Plan for the reading course on ray categories

In representation theory it is important to know when a ray category X has the following properties:
a) the fundamental group $\pi_1(X)$ is free;
b) the cohomolgy group $H^2(X, G)$ vanishes (G an abelian group);
c) the universal cover $\tilde{X}$ is interval-finite.

Fischbacher proved a),b),c) in [Fi] for zigzag-finite X, simplifying and generalizing results of Bautista-Gabriel-Roiter-Salmerón in [BGRS]. Geiss observed in [Ge] that there was some room in Fischbacher's arguments and extended a),b) to weakly zigzag-finite X, thus including minimal representation-infinite X. Recently, Bongartz in [Bo2] proved a),b) *and* c) for minimal representation-infinite X by a different approach, namely by a detailed analysis of crowns in such an X and the transfer of knowledge from a suitable quotient to X itself.

In our reading course, we will try to cover [Fi] and the improvements in [Ge], section 14. Here is the list of topics for talks together with some comments and some suggestions regarding the presentation:

1) *Invariance of the fundamental group under reduction*

To get started, one may call *mesh* (alternative terminology is welcome!) of a ray category X a system of arrows α_i and paths u_i, u satisfying the conditions (1),(2),(3) (and (2')) in [Fi], 2.1. A *reducing ideal* of X is then any ideal generated by the initial arrows α_i of a mesh or the final arrows of a dually defined comesh. One shows that for such a reducing ideal I, the canonical homomorphism $\pi_1(X) \leftarrow \pi_1(X/I)$ is bijective ([Fi], 2.2).

2) *Tackles*

To prepare for the existence of meshes or comeshes under suitable finiteness conditions, one needs the concept of a tackle ([BGRS], 8.3; [Fi], 2.1; [Ge], 14.4) and a lemma on efficient tackles ([BGRS], 8.4; [Fi], 2.1, step a); [Ge], 14.4). The shifts in the various definitions of a tackle and the related notion of the range of an object should be discussed carefully.

3) *Reduction lemma*

This is [Fi], 2.1, steps b) – d) and [Ge], 14.5.

4) *Reducing filtrations, freeness of the fundamental group*

For a filtration of a ray category X, i.e., a decreasing sequence of ideals

$$I^0 \supseteq \ldots \supseteq I^p \supseteq I^{p+1} \supseteq \ldots \quad \text{such that} \quad \bigcap I^p = 0,$$

one has isