory which we need. The final section 7 is devoted to our algorithms using partially ordered sets of thin start modules. We relate the module category of an algebra to the subspace category of a vector space category using fiber sum functors and explain that linear vector space categories are completely described by partially ordered sets. After recalling the theorems of Kleiner, Nazarova and Zavadskij on the representation type of partially ordered sets, we present the classification result about tame incidence algebras of polynomial growth mentioned above as a current application.

For all definitions and results which are used explicitely but not introduced or stated completely, we will give precise references. On the other hand, we will use several arguments between the lines and refer to the books mentioned above for proofs of these.

2 Algebras, aggregates and modules

2.1 Finite-dimensional algebras and their basic algebras

Algebras appearing in 'nature' tend not to be basic. In the representation theory of algebras we usually pass to a Morita equivalent basic algebra which can be described by a finite quiver with relations provided that the field k is algebraically closed. Later we will explain how this passage is performed on the categorical level, but for the convenience of the reader we start out by recalling the classical approach using idempotents.

Let A be a unital associative algebra over a field k. The *Jacobson radical* $\operatorname{rad} A$ of A is by definition the intersection of all maximal right ideals of A. One obtains the same set by intersecting all maximal left ideals. Consequently, $\operatorname{rad} A$ is a two-sided ideal of A. If A is finite-dimensional, this ideal is nilpotent and the factor algebra $A/\operatorname{rad} A$ is semisimple, hence isomorphic to a finite product of matrix algebras of skew fields which are finite-dimensional k-algebras.

The set of non-invertible elements of an algebra A always contains the Jacobson radical. If this inclusion is an equality and the unit element of A is non-zero, then the algebra is said to be *local.* For a local algebra the factor algebra $A/\operatorname{rad} A$ is a skew field.

An element e of an algebra A is called *idempotent* if $e = e^2$. Two idempotents e, f are said to be *orthogonal* if $0 = ef = fe$. An idempotent e is called *primitive* if $e \neq 0$ and e cannot be written as $e = f + g$ where $f \neq 0 \neq g$ and f, g are orthogonal idempotents.

If A is finite-dimensional over the field k, then the unit element 1_A of A can be written as a sum $1_A = \sum_{i=1}^{n} e_i$ of primitive idempotents e_i which are mutually orthogonal. For each such idempotent e_i the subset e_iAe_i together with the induced addition and multiplication becomes a finite-dimensional algebra with unit element e_i. The primitivity of e_i implies that e_iAe_i is a local algebra.

On the set $\{e_1, \dots, e_n\}$ we obtain an equivalence relation by defining $e_i \sim e_j$ if the ideal $e_iAe_jAe_i$ of the algebra e_iAe_i coincides with e_iAe_i. We choose a minimal set $\{f_1, \dots, f_m\}$ of representatives of the equivalence classes. The sum $f := \sum_{j=1}^{m} f_j$ is again an idempotent of A and $B := fAf$ is a finite-dimensional k-algebra with unit element f. Any algebra isomorphic to B is called a *basic algebra* of A. The subset fA of A becomes a B-A-bimodule in the obvious way. Tensoring with this bimodule over B furnishes equivalences $\mathrm{Mod}-B \to \mathrm{Mod}-A$ and $\mathrm{mod}-B \to \mathrm{mod}-A$ which are called *Morita equivalences*. These equivalences allow us to consider only basic algebras A i.e. algebras which coincide with their basic algebra B for the conceptual study of $\mathrm{Mod}-A$ and $\mathrm{mod}-A$. Nevertheless, it has to be emphasized that it is usually non-trivial to construct B from a practically given A. One may consider group algebras to get a feeling for the difficulties.

2.2 Aggregates and spectroids

Let us now reformulate and generalize the construction of the basic algebra in categorical terms following [GR]. Remember that k is a field. A category $\mathcal{A}$ is called *k-linear* if all morphism spaces $\mathcal{A}(x, y)$ are vector spaces over k and the composition is k-bilinear. In analogy to the algebra case one defines the *Jacobson radical* $\operatorname{rad}\mathcal{A}$ of a k-linear category as the intersection of all maximal right (or maximal left) ideals of $\mathcal{A}$. The category $\mathcal{A}$ is called *k-additive* if it is k-linear and additive i.e. has finite coproducts. A skeletally small k-additive category $\mathcal{A}$ is said to be an *aggregate* if all morphism spaces $\mathcal{A}(x, y)$ are finite-dimensional over k and each object of $\mathcal{A}$ is a coproduct of finitely many objects with local endomorphism algebras. Due to the theorem of Krull-Remak-Schmidt the objects forming such a coproduct are unique up to isomorphism. Note that in an aggregate the *indecomposable* objects, which are those non-zero objects that cannot be written as a coproduct of two non-zero objects, coincide with the objects having a local endomorphism algebra. It is well-known that the category $\mathrm{mod}-A$ of finite-dimensional right modules over a finite-dimensional algebra A is an aggregate.

Given an arbitrary k-linear category $\mathcal{A}$ we can consider the smallest k-additive category $\operatorname{add}\mathcal{A}$ which contains $\mathcal{A}$. Usually, idempotent morphisms in a k-linear category $\mathcal{A}$ will not split i.e. will not factorize as $\iota\pi$ where π is a retraction and ι is a section. But again we can form the smallest category $\operatorname{splt}\mathcal{A}$ containing $\mathcal{A}$ in which all idempotent morphisms split. In particular, if we start out with a skeletally small k-linear category $\mathcal{A}$ where all morphism spaces are finite-dimensional over k, then $\operatorname{splt}\operatorname{add}\mathcal{A}$ will be an aggregate.

Each finite-dimensional k-algebra A furnishes a k-linear category which has only one object ω whose endomorphism algebra is just A and the composition is the multiplication. By abuse of language this category is also denoted by A. Obviously, it is skeletally small and has finite-dimensional morphism spaces. Thus $\operatorname{splt}\operatorname{add}A$ is an aggregate.

A small k-linear category $\mathcal{S}$ is called a *spectroid* if it has finite-dimensional morphism spaces, the endomorphism algebra of each object is local and two objects are isomorphic if and only if they are equal. Note that for a spectroid $\mathcal{S}$ the Jacobson radical $\operatorname{rad}\mathcal{S}(x, y)$ coincides with $\mathcal{S}(x, y)$ if $x \neq y$ and coincides with the set of non-invertible elements of the local algebra $\mathcal{S}(x, x)$ for $x = y$. If for an aggregate $\mathcal{A}$ we choose a complete unshortenable set of representatives of the indecomposable objects and denote by $\mathcal{S}(\mathcal{A})$ the full subcategory of $\mathcal{A}$ associated with this set of objects, then $\mathcal{S}(\mathcal{A})$ is a spectroid and $\operatorname{add}\mathcal{S}(\mathcal{A})$ is equivalent to $\mathcal{A}$. Conversely, if $\mathcal{S}$ is a spectroid, then $\operatorname{add}\mathcal{S}$ is an aggregate and $\mathcal{S}(\operatorname{add}\mathcal{S})$ is isomorphic to $\mathcal{S}$. Hence the equivalence classes of aggregates are in one-to-one correspondence with the isomorphism classes of spectroids.

If we start out with a finite dimensional algebra A, then $\mathcal{S}(\operatorname{add}A)$ is a finite spectroid i.e. has a finite set $\{x_1, \dots, x_n\}$ of objects. The space B of matrices $(b_{ij})_{i,j=1,\dots,n}$ where $b_{ij} \in \mathcal{S}(\operatorname{add}A)(x_j, x_i)$ becomes a finite-dimensional k-algebra via the usual matrix multiplication. This algebra B is a basic algebra of A and $\mathcal{S}(\operatorname{add}B)$ can be identified with $\mathcal{S}(\operatorname{add}A)$. Thus basic algebras and finite spectroids are just two aspects of the same concept. We think that finite spectroids are more appropriate for computational purposes as we will see that they are closely related to quivers. Therefore we will continue using this terminology.

2.3 Modules over spectroids

Given a skeletally small k-linear category $\mathcal{A}$, we denote by $\mathrm{Mod}-\mathcal{A}$ the category of all contravariant k-linear functors $M : \mathcal{A} \to \mathrm{Mod}-k$ where a functor is called k-linear if it induces k-linear maps on the morphism spaces. By $\mathrm{mod}-\mathcal{A}$ we denote the full subcategory of all locally finite-dimensional objects M i.e. $\dim_k M(x) < \infty$ for each object x of $\mathcal{A}$. Since k-linear functors preserve finite (co)products, we obtain that $\mathrm{Mod}-\mathcal{A}$ is equivalent to $\mathrm{Mod}-(\mathrm{add}\,\mathcal{A})$ and $\mathrm{Mod}-(\mathrm{splt}\,\mathcal{A})$. In particular, we obtain that for an aggregate $\mathcal{A}$ the module categories $\mathrm{Mod}-\mathcal{A}$ and $\mathrm{Mod}-\mathcal{S}(\mathcal{A})$ are equivalent which yields an equivalence of $\mathrm{mod}-\mathcal{A}$ and $\mathrm{mod}-\mathcal{S}(\mathcal{A})$. This is the reformulation of Morita equivalence in the categorical context.

For a finite-dimensional algebra A now the symbol $\mathrm{Mod}-A$ seems to have two meanings. But obviously, the two categories associated with this symbol actually coincide. Therefore we obtain that $\mathrm{Mod}-A \cong \mathrm{Mod}-\mathcal{S}(\mathrm{add}\,A)$ and also $\mathrm{mod}-A \cong \mathrm{mod}-\mathcal{S}(\mathrm{add}\,A)$. This justifies to replace the study of $\mathrm{mod}-A$ for finite-dimensional algebras A by the study of $\mathrm{mod}-\mathcal{S}$ for finite spectroids $\mathcal{S}$ which as seen above is nothing else but passing to a basic algebra. Moreover, thanks to the Krull-Remak-Schmidt theorem most of the information about the aggregate $\mathrm{mod}-\mathcal{S}$ will be contained in the spectroid $\mathcal{S}(\mathrm{mod}-\mathcal{S})$ which we will denote by $\mathrm{ind}-\mathcal{S}$.

Let $\mathcal{S}$ be a finite spectroid. The full subcategory $\mathrm{proj}-\mathcal{S}$ of *projective* objects in $\mathrm{mod}-\mathcal{S}$ consists of those modules P for which the functor $\mathrm{Hom}_{\mathcal{S}}(P,-) : \mathrm{mod}-\mathcal{S} \to \mathrm{mod}-k$ is exact i.e. preserves short exact sequences. Since it is even a full subaggregate we can ask for a spectroid of $\mathrm{proj}-\mathcal{S}$. For this reason let us consider the functor $-^{\wedge} : \mathcal{S} \to \mathrm{mod}-\mathcal{S}$ which sends the object x of $\mathcal{S}$ to the representable functor $x^{\wedge} := \mathcal{S}(-,x)$ and a morphism $f : x \to y$ to $f^{\wedge} := \mathcal{S}(-,f) : \mathcal{S}(-,x) \to \mathcal{S}(-,y)$. The functor is actually fully faithful. This follows from the Yoneda-Lemma which states that the functor $\mathrm{Mod}-\mathcal{S} \to \mathrm{mod}-k$ sending a module M to $M(x)$ is isomorphic to the representable functor $\mathrm{Hom}_{\mathcal{S}}(x^{\wedge},-)$. Thus the representable functors $\mathrm{Hom}_{\mathcal{S}}(x^{\wedge},-) : \mathrm{mod}-\mathcal{S} \to \mathrm{mod}-k$ are exact for all $x \in \mathcal{S}$. Moreover, since every module in $\mathrm{mod}-\mathcal{S}$ is an epimorphic image of a finite coproduct of modules of the shape $x^{\wedge}$, the full subcategory of $\mathrm{proj}-\mathcal{S}$ whose objects are the $x^{\wedge}$ is a spectroid of $\mathrm{proj}-\mathcal{S}$.

Of course the categories $\mathcal{S}-\mathrm{Mod}$ resp. $\mathcal{S}-\mathrm{mod}$ of covariant k-linear functors $\mathcal{S} \to \mathrm{Mod}-k$ resp. $\mathcal{S} \to \mathrm{mod}-k$ behave completely dually to $\mathrm{Mod}-\mathcal{S}$ and $\mathrm{mod}-\mathcal{S}$. If we denote by $\mathrm{D} = \mathrm{Hom}_k(-,k)$ the usual vector space duality, then $\mathcal{S}-\mathrm{mod} \to \mathrm{mod}-\mathcal{S}$, $M \mapsto \mathrm{D}\,M$ is also a duality functor which will be called D as well. We again obtain a fully faithful functor $\mathcal{S} \to \mathrm{mod}-\mathcal{S}$ by sending an object x of $\mathcal{S}$ to the functor $x^{\vee} := \mathrm{D}\,\mathcal{S}(x,-)$. Let us invoke the dual version of the Yoneda-Lemma saying that the functor $\mathrm{Mod}-\mathcal{S} \to \mathrm{mod}-k$ which sends a module M to $\mathrm{D}\,M(x)$ is isomorphic to the functor $\mathrm{Hom}_{\mathcal{S}}(-,x^{\vee})$. Consequently, if we denote by $\mathrm{inj}-\mathcal{S}$ the full subcategory of $\mathrm{mod}-\mathcal{S}$ formed by the *injective* objects, which are those Q in $\mathrm{mod}-\mathcal{S}$ such that the representable functor $\mathrm{Hom}_{\mathcal{S}}(-,Q) : \mathrm{mod}-\mathcal{S} \to \mathrm{mod}-k$ is exact, then $\mathrm{inj}-\mathcal{S}$ is a full subaggregate which has a spectroid whose objects are the $x^{\vee}$ for x in $\mathcal{S}$.

A module M in $\mathrm{mod}-\mathcal{S}$ is said to be *semisimple* if $0 = \mathrm{rad}\,M := M(\mathrm{rad}\,\mathcal{S})$. Clearly $\mathrm{top}\,M := M/\mathrm{rad}\,M$ is semisimple for all M in $\mathrm{mod}-\mathcal{S}$. The full subcategory of semisimple modules forms a subaggregate of $\mathrm{mod}-\mathcal{S}$. If we put $x^{-} := \mathrm{top}\,x^{\wedge}$ for all x in $\mathcal{S}$, then the full subcategory of $\mathrm{mod}-\mathcal{S}$ whose objects are the modules x^{-} forms a spectroid of the aggregate of semisimple modules. Modules isomorphic to some x^{-} are called simple. They are precisely the modules in $\mathrm{mod}-\mathcal{S}$ which have exactly two submodules. For each module M in $\mathrm{mod}-\mathcal{S}$ the largest semisimple submodule is called the *socle* of M and denoted by $\mathrm{soc}\,M$. As $\mathrm{rad}\,x^{\wedge}$ is the unique maximal submodule of $x^{\wedge}$ by duality the module $x^{\vee}$ has a simple socle which is isomorphic to x^{-} again.

3 Quivers and posets

3.1 The path category of a quiver and the k-linear category of a poset

A *quiver* Δ consists of a set of vertices Δ_0 and a set of arrows Δ_1 which is the disjoint union of sets $\Delta(x,y)$ where the elements of $\Delta(x,y)$ are the arrows $\alpha : x \to y$ from the vertex x to the vertex y. Thus a quiver is nothing but an oriented graph. Given a quiver Δ its *path category* $\mathrm{Pth}\,\Delta$ has as objects the vertices of Δ and the morphisms $x \to y$ are the *paths* from x to y which are by definition the formal compositions $\alpha_n \cdots \alpha_1$ where α_1 starts in x, α_n ends in y and the end point of α_i coincides with the start point of α_{i+1} for all $i = 1, \ldots, n-1$. The positive integer n is called the *length* of the path. We also introduce a path 1_x of length $= 0$ from each vertex x to itself. The composition in $\operatorname{Pth}\Delta$ of paths of positive length is just concatenation whereas the 1_x act as identities.

With any category $\mathcal{C}$ we can associate a k-linear category $k\mathcal{C}$ by taking as objects the objects of $\mathcal{C}$ and by using the elements of $\mathcal{C}(x,y)$ as k-basis of the morphism space $k\mathcal{C}(x,y)$. The composition in $k\mathcal{C}$ is of course obtained by k-linear extension of the composition in $\mathcal{C}$. In particular, if we start with a quiver Δ, then the category $k\Delta := k(\operatorname{Pth}\Delta)$ is called the *k-linear path category* of Δ.

Let $k^1\Delta$ be the ideal of $k\Delta$ generated by the arrows of Δ which are considered as paths of length $= 1$ and for all $n \in \mathbb{N}_0$ let $k^n\Delta$ be the n-th power of the ideal $k^1\Delta$. Usually the category $k\Delta$ will not be a spectroid. But we can obtain a spectroid form $k\Delta$ by factoring out a suitable ideal and imposing suitable finiteness conditions on Δ. Namely, let us require that Δ is *locally bounded* i.e. only finitely many arrows start and stop in each vertex. Let us call an ideal I of $k\Delta$ *weakly admissible* if $I \subseteq k^2\Delta$ and for any two objects x, y of Δ there is a positive integer $n(x,y)$ such that $k^{n(x,y)}(x,y) \subseteq I(x,y)$. Then $k\Delta/I$ is a spectroid, if Δ is a locally bounded quiver and I is a weakly admissible ideal of $k\Delta$. Note that in this case the Jacobson radical of $k\Delta/I$ coincides with $k^1\Delta/I$.

Another instructive class of spectroids can be constructed from partially ordered sets which we briefly call *posets*. We consider a poset S as category by taking as usual the underlying set as set of objects and introducing a morphism $(y \mid x)$ for all elements x, y of the poset satisfying $x \geq y$. The composition $(z \mid y) \cdot (y \mid x)$ is defined to be $(z \mid x)$. For a poset S the k-linear category kS is always a spectroid. The radical of kS is formed by all k-multiples of morphisms $(y \mid x)$ such that $x > y$. If we factor out an ideal I of kS with $I \subseteq \operatorname{rad} kS$, then kS/I is again a spectroid. In case all intervals in the poset S are finite, we can construct kS from the Hasse quiver Δ_S of S by forming $k\Delta_S/I$ where I is the ideal of $k\Delta_S$ formed by all differences $u - v$ where u, v are paths in Δ_S with the same start and end point.

3.2 Modules over path categories and posets

Quivers and posets are discrete structures. Therefore finite quivers and posets are easily encoded by matrices with integer coefficients which are well-adapted for computers. However, ideals of a path category kQ will usually be generated by linear combinations of paths involving scalars from the field k. Thus a typical element ρ of a generating set of an ideal I of $k\Delta$ is of the shape $\rho = \sum_{i=1}^n \lambda_i u_i$ where $\lambda_i \in k$ and the $u_i = \alpha_{n(i)}^{(i)} \ldots \alpha_1^{(i)}$ are paths from x to y in Δ.

Modules over a path category $k\Delta$ can be studied via mere linear algebra. Namely, they are the *k-linear representations* of the quiver Δ. This means that such a module M is given by vector spaces $M(x)$ over k for each vertex x of Δ and a k-linear map $M(\alpha) : M(y) \to M(x)$ for each arrow $\alpha : x \to y$ in Δ. This is inherited by modules over $k\Delta/I$ where we assume I to be generated by a set $\{\rho\}$ of generators. The modules in $\operatorname{Mod}-k\Delta/I$ are the k-linear representations M of the quiver Δ *satisfying the relations* $\{\rho\}$ i.e. $0 = M(\rho) = \sum_{i=1}^n \lambda_i M(u_i)$ for all ρ where $M(u_i) := M(\alpha_1^{(i)}) \ldots M(\alpha_{n(i)}^{(i)})$.

In contrast to path categories, for a poset S an ideal of kS can always be generated by morphisms $\rho = (y \mid x)$. Modules M over kS are *k-linear representations of S* i.e. are given by vector spaces $M(x)$ over k for each element x of S and k-linear maps $M(y \mid x) : M(y) \to M(x)$ for all $x \geq y$ which satisfy $M(z \mid x) = M(y \mid x) M(z \mid y)$ for all $x \geq y \geq z$. Consequently, modules over kS/I where I is generated by a set of morphisms $\rho = (y \mid x)$ are k-linear representations of S *satisfying the relations* $\{\rho\}$ i.e. $M(y \mid x) = 0$ for all $\rho = (y \mid x)$.

3.3 The quiver of a spectroid

From now on we assume that the field k is algebraically closed! The quiver $\operatorname{Qv}\mathcal{S}$ of a spectroid $\mathcal{S}$ has as set of vertices the set of objects of $\mathcal{S}$. For two objects x, y there are precisely $\dim_k \operatorname{irr}\mathcal{S}(x,y)$ arrows from x to y where $\operatorname{irr}\mathcal{S}(x,y)$ is the factor space $\operatorname{rad}\mathcal{S}(x,y)/(\operatorname{rad}\mathcal{S})^2(x,y)$. The morphisms in $\operatorname{rad}\mathcal{S}(x,y)$ whose classes are non-zero in $\operatorname{irr}\mathcal{S}(x,y)$ are called *irreducible*. Since $\mathcal{S}$ has finite-dimensional morphism spaces the quiver $\operatorname{Qv}\mathcal{S}$ is *locally finite* i.e. there are only finitely many arrows between any two vertices.

If we consider the spectroid $\mathcal{S}$ as quiver in the obvious way, then a quiver morphism $\Phi : \mathrm{Qv}\,\mathcal{S} \to \mathcal{S}$ is said to be a *presentation* if it acts as identity on the vertices and sends the arrows from x to y in $\mathrm{Qv}\,\mathcal{S}$ to morphisms in $\mathcal{S}(x, y)$ whose classes form a k-basis of $\mathrm{irr}\,\mathcal{S}(x, y)$. A presentation $\Phi : \mathrm{Qv}\,\mathcal{S} \to \mathcal{S}$ extends uniquely to a k-linear functor $k(\mathrm{Qv}\,\mathcal{S}) \to \mathcal{S}$ which we will call a presentation of $\mathcal{S}$ as well.

Unfortunately, a presentation needs not to be full. For instance, consider the spectroid $k\mathbb{Q}$ where $\mathbb{Q}$ are the rationals totally ordered in the usual way. Since $\mathrm{rad}\,k\mathbb{Q} = (\mathrm{rad}\,k\mathbb{Q})^2$, the quiver of $k\mathbb{Q}$ does not have any arrows. Therefore no presentation of $k\mathbb{Q}$ can be full. This example suggests to consider the ideal $\mathrm{rad}^\infty\,\mathcal{S}$ of the spectroid $\mathcal{S}$ which is defined as the intersection of all powers $(\mathrm{rad}\,\mathcal{S})^n$ where $n \in \mathbb{N}$. Since $\mathrm{rad}^\infty\,\mathcal{S} \subseteq \mathrm{rad}^2\,\mathcal{S}$, we obtain that a presentation $\Phi : k(\mathrm{Qv}\,\mathcal{S}) \to \mathcal{S}$ induces a presentation $\Phi : k(\mathrm{Qv}\,\mathcal{S}) \to \mathcal{S}/rad^\infty\mathcal{S}$ and $\mathcal{T} := \mathcal{S}/rad^\infty\mathcal{S}$ is still a spectroid. Moreover, this spectroid $\mathcal{T}$ is *generalized standard* i.e. $\mathrm{rad}^\infty\,\mathcal{T} = 0$. It is easy to prove that a presentation $\Phi : k(\mathrm{Qv}\,\mathcal{S}) \to \mathcal{S}$ of a spectroid $\mathcal{S}$ is full and the kernel of Φ is a weakly admissible ideal of $k(\mathrm{Qv}\,\mathcal{S})$ if and only if the spectroid $\mathcal{S}$ is generalized standard.

Since finite spectroids $\mathcal{S}$ are of particular interest for us, we remember that this finiteness forces $\mathrm{rad}\,\mathcal{S}$ to be nilpotent. Hence $\mathcal{S}$ is generalized standard and consequently can be identified with $k(\mathrm{Qv}\,\mathcal{S})/I$ where I is some weakly admissible ideal of $k(\mathrm{Qv}\,\mathcal{S})$.

Roughly speaking, the basic aim of Combinatorial REPresentation theory is to study the quiver of $\mathrm{ind}{-}\mathcal{S}$ for a given finite spectroid $\mathcal{S}$. However, usually this quiver gives only rather poor information about the category $\mathrm{ind}{-}\mathcal{S}$ because $\mathrm{ind}{-}\mathcal{S}$ fails to be generalized standard in most cases. In the next two sections we will outline how to get deeper insight on one hand by looking at additional data and on the other hand by passing to special situations.

4 The Auslander-Reiten quiver

4.1 Almost split sequences

A morphism $g : Y \to Z$ in an aggregate $\mathcal{A}$ is called *right almost split* if it lies in $\mathrm{rad}\,\mathcal{A}$ and any other morphism $g' : Y' \to Z$ in $\mathrm{rad}\,\mathcal{A}$ can be factorized as $g' = gh$ for some $h : Y' \to Y$ in $\mathcal{A}$. The morphism g is said to be *right minimal*, if every endomorphism s of Y satisfying $gs = g$ has to be an automorphism. Left almost split and left minimal morphism are defined in the dual fashion. Morphisms which are both right almost split and right minimal are briefly called *right minimal almost split*. Minimal almost split morphism are basically unique: if $g : Y \to Z$ is a minimal right almost split morphism, then any other minimal right almost split morphism ending in Z is of the shape gs for some isomorphism s. A right minimal almost split morphism $g : Y \to Z$ in an aggregate $\mathcal{A}$ with Z in $\mathcal{S}(\mathcal{A})$ stores all information about the predecessors of Z in $\mathrm{Qv}\,\mathcal{S}(\mathcal{A})$. Namely, for any object U in $\mathcal{S}(\mathcal{A})$ the number of arrows from $U \to Z$ in $\mathrm{Qv}\,\mathcal{S}(\mathcal{A})$ coincides with the multiplicity of U as direct summand of Y.

In a module category left minimal almost split monomorphisms and right minimal almost split epimorphisms starting resp. ending in indecomposable objects are closely related. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathrm{mod}{-}\mathcal{S}$ for a finite spectroid $\mathcal{S}$. Then the following two properties are equivalent:

(i) The module Z is indecomposable and g is right minimal almost split.

(ii) The module X is indecomposable and f is left minimal almost split.

Short exact sequences as above are called *almost split sequences* or *Auslander-Reiten sequences.* They may be interpreted as a device to construct a 'new' indecomposable module X by forming the kernel of the right minimal almost split epimorphism g ending in the 'known' indecomposable module Z. That a right minimal almost split epimorphism g exists in all possible cases follows from the theorem below which is most important for modern representation theory.

Theorem (Auslander,Reiten[AR]). *Let $\mathcal{S}$ be a finite spectroid. For each module Z in* $\mathrm{ind}{-}\mathcal{S}$ *there exists a right minimal almost split morphism $g : Y \to Z$ in* $\mathrm{mod}{-}\mathcal{S}$ *which is an epimorphism if Z is not projective and can be chosen as the canonical inclusion* $\mathrm{rad}\,x^\wedge \hookrightarrow x^\wedge$ *if $Z = x^\wedge$ is projective.*

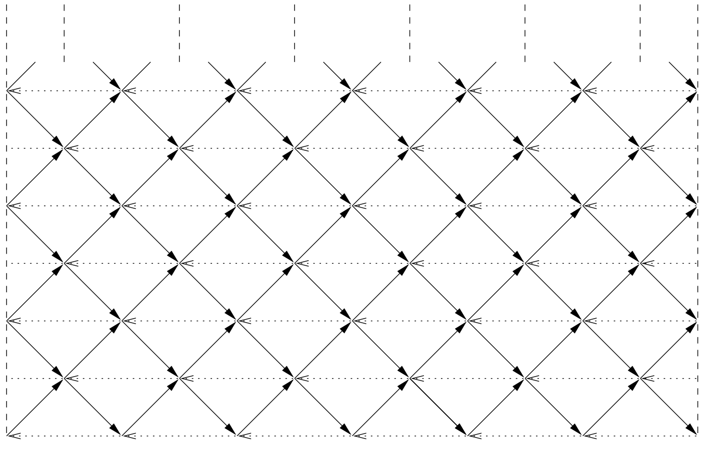
By the uniqueness of almost split morphisms an almost split sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ is uniquely determined up to isomorphism by Z and dually also by X. Thus by mapping Z to $\tau Z := X$ we obtain a bijection from the set of non-projective modules in ind$-\mathcal{S}$ to the set of non-injective modules in ind$-\mathcal{S}$. The inverse of this bijection is usually denoted by τ^-.

Using the map τ the Auslander-Reiten formula $\mathrm{D}\,\mathrm{Ext}^1_{\mathcal{S}}(Z,U) \cong \overline{\mathrm{Hom}}_{\mathcal{S}}(U,\tau Z)$ allows to calculate Ext-functors via Hom-functors. This will be very useful for calculations since Hom-functors will turn out to be computable. Note, that $\overline{\mathrm{Hom}}_{\mathcal{S}}(U,\tau Z)$ is the factor of $\mathrm{Hom}_{\mathcal{S}}(U,\tau Z)$ by the subspace of all morphisms factoring through an injective module. In fact, the Auslander-Reiten formula can be used for proving the existence of almost split sequences. Of course for this purpose the map τ has to be introduced before as the 'dual of the transpose'.

4.2 The Auslander-Reiten quiver

The existence of almost split sequences shows that for a finite spectroid $\mathcal{S}$ the quiver Qv(ind$-\mathcal{S}$) is endowed with an additional structure, namely the translation map τ which has the property that for each non-projective vertex Z and for any vertex U the number of arrows from U to Z equals the number of arrows from τZ to U. Quivers endowed with such a translation are called *translation quivers.* The translation quiver of ind$-\mathcal{S}$ for a finite spectroid $\mathcal{S}$ is called the *Auslander-Reiten quiver* of $\mathcal{S}$. It is usually denoted by $\Gamma_{\mathcal{S}}$ rather than Qv(ind$-\mathcal{S}$) and we will follow this tradition in the following.

In general the Auslander-Reiten quiver $\Gamma_{\mathcal{S}}$ will not be connected and various kinds of components appear. We will mention some special kinds of components for later use. Note, that a quiver is said to be *directed* it it does not admit an oriented cycle i.e. a path of positive length which starts and ends in the same vertex. A component of $\Gamma_{\mathcal{S}}$ is called preprojective if it is directed and any vertex is of the shape $\tau^{-m}x^{\wedge}$ for some indecomposable projective vertex $x^{\wedge}$. Preinjective components are defined in the dual way. Preprojective components can be 'knitted' inductively starting from their projective vertices. An infinite component of $\Gamma_{\mathcal{S}}$ as in the figure below (the right and the left rim have to be identified) is said to be a *stable tube.* As usual the translation τ is indicated by dotted arrows. The number $r \in \mathbb{N}$ of vertices at the 'mouth' (which is 6 in the example below) is called the *rank* of the tube. Stable tubes of rank $= 1$ are said to be *homogeneous.*



The translation structure of the Auslander-Reiten quiver imposed by the existence of almost split sequences can be used for the calculation of dimensions of Hom-functors. If Z is a non-projective vertex from $\Gamma_{\mathcal{S}}$, the modules $Y_1, \dots, Y_l$ are the predecessors of Z and there are n_i arrows from Y_i to Z, then for any U in ind$-\mathcal{S}$ the formula

$$\dim_k \mathrm{Hom}_{\mathcal{S}}(U,\tau Z) + \dim_k \mathrm{Hom}_{\mathcal{S}}(U,Z) = \sum_{i=1}^{l} n_i \dim_k \mathrm{Hom}_{\mathcal{S}}(U,Y_i) + \delta_{U,Z}$$

holds. This allows to compute the values of dimensions of Hom-functors on stable tubes from the values on the 'mouth'. Also one can calculate dimensions of Hom-functors on preinjective components by induction starting with the injective modules where the value is given by the dual Yoneda lemma.

4.3 The Grothendieck group

For a finite spectroid $\mathcal{S}$ the *Grothendieck group* $K_0(\mathcal{S})$ with respect to short exact sequences is the factor of the free abelian group with basis mod$-\mathcal{S}$ by the subgroup generated by all differences $Y - X - Z$ such that there is a short exact sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$. The structure of this group is easy to determine. For any module M in mod$-\mathcal{S}$ we define its *dimension vector* $\mathbf{dim}\, M$ as the vector in $\mathbb{Z}^{\mathcal{S}}$ whose entry at x is the dimension $\dim_k M(x)$. Then the map $\mathbf{dim} : \text{mod}-\mathcal{S} \to \mathbb{Z}^{\mathcal{S}}$ induces an isomorphism of groups $K_0(\mathcal{S}) \to \mathbb{Z}^{\mathcal{S}}$. In particular, this shows that for any short exact sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ in mod$-\mathcal{S}$ the equation $\mathbf{dim}\, X + \mathbf{dim}\, Z = \mathbf{dim}\, Y$ holds. Applying this to almost split sequences allows to calculate the dimension vectors of modules in stable tubes from those at the mouth and inductively the dimension vectors of modules in preprojective (resp. preinjective) components from the dimension vectors of the indecomposable projective (resp. injective) modules. Note that $(\mathbf{dim}\, x^{\wedge})_y = \dim_k \mathcal{S}(y, x)$ and $(\mathbf{dim}\, x^{\vee})_y = \dim_k \mathcal{S}(x, y)$ for all x, y in $\mathcal{S}$.

The dimension vector isomorphism can also be used to calculate certain dimensions of Hom- and Ext-functors by multiplication of integer matrices: The *Cartan matrix* $C_{\mathcal{S}}$ of the finite spectroid $\mathcal{S}$ is an integer matrix indexed over $\mathcal{S} \times \mathcal{S}$ whose entry at (x, y) is given by the dimension $\dim_k \mathcal{S}(x, y)$. Thus, if $\mathcal{S}$ has n objects, then $C_{\mathcal{S}}$ is an $n \times n$-matrix. In case the spectroid $\mathcal{S}$ has finite global dimension, on one hand the Cartan matrix is invertible over the integers and on the other hand we obtain a well-defined bilinear form $K_0(\mathcal{S}) \times K_0(\mathcal{S}) \to \mathbb{Z}$ by putting $\langle M, N \rangle_{\mathcal{S}} := \sum_{\nu=0}^{\infty} (-1)^{\nu} \dim_k \mathrm{Ext}^{\nu}_{\mathcal{S}}(M, N)$. These two constructions are related by the formula $\langle M, N \rangle_{\mathcal{S}} = (\mathbf{dim}\, M) C_{\mathcal{S}}^{-T} (\mathbf{dim}\, N)^T$ valid for all M, N in mod$-\mathcal{S}$. Later we will encounter the quadratic form $\chi_{\mathcal{S}}$ associated with this bilinear form which is called the *Euler form* of $\mathcal{S}$.

5 Representation types

5.1 Finite representation type

A finite spectroid $\mathcal{S}$ is said to be *of finite representation type* (or shorter *representation finite*) if its associated spectroid ind$-\mathcal{S}$ is finite i.e. there exist up to isomorphism only finitely many indecomposable modules in mod$-\mathcal{S}$. The Auslander-Reiten quiver gives a handy criterion to check finite representation type.

Theorem (Auslander[Au]). *If the Auslander-Reiten quiver $\Gamma_{\mathcal{S}}$ of a finite spectroid $\mathcal{S}$ with connected quiver* $\mathrm{Qv}\, \mathcal{S}$ *has a component such that there is a common bound for the dimensions of the modules in this component, then the component is finite and coincides with $\Gamma_{\mathcal{S}}$.*

This leaves the problem to find at least one component of the Auslander-Reiten quiver. Later we will see a class of spectroids which always have at least one preprojective component. For spectroids with small global dimension deciding finite representation type is reduced to a combinatorial problem by the following result.

Theorem (Ringel[Ri2]; Bongartz[Bo2]). *Suppose $\mathcal{S}$ is a finite spectroid whose quiver* $\mathrm{Qv}\, \mathcal{S}$ *is directed and connected, the global dimension of $\mathcal{S}$ is at most 2 and the Auslander-Reiten quiver $\Gamma_{\mathcal{S}}$ has a preprojective component. Then $\mathcal{S}$ is representation finite if and only if the Euler form $\chi_{\mathcal{S}}$ is weakly positive. Moreover, in this case the dimension vector map is a bijection between* ind$-\mathcal{S}$ *and the set of positive 1-roots of $\chi_{\mathcal{S}}$.*

We have to make up for some notations used in the above theorem. The Euler form $\chi_{\mathcal{S}}$ is said to be *weakly positive* if $\chi_{\mathcal{S}}(x) > 0$ for all $x \in \mathbb{Z}, x > 0$ where we endow $\mathbb{Z}^{\mathcal{S}}$ with the usual product order. The *positive 1-roots* of $\chi_{\mathcal{S}}$ are the vectors $x \in \mathbb{Z}^{\mathcal{S}}$ satisfying $x > 0$ and $\chi_{\mathcal{S}}(x) = 1$. To find out in an efficient algorithmic way, whether $\chi_{\mathcal{S}}$ and similar quadratic forms are weakly positive, will be the topic of other contributions in this collection. Also generalizations of weak positivity will be treated. Therefore we will not go deeper into this subject. Let us mention however that the above theorem applies to hereditary spectroids because their Auslander-Reiten quivers always possess a preprojective component. We recall that a finite spectroid is hereditary if and only if its quiver $\mathrm{Qv}\, \mathcal{S}$ is directed and $\mathcal{S} \cong k(\mathrm{Qv}\, \mathcal{S})$. Thus we obtain the following classical result which is one of the starting points of the modern representation theory of algebras.

Theorem (Gabriel[Ga1]). *Let $\mathcal{S} = k\Delta$ be a finite hereditary spectroid with connected quiver Δ. Then $\mathcal{S}$ is of finite representation type if and only if the underlying graph of the quiver Δ is one of the Dynkin graphs $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(8 \geq n \geq 6)$.*

5.2 Tame and wild representation type

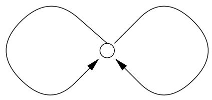
To define tame and wild representation type we have to introduce some geometric concepts. Let $\mathcal{S} = k\Delta/I$ be a finite spectroid and $\{\rho\}$ be a finite set of generators of the ideal I. For fixed $d \in \mathbb{N}_0^{\mathcal{S}}$ the set $\mathrm{mod_d}{-}\mathcal{S}$ of modules M in $\mathrm{mod}{-}\mathcal{S}$ satisfying $\mathbf{dim}\, M = d$ can be considered as a Zariski closed subvariety of the linear variety $\prod_\alpha k^{d_x \times d_y}$ where α runs through all arrows $\alpha : x \to y$ in the quiver Δ. To see this, we recall first that such a module M can be identified with a representation of Δ such that $M(\rho) = 0$ for each relation ρ and $\dim_k M(x) = d_x$ for all x in $\mathcal{S}$. By fixing bases in the spaces $M(x)$ any arrow $\alpha : x \to y$ in Δ is represented by a $d_x \times d_y$-matrix $M(\alpha)$ over k and the equation $M(\rho) = 0$ amounts to a finite collection of polynomial equation on the matrix entries.

There is an analogous way to associate varieties with finite spectroids whose object set without loss of generality is chosen as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$. For this purpose let us now fix a $n \times n$-matrix B over the non-negative integers and define $C := B + I$ where I is a unit matrix of the same shape as B. The set alg_C of spectroids with Cartan matrix C can be considered as a Zariski closed subvariety of the linear variety $\prod_{(x,y,z)} k^{B_{y,z} \times B_{x,y} \times B_{x,z}}$ where the triples (x, y, z) run through $\mathcal{S}^3$. To see this, one chooses bases in all spaces $\mathrm{rad}\,\mathcal{S}(x, y)$ and represents the composition $\mathrm{rad}\,\mathcal{S}(y, z) \times \mathrm{rad}\,\mathcal{S}(x, y) \to \mathrm{rad}\,\mathcal{S}(x, z)$ by an $B_{x,z}$-tuple of $B_{y,z} \times B_{x,y}$-matrices. The associativity of the composition and the nilpotency of $\mathrm{rad}\,\mathcal{S}$ impose polynomial equations on the entries of these matrices.

A finite spectroid $\mathcal{S}$ is said to be of *tame representation type* (or shorter *tame*) if for each possible dimension vector $d \in \mathbb{N}_0^{\mathcal{S}}$ there exist finitely many regular 1-parameter families $\gamma_1, \ldots, \gamma_{\mu(d)}$ with the property that up to isomorphism and finitely many failures each indecomposable module in $\mathrm{mod_d}{-}\mathcal{S}$ lies in the image of one of the maps $\gamma_i : k \to \mathrm{mod_d}{-}\mathcal{S}$.

Let us choose these 1-parameter families in a way such that $\mu(d)$ is minimal for all $d \in \mathbb{N}_0^{\mathcal{S}}$ and let us define a function $\bar{\mu} : \mathbb{N}_0 \to \mathbb{N}_0$ by $\bar{\mu}(n) := \sum_d \mu(d)$ where we sum over all d satisfying $\sum_{x\in\mathcal{S}} d_x = n$. Thus $\bar{\mu}$ measures the number of 1-parameter families needed to parameterize the indecomposables M of total dimension $n = \sum_{x\in\mathcal{S}} \dim_k M(x)$. If the function $\bar{\mu}$ grows at most polynomially, then the spectroid $\mathcal{S}$ is said to be *tame of polynomial growth.* In particular, $\mathcal{S}$ is called *domestic* if μ is of constant growth. Note, that representation finite spectroids are tame. In fact, the confirmation of the second Brauer-Thrall conjecture means that the representation finite spectroids are precisely the tame spectroids such that $\bar{\mu} = 0$.

Let Ω be the double loop quiver which is shown below. A finite spectroid $\mathcal{S}$ is called *of wild representation type* (or shorter *wild*) provided there is an embedding functor $\mathrm{mod}{-}k\Omega \to \mathrm{mod}{-}\mathcal{S}$ which preserves indecomposability and isomorphism classes.



Recall that the modules in $\mathrm{mod}{-}k\Omega$ can be identified with pairs of square matrices of the same size. Furthermore, isomorphism means equivalence under simultaneous conjugation.

Theorem (Drozd[Dd] see also [C-B1]). *A finite spectroid is tame or wild but not both.*

5.3 Degenerations and coverings

A well-tried strategy to investigate complicated structures in mathematics is to 'simplify' the structure without changing too much the property one wants to study. This applies as well to finite spectroids as to their representation type. Let us first make use of the geometry of alg_C.

Theorem (Gabriel[Ga2]). *The subset of representation finite spectroids in the variety* alg_C *is Zariski open.*

Recall that a spectroid $\mathcal{T}$ in alg_C is said to be a *degeneration* of another spectroid $\mathcal{S}$ in alg_C provided $\mathcal{T}$ lies in the Zariski closure of the subset of alg_C formed by all spectroids isomorphic to $\mathcal{S}$. Thus the above theorem implies that for checking finite representation type one may pass to a degeneration of the given spectroid. For the time being it is only conjectured that also the subset of tame spectroids is open, however it has already been proved by Geiß (see [Ge]) that degenerations of wild spectroids are wild which shows that also tame representation type can be decided via degenerations.

If a spectroid is given by a quiver with relations, then degenerations usually allow to simplify the relations. In contrast, covering theory allows to simplify the quiver by 'untwisting oriented cycles'. Let us make this a little more precise. Let $\mathcal{T}$ be a *locally bounded* spectroid i.e. for all x in $\mathcal{S}$ the sum $\sum_y(\dim_k \mathcal{S}(x,y) + \dim_k \mathcal{S}(y,x))$ is finite where y runs through all objects y of $\mathcal{T}$. Moreover, let G be a group of automorphism of $\mathcal{T}$ which acts freely on the objects of $\mathcal{T}$. Then the orbits of objects and morphisms of $\mathcal{T}$ under G are made into a spectroid $\mathcal{T}/G$ in the natural way. The canonical functor $\mathcal{T} \to \mathcal{T}/G$ is called a *covering functor*. There are various theorems asserting that $\mathcal{T}/G$ inherits the representation type of $\mathcal{T}$ under additional assumptions on $\mathcal{T}$ and G. We refer e.g. to [Ga4], [DS1] and [DS2] for more details.

Whereas for finite representation type coverings also allow complete control over the spectroid of indecomposable representations, this is still an open problem for tame type. Moreover, the covering $\mathcal{T}$ usually fails to be finite. Although the representation type is determined by the finite full subspectroids of $\mathcal{T}$, for algorithmic purposes one of course does not want to consider all of them but looks for some finite subcollection. This can be done for finite representation type. In this case one also has a canonical degeneration the so-called 'standard form' which admits a strongly simply connected 'universal covering'. Therefore in the next section we will study strongly simply connected spectroids from the computational point of view. Examples show that there is good hope that also for tame type a similar simplification using a covering of a standard form may be possible. Nevertheless, the right concept of a standard form has to be developed and the notion of 'strong simple connectedness' needs some further generalization.

Coming back to finite representation type, the formation of the standard form and of the universal covering lead to quite clear combinatorial algorithms. Unfortunately, up to now these algorithms are only partially implemented.

6 Strongly simply connected spectroids of finite and tame representation type

6.1 Completely separating spectroids

There are various ways to introduce strongly simply connected spectroids. We will use one which is best suited for computational purposes. Let $\mathcal{S}$ be a finite spectroid whose quiver $\mathrm{Qv}\,\mathcal{S}$ is directed. For x in $\mathcal{S}$ we decompose the radical $\mathrm{rad}\, x^\wedge = \oplus_{i=1}^r R_i$ of the indecomposable projective modules $x^\wedge$ into a direct sum of indecomposable modules R_i in $\mathrm{ind}{-}\mathcal{S}$. The object x is said to be *separating* if the supports $\mathrm{supp}\, R_i := \{y \in \mathcal{S} \mid R_i(y) \neq 0\}$ lie in r different connected components of the full subquiver of $\mathrm{Qv}\,\mathcal{S}$ supported by the vertices z which are not terminus of a path in $\mathrm{Qv}\,\mathcal{S}$ starting in x. The spectroid $\mathcal{S}$ itself is called *separating* if all objects x in $\mathcal{S}$ are separating. Finally, the spectroid $\mathcal{S}$ is defined to be *strongly simply connected* provided each full convex subcategory is separating where convexity is required with respect to the partial ordering imposed on $\mathcal{S}$ by the directed quiver $\mathrm{Qv}\,\mathcal{S}$.

We have seen before that the existence of a preprojective component allows the checking of the representation type with combinatorial tools. From this follows the importance of the following result for computational purposes.

Theorem (Bautista,Larrión[BL]; Bongartz[Bo4]). *If the spectroid $\mathcal{S}$ is separating, then the Auslander-Reiten quiver $\Gamma_{\mathcal{S}}$ has a preprojective component.*

It is easy to see that a finite spectroid $\mathcal{S}$ with directed quiver $\mathrm{Qv}\,\mathcal{S}$ cannot be representation finite if there are two objects x, y satisfying $\dim_k \mathcal{S}(x,y) > 1$. Analogously, $\mathcal{S}$ cannot be tame if there exist x, y such that $\dim_k \mathcal{S}(x,y) > 2$. Therefore concerning finite type we do not loose generality if we consider only spectroids $\mathcal{S}$ which are *schurian* i.e. $\mathcal{S}(x,y) \leq 1$ for each pair x, y of objects. Schurian strongly simply connected spectroids are called *completely separating*. As tame strongly simply connected spectroids $\mathcal{S}$ such that $\dim_k \mathcal{S}(x,y) = 2$ appear, we really loose some generality by considering only the schurian case. Nevertheless, all the known interesting phenomena for tame type already occur in the completely separating situation. On the other hand there is not yet a good data structure for strongly simply connected spectroids in general whereas a computer can easily deal with completely separating spectroids due to the following result.

Theorem ([Dr4]). *If $\mathcal{S}$ is a finite completely separating spectroid, then there is a finite poset S and an admissible ideal I of kS such that $\mathcal{S} \cong kS/I$.*

Thus one has a simple combinatorial data structure given by the Hasse quiver of S and a finite set of relations $\rho = (y \mid x)$ for I. The above result is the reason why many CREP procedures deal with spectroids given by a finite poset with relations. Clearly this concept will have to undergo generalizations in the future.

6.2 Tilting spectroids

Before coming to classifications we make a small detour to an area in representation theory called tilting theory. Let $\mathcal{S}$ be a finite spectroid with n objects. A full subspectroid $\mathcal{T}$ of ind$-\mathcal{S}$ is called *tilting* if the following three conditions are satisfied:

(i) The projective dimension of each module in $\mathcal{T}$ is at most 1.

(ii) $\mathrm{Ext}^1_{\mathcal{S}}(X,Y) = 0$ for any pair of objects X, Y in $\mathcal{T}$.

(iii) $\mathcal{T}$ has precisely n objects.

The importance of tilting theory comes from the fact that the representation theory of $\mathcal{S}$ and $\mathcal{T}$ are closely related. In fact, the derived categories of $\mathcal{S}$ and $\mathcal{T}$ even coincide. We will only need the following special aspect of these facts.

Theorem (Brenner,Butler[BB]; Happel,Ringel[HR]). *Let $\mathcal{S} = k\Delta$ be a hereditary spectroid and $\mathcal{T}$ a tilting spectroid in* ind$-\mathcal{S}$. *Then there are two disjoint subsets $\mathcal{G}$ and $\mathcal{F}$ of* ind$-\mathcal{S}$ *such that* ind$-\mathcal{T}$ *is the disjoint union of the the set of modules* $\mathrm{Hom}_{\mathcal{S}}(-,X)$, *$X \in \mathcal{G}$ and the set of modules* $\mathrm{Ext}^1_{\mathcal{S}}(-,Y)$, *$Y \in \mathcal{F}$.*

In particular, this means that ind$-\mathcal{T}$ has less objects than ind$-\mathcal{S}$. More precisely, $\mathcal{T}$ is representation finite provided $\mathcal{S}$ is representation finite and $\mathcal{T}$ is tame or representation finite provided $\mathcal{S}$ is tame.

We have seen before that we can calculate dimensions of Hom-functors on certain components of the Auslander-Reiten quiver in a combinatorial way. Thus the Auslander-Reiten formula can be used to check condition (ii) by a computer. Fortunately, the same applies to condition (i) because X in ind$-\mathcal{S}$ has projective dimension at most 1 if and only if $\mathrm{Hom}_{\mathcal{S}}(y^{\vee}, \tau X) = 0$ for all y in $\mathcal{S}$. Finally, the calculation of Hom-dimensions provides the Cartan matrix $C_{\mathcal{T}}$. If $\mathcal{T} = kT/I$ is completely separating, then it is easy to restore the combinatorial data namely the poset T and the combinatorial generators for I from $C_{\mathcal{T}}$.

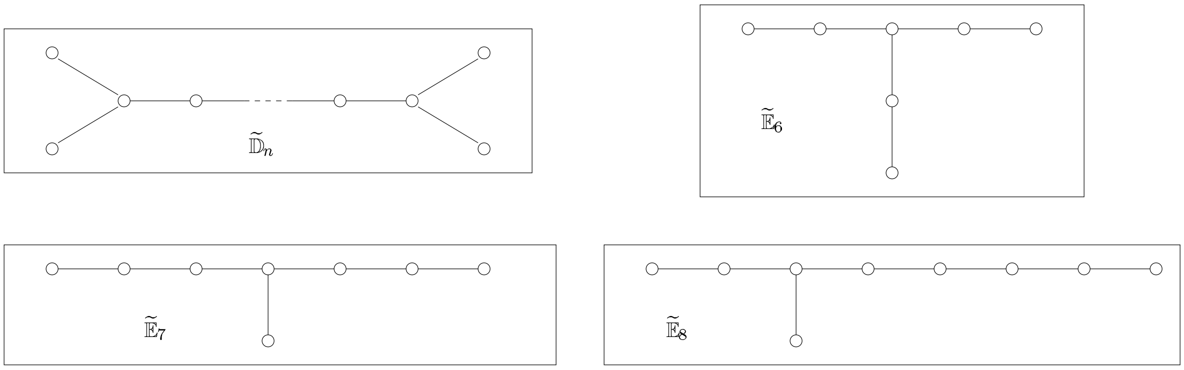
6.3 Classifications

The proper substructures to consider of strongly simply connected spectroids are the convex full subspectroids because they inherit the defining property. This is the motivation for the following definitions. A completely separating spectroid $\mathcal{S} = kS/I$ is called

- *critical* if $\mathcal{S}$ is not representation finite but each proper convex full subspectroid is representation finite.
- *hypercritical* if $\mathcal{S}$ is wild but each proper convex full subspectroid is tame.
- *sincere* if there exists M in ind$-\mathcal{S}$ which is sincere i.e. $M(x) \neq 0$ for all x in S.
- *weakly sincere* if there exists M in ind$-\mathcal{S}$ which is weakly sincere i.e. $M(x) \neq 0$ for all extremal x in S.

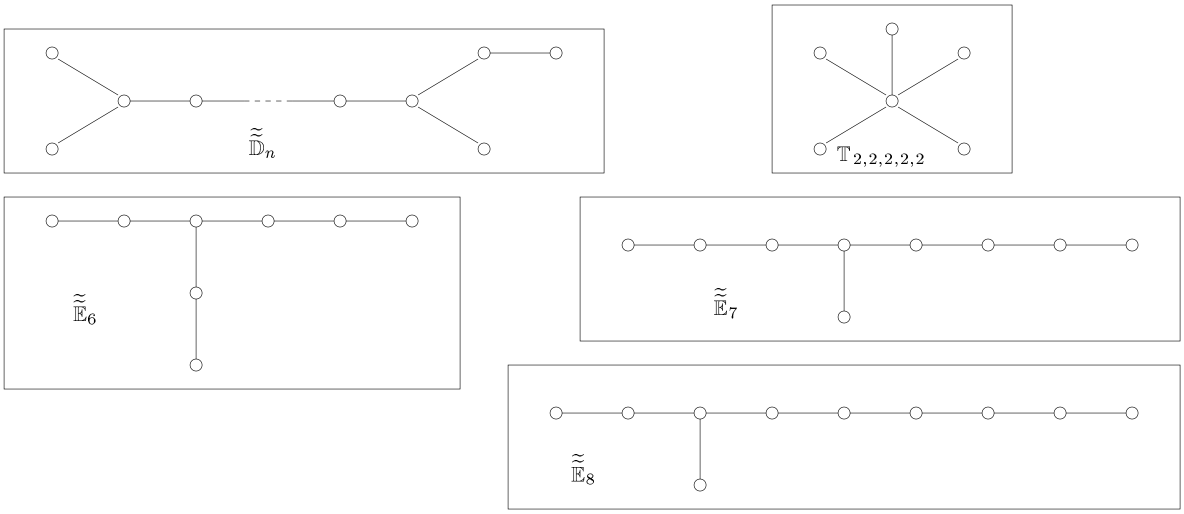
Obviously, for showing that a given completely separating spectroid $\mathcal{S}$ is representation finite (resp. tame) it is necessary and sufficient that no critical (resp. hypercritical) completely separating spectroid appears as convex full subspectroid. Moreover, once one knows all weakly sincere completely separating spectroids and their weakly sincere indecomposables, then one knows all indecomposables over any completely separating spectroid. The reason for this is that for a convex full subspectroid $\mathcal{T}$ of $\mathcal{S}$ there is a canonical full embedding mod$-\mathcal{T} \to$ mod$-\mathcal{S}$ which extends a module M in mod$-\mathcal{T}$ to a module in mod$-\mathcal{S}$ by putting $M(x) := 0$ for all x in $\mathcal{S} \setminus \mathcal{T}$.

Theorem (Bongartz[Bo3]). *The critical completely separating spectroids coincide with the preprojective schurian tilting spectroids $\mathcal{T}$ in* ind$-k\Delta$ *where Δ is an extended Dynkin tree i.e. the underlying graph of Δ is one of $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$ or $\widetilde{\mathbb{E}}_n$ $(8 \geq n \geq 6)$.*



Using this characterization the critical completely separating spectroids were classified by Happel and Vossieck in [HV]. There are three series arising from $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$ and 4302 isomorphism classes of exceptional cases coming from $\widetilde{\mathbb{E}}_n$ $(8 \geq n \geq 6)$ which were classified using a computer program based on the combinatorial techniques which we outlined before.

Theorem (de la Peña[Pe1]). *The hypercritical completely separating spectroids contain the preprojective schurian tilting spectroids $\mathcal{T}$ in* ind$-k\Delta$ *where Δ is a hyperextended Dynkin tree i.e. the underlying graph of Δ is one of $\widetilde{\widetilde{\mathbb{D}}}_n$ $(8 \geq n \geq 4)$, $T_{2,2,2,2,2}$ or $\widetilde{\widetilde{\mathbb{E}}}_n$ $(8 \geq n \geq 6)$.*



It is conjectured that the the hypercritical completely separating spectroids really coincide with the tilting spectroids from the above theorem. These tilting spectroids were classified by Unger in [Un]. There are 13747 isomorphism classes coming from $\widetilde{\widetilde{\mathbb{E}}}_n$ $(8 \geq n \geq 6)$ and a small list from $\widetilde{\widetilde{\mathbb{D}}}_n$ $(8 \geq n \geq 4)$.

Only for the representation finite case there exists a complete classification for the weakly sincere spectroids. Note, that an indecomposable module over a completely separating spectroid of finite representation type is weakly sincere if and only if it is sincere and if and only if it is faithful.

Theorem (Bongartz[Bo1]; Ringel[Ri2]). *The sincere completely separating representation finite spectroids are contained in the schurian tilting spectroids $\mathcal{T}$ in* ind$-k\Delta$ *having preprojective and preinjective objects where Δ is a tree with at most four endpoints.*

In [Bo1] Bongartz gave a bound N such that the sincere completely separating representation finite spectroids with more than N objects lie in 24 infinite series. This bound was optimized to 13 by Ringel in [Ri2]. The 16344 isomorphism classes of exceptional cases are constructed in [Dr1] (see also [Dr2]). Note, that the above theorem did not help too much in that construction because there are too many of these trees. An inductive construction proposed in [Ri2] was used for [Dr1].

It would be an exercise to repeat the above classifications using CREP. In practice one is usually interested in the lists themselves e.g. in order to test some conjectures on them. The lists are accessible in CREP by calling the procedures **critical_alge**, **hypercritical_algd**, **hypercritical_alge** and **esrdalg**. Note that in these data bases not all isomorphism classes but only representatives in certain equivalence classes called 'frames' are stored. Roughly speaking two spectroids belong to the same frame if up to duality they differ only by 'trivial branches'. CREP also offers procedures **iscriticald**, **iscriticale**, **ishypercriticald**, **ishypercriticale** and **esrdtest** to verify if a given spectroid is in one of the classification lists. We refer to [DN2] for details about the use of the programs mentioned.

In particular, by checking the criticality of all convex full subspectroids of a given completely separating spectroid using **iscriticald** and **iscriticale** one would get an algorithm for deciding finite representation type. We will see in the next section how this can be done in a more efficient way.

7 Thin modules over completely separating spectroids

7.1 Vector space categories

As seen above, modules over a finite spectroid can be identified with representations of quivers. After choosing bases, one obtains finite sequences of matrices satisfying some relations. In this language morphisms in the category mod$-\mathcal{S}$ correspond to certain admissible transformations on the rows and columns of the matrices. Clearly there are many similar 'matrix problems' and some of them actually play an important role in representation theory. Putting them into categorical language these problems deal with 'representations of bimodules'. We refer to [C-B2] for an account on the various concepts which are used.

A special class of bimodules which is very well studied is furnished by vector space categories. A *vector space category* is a pair $(\mathcal{A}, M)$ where $\mathcal{A}$ is an aggregate and $M : \mathcal{A} \to \text{mod}-k$ is a k-linear functor. The vector space category $(\mathcal{A}, M)$ is said to be *faithful* if the functor M acts faithful. The corresponding category of representations is called the *subspace category* of $(\mathcal{A}, M)$ and denoted by $\check{\mathcal{U}}(\mathcal{A}, M)$. It is again an aggregate whose objects are the pairs $V = (V_\omega, \gamma_V, V_0)$ such that V_ω is a finite-dimensional k-space, V_0 is an object in $\mathcal{A}$ and $\gamma_V \in \text{Hom}_k(V_\omega, M(V_0))$. A morphism $V \to W$ is a pair $f = (f_\omega, f_0)$ such that $f_\omega : V_\omega \to W_\omega$ is a k-linear map, $f_0 : V_0 \to W_0$ is a morphism in $\mathcal{A}$ and $\gamma_W f_\omega = M(f_0)\gamma_V$. For finite vector space categories $(\mathcal{A}, M)$, which means that the spectroid $\mathcal{S}(\mathcal{A})$ is finite, the representation types are defined in an analogous way as for spectroids .

Vector space categories arise in representation theory because there are various 'reduction functors' relating mod$-\mathcal{S}$ to a suitable subspace category. We only present one example of such a reduction functor which we will need later. Suppose the finite spectroid $\mathcal{S}$ has an objects x such that the algebra $\mathcal{S}(x, x)$ is isomorphic to k. This happens for all vertices of spectroids whose quiver Qv$\,\mathcal{S}$ is directed thus in particular for completely separating spectroids. Let us denote by K_x the full subaggregate of mod$-\mathcal{S}$ given by all M satisfying $\tau M(x) = 0$. Then the functor $F : \check{\mathcal{U}}(K_x, \text{Hom}_{\mathcal{S}}(x^\wedge, -)) \to \text{mod}-\mathcal{S}$ which maps an object $V = (V_\omega, \gamma_V, V_0)$ to the cokernel of the adjoint homomorphism $V_\omega \otimes_k x^\wedge \to V_0$ of the map $\gamma_V : V_\omega \to \text{Hom}_{\mathcal{S}}(x^\wedge, V_0)$ is called *fiber sum functor.*

Theorem ([Dr5]). *The fiber sum functor $F : \check{\mathcal{U}}(K_x, \text{Hom}_{\mathcal{S}}(x^\wedge, -)) \to$ mod$-\mathcal{S}$ is full, dense and its kernel is an ideal generated by finitely many indecomposable objects.*

Note, that the classical one-point extension (resp. one-point coextension) technique (see e.g. [Ri2]) is a special case of the reduction via fiber sum functors for the case that x is a source (resp. sink).

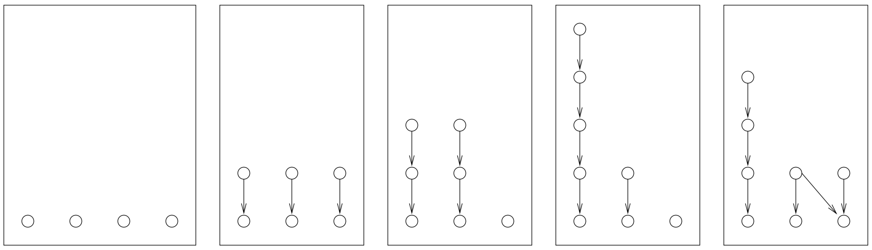
7.2 Linear vector space categories

It follows from the above theorem that F preserves and reflects finite and tame representation type. Nevertheless, the result is only helpful provided $\check{\mathcal{U}}(K_x, \mathrm{Hom}_{\mathcal{S}}(x^\wedge, -))$ is easier to study than mod$-\mathcal{S}$. We will outline an interesting situation where this is actually the case.

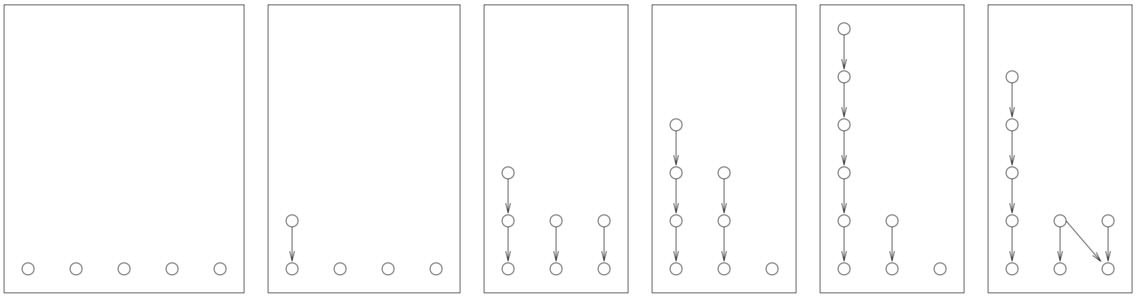
A faithful vector space category $(\mathcal{A}, M)$ is said to be *linear* if $\dim_k M(X) = 1$ for all X in $\mathcal{S}(\mathcal{A})$. Linear vector space categories are easy to understand. They can always be identified with $(\mathrm{add}\, k\mathcal{P}, can)$ where $\mathcal{P}$ is a poset and *can* is the canonical functor sending each point x of $\mathcal{P}$ to k and each morphism $(y \mid x)$ to the identity of k. It is trivial to construct the poset $\mathcal{P}$ from $\mathcal{A}$. Namely, the object set of $\mathcal{P}$ is $\mathcal{S}(\mathcal{A})$ and $x \geq y$ holds whenever $\mathcal{A}(x, y) \neq 0$.

The importance of linear vector space categories comes mainly from the fact that it is much easier to determine their representation type. This is due to theorems originating from the 'Kiev school' of representation theory.

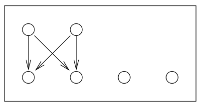
Theorem (Kleiner[Kl]). *Suppose $\mathcal{P}$ is a finite poset. Then the linear vector space category $(\mathrm{add}\, k\mathcal{P}, can)$ is of finite representation type if and only if $\mathcal{P}$ does not contain a full subposet from the following list $\mathcal{K}$ of 5 posets.*



Theorem (Nazarova[Na]). *Suppose $\mathcal{P}$ is a finite poset. Then the linear vector space category $(\mathrm{add}\, k\mathcal{P}, can)$ is of tame representation type if and only if $\mathcal{P}$ does not contain a full subposet from the following list $\mathcal{N}$ of 6 posets.*



Theorem (Nazarova,Zavadskij[NZ]). *Suppose $\mathcal{P}$ is a finite poset. Then the linear vector space category $(\mathrm{add}\, k\mathcal{P}, can)$ is tame of polynomial growth if and only if $\mathcal{P}$ does not contain as full subposet a poset from the list $\mathcal{N}$ or the poset $\mathcal{Z}$ shown below.*



7.3 Thin modules

In the representation finite situation the vector space category obtained from $(K_x, \mathrm{Hom}_{\mathcal{S}}(x^\wedge, -))$ by factoring out the kernel of $\mathrm{Hom}_{\mathcal{S}}(x^\wedge, -)$ from K_x (in order to have a faithful functor) is linear for arbitrary objects x of $\mathcal{S}$. In the tame situation this is not true any more. Moreover, the spectroid $\mathrm{ind}\, K_x$ may also be infinite in this case. Finally, it seems to be impossible to construct K_x from $\mathcal{S}$ without good knowledge about ind$-\mathcal{S}$. In the following we will introduce a subaggregate of K_x which can be computed directly from $\mathcal{S}$ and gives rise to a finite linear vector space category.

Call a module M in mod$-\mathcal{S}$ *thin* if it satisfies $\dim_k M(x) \leq 1$ for all x in $\mathcal{S}$. Indecomposable thin modules over a completely separating spectroid $\mathcal{S} = kS/I$ are easy to describe. We say that a convex subset T of S is *strictly convex* if $\mathcal{S}(x,y) \neq 0$ for all $x \geq y$ in T. In terms of generators of I this conditions means that $x \geq s \geq t \geq y$ does not happen for any generator $\rho = (t|s)$. With this notation the map $M \mapsto \operatorname{supp} M$ furnishes a bijection from the subset of thin modules in ind$-\mathcal{S}$ to the set of strictly convex connected subsets of S. How can one check via $\operatorname{supp} M$ if an indecomposable thin modules M belongs to K_x? We can assume $M(x) \neq 0$ since otherwise $\operatorname{Hom}_{\mathcal{S}}(x^\wedge, -)$ vanishes on M. It turns out that M with $T := \operatorname{supp} M$ lies in K_x if and only if T is *x-irreducible*. This means by definition that the sum of the number of connected components of $T \setminus \{y \in T \mid x \geq y\}$ and the number of x-extension points of T is at most 1. An *x-extension point* of T is an element y of $S \setminus T$ such that $y \geq x$ and $T \cup \{y\}$ is still strictly convex. Denoting by $\mathcal{P}_x$ the set of x-irreducible strictly convex connected subsets of S containing x, we have seen that we can embed the linear vector space category $(\operatorname{add} k\mathcal{P}_x, can)$ into $(K_x, \operatorname{Hom}_{\mathcal{S}}(x^\wedge, -))$ (at least after factoring out the kernel of the functor $\operatorname{Hom}_{\mathcal{S}}(x^\wedge, -)$ from K_x). It is still necessary to express the order on $\mathcal{P}_x$ in a easily computable way: we have to put $T_1 \geq T_2$ if and only if the connected component C of $T_1 \cap T_2$ which contains x is an ideal in T_1 (i.e. closed against bigger elements in T_1) and a coideal in T_2 (i.e. closed against smaller elements in T_2). Thus with each objects x of a completely separating spectroid $\mathcal{S} = kS/I$ we have associated a finite partially ordered set $\mathcal{P}_x$ which can be calculated form S and the generators of I in an algorithmic way.

Theorem ([DN1]). *Let $\mathcal{S}$ be a finite completely separating spectroid.*

(a) $\mathcal{S}$ is representation finite if and only if for each object x of $\mathcal{S}$ the finite poset $\mathcal{P}_x$ does not contain a full subposet from the list $\mathcal{K}$.

(b) If $\mathcal{S}$ is tame, then for each object x of $\mathcal{S}$ the finite poset $\mathcal{P}_x$ does not contain a full subposet from the list $\mathcal{N}$.

The value of the result comes from the fact that the lists $\mathcal{K}$ and $\mathcal{N}$ are remarkably shorter than the list of critical and hypercritical spectroids so that one can implement efficient procedures which construct the posets $\mathcal{P}_x$ and look for full subposets from the lists $\mathcal{K}$ and $\mathcal{N}$. This is done in CREP with the procedures **mchrepfin** and **mchwild**.

Unfortunately part (b) does not give a necessary and sufficient conditions. The situation improves if one considers tameness of polynomial growth.

Theorem. *A finite completely separating spectroid $\mathcal{S}$ is tame of polynomial growth if and only if for each object x of $\mathcal{S}$ the finite poset $\mathcal{P}_x$ does not contain as full subposet a poset from the list $\mathcal{N}$ or the poset $\mathcal{Z}$.*

The proof of this rests on a recent theorem of Skowroński (see [Sk5]) who showed that a strongly simply connected spectroid is tame of polynomial growth if it does not contain a convex full subspectroid which is hypercritical or pg-critical. We already encountered the hypercritical spectroids. The pg-critical spectroids are certain extensions of critical spectroids of type $\widetilde{\mathbb{D}}_n$ which were classified in [NS]. To get the above result it was checked that each pg-critical completely separating spectroid has an object x such that $\mathcal{P}_x$ contains $\mathcal{Z}$ as a full subposet.

The algorithm suggested by the above theorem is implemented in CREP as the procedure **mchpg**. Let us finish by an exercise using this program. We wish to classify the tame completely separating connected spectroids of the shape $\mathcal{S} = kS$ containing a convex full subspectroid which is critical of type $\widetilde{\mathbb{E}}_n$ $(n = 6, 7, 8)$. This classification was announced in [DN1]. Note, that any such spectroid has a characteristic sincere indecomposable module M where $M(x) = k$ for all objects x and all $(y \mid x)$ are mapped to id_k. Following [PS] there should be only finitely many isomorphism classes of these spectroids and all the appearing ones should be tame of polynomial growth. We try to construct the corresponding posets S be induction starting with the critical ones as list L_0. For constructing L_{n+1} from L_n we write a program which calculates all possible posets which are a connected one-point extension of the ones in the list L_n obtaining a list L''. From L'' we eliminate all those which are not completely separating using the CREP-procedure **complsep** and obtain a list L'. We apply the function **mchwild** to all elements of L' and put into L_{n+1} only those where the answer is 'false'. We observe that for $n = 9$ the list L_n is empty. Hence all L_n for $n \geq 9$ are empty. Finally we apply **mchpg** to all elements of the lists $L_1, \ldots, L_8$ and always get the answer 'true'.

The classification list obtained can be seen in the latest relase of CREP using the procedure **tpe_alg**. Let us present in a table the number of frames of spectroids in this list which are not themselves tame concealed.

number of objects	8	9	10	11	12	13	14	15
number of frames	28	155	482	224	47	14	5	1

References

[Au] M. Auslander, *Applications of morphisms determined by objects,* Proc. Conf. Represent. Theory, Philadelphia (1976), Lect. Notes Pure App. Math. 37 (1978), 245-327.

[AR] M. Auslander, I. Reiten, *Representation theory of Artin algebras III,* Commun. Algebra 3 (1975), 239-294.

[ARS] M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of Artin algebras,* Cambridge, 1995.

[BB] S. Brenner, M.C.R. Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors,* Lect. Notes Math. 832 (1980),103-169.

[BL] R. Bautista, F. Larrión, *Auslander-Reiten quivers for certain algebras of finite representation type,* J. London Math. Soc. 26 (1982), 43-52.

[BGRS] R. Bautista, P. Gabriel, A.V. Roiter, L. Salmerón, *Representation-finite algebras and multiplicative bases,* Invent. Math. 81 (1985), 217-285.

[Bo1] K. Bongartz, *Treue einfach zusammenhängende Algebren I,* Comment Math. Helv. 57 (1982), 282-330.

[Bo2] K. Bongartz, *Algebras and quadratic forms,* J. London Math. Soc. 28 (1983), 461-469.

[Bo3] K. Bongartz, *Critical simply connected algebras,* Manuscr. Math. 46 (1984), 117-136.

[Bo4] K. Bongartz, *A criterion for finite representation type,* Math Ann. 269 (1984), 1-12.

[BG] K. Bongartz, P. Gabriel: *Covering spaces in representation theory,* Invent. Math. 65 (1982), 331-378.

[Br] S. Brenner, *Decomposition properties of some small diagrams of modules,* Symposia Math. Inst. Naz. Alta Mat. 13 (1974), 127-141.

[BrG] O. Bretscher, P. Gabriel, *The standard form of a representation-finite algebra,* Bull. Soc. math. France 111 (1983), 21-40.

[C-B1] W. Crawley-Boevey, *On tame algebras and bocses,* Proc. London Math. Soc. (3) 56 (1988), 451-483.

[C-B2] W. Crawley-Boevey, *Matrix problems and Drozd's theorem,* in: Topics in Algebra, Banach Center Publ. 26, part I (1990), 199-222.

[DR] V. Dlab, C.M. Ringel, *Indecomposable representations of graphs and algebras,* Mem. Amer. Math. Soc. 173 (1973).

[DS1] P. Dowbor, A. Skowroński: *On the representation type of locally bounded categories,* Tsukuba J. Math. 10 (1986), 63-72.

[DS2] P. Dowbor, A. Skowroński, *Galois coverings of representation infinite algebras,* Comment. Math. Helv. 62 (1987), 311-337.

[Dr1] P. Dräxler, *Aufrichtige gerichtete Ausnahmealgebren,* Bayreuther Math. Schriften 29 (1989).

[Dr2] P. Dräxler, *Sur les algèbres exceptionalles de Bongartz,* C.R. Acad. Sci. Paris, t. 311, Série I, (1990), 495-498.

[Dr3] P. Dräxler, *Fasersummen über dünnen s-Startmoduln,* Arch. Math. 54 (1990), 252-257.

[Dr4] P. Dräxler, *Completely separating algebras,* J. Algebra 165 (1994), 550-565.

[Dr5] P. Dräxler, *On the density of fiber sum functors,* Math. Z. 216 (1994), 645-656.

[Dr6] P. Dräxler, *Combinatorial vector space categories,* Comm. Alg. 22 (1994), 5803-5815.

[DN1] P. Dräxler, R. Nörenberg, *Thin start modules and tame representation type,* in: Representations of Algebras, CMS Conference Proceedings 14 (1993), 149-163.

[DN2] P. Dräxler, R. Nörenberg, *CREP Manual - Version 1.0 using Maple as surface,* Preprint E 96-002 of the SFB 343 Bielefeld.

[Dd] Ju. A. Drozd: *Tame and wild matrix problems,* Lecture Notes in Math. 832 (1980), 242-258.

[Er] K. Erdmann, *Blocks of tame representation type and related topics,* Lecture Notes in Math. 1428 (1980).

[Ga1] P. Gabriel, *Unzerlegbare Darstellungen I,* Manuscr. Math. 6 (1972), 71-103.

[Ga2] P. Gabriel, *Finite representation type is open* Lecture Notes Math. 488 (1975), 132-155.

[Ga3] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras,* Lecture Notes in Math. 831 (1980), 1-71.

[Ga4] P. Gabriel, *The universal cover of a representation-finite algebra,* Lecture Notes in Math. 903 (1981), 68-105.

[G-V] P. Gabriel, L.A. Nazarova, A.V. Roiter, A.V, V.V.Sergejchuk, D. Vossieck, *Tame and wild subspace problems,* Ukr. Math. J. 45 (1993), 313-352.

[GP] P. Gabriel, J.A. de la Peña, *On algebras, wild and tame,* in: Duration and change, fifty years at Oberwolfach, Springer (1994), 177-210.

[GR] P. Gabriel, A.V. Roiter, *Representations of finite-dimensional algebras,* Encyclopedia of the Mathematical Sciences, Vol. 73, Algebra VIII, A.I. Kostrikin and I.V. Shafarevich (Eds.), Berlin, Heidelberg, New York, 1992.

[Ge] Ch. Geiß, *On degenerations of tame and wild algebras,* Arch. Math. 64 (1995), 11-16.

[Ha] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras,* LMS lecture notes series 119 (1988).

[HR] D. Happel, C.M. Ringel, *Tilted algebras* Trans. Amer. Math. Soc. 274 (1982), 399-443.

[HV] D. Happel, D. Vossieck, *Minimal algebras of infinite representation type with preprojective component,* Manuscr. Math. 42 (1983), 221-243.

[Kl] M.M. Kleiner, *Partially ordered sets of finite type,* J. Sov. Math. 23 (1975), 607-615.

[Mi] B. Mitchell, *Rings with several objects,* Advances in Math. 8 (1972), 1-161.

[Na] L.A. Nazarova, *The representations of partially ordered sets of infinite type,* Izv. Akad. Nauk SSSR, Ser. Math. 39 (1975), 963-991.

[NZ] L.A. Nazarova, A.G. Zavadskij, *Partially ordered sets of tame type,* in: Matrix Problems, Kiev (1977), 122-143 (in Russian).

[NS] R. Nörenberg, A. Skowroński, *Tame minimal non-polynomial growth strongly simply connected algebras,* Colloq. Math. 73 (1997), 301-330.

[Pe1] J.A. de la Peña, *Algebras with hypercritical Tits form,* in: Topics in Algebra, Banach Center Publ. 26, part I (1990), 353-369.

[Pe2] J.A. de la Peña, *The Tits form of a tame algebra,* in: Representation Theory of Algebras and related Topics, CMS Conf. Proc. 19 (1996), 159-183.

[PS] J.A. de la Peña, A. Skowroński, *Forbidden subcategories of non-polynomial growth tame simply connected algebras,* Can. J. Math. 48 (1995), 1018-1043.

[Ri1] C.M. Ringel, *Tame algebras,* in: Representation Theory I, Lecture Notes in Math. 831 (1980), 134-287.

[Ri2] C.M. Ringel, *Tame algebras and integral quadratic forms,* Lecture Notes in Math. 1099 (1984).

[Sk1] A. Skowroński, *Algebras of polynomial growth,* in: Topics in Algebra, Banach Center Publ. 26, Part I, (1990), 535-568.

[Sk2] A. Skowroński, *Simply connected algebras and Hochschild cohomologies,* in: Representations of Algebras, CMS Conf. Proc. 14 (1993), 431-447.

[Sk3] A. Skowroński, *Cycles in module categories*, in: Finite Dimensional Algebras and Related Topics, ed.: V. Dlab, L.L Scott, Dordrecht/Boston/London (1994), 309-345.

[Sk4] A. Skowroński, *Module categories over tame algebras*, in: Representation Theory of Algebras and related Topics, CMS Conf. Proc. 19 (1996), 281-313.

[Sk5] A. Skowroński, *Simply connected algebras of polynomial growth*, to appear in Compositio Math.

[Un] L. Unger, *The concealed algebras of the minimal wild hereditary algebras,* Bayreuther Math. Schr. 31 (1990), 145-154.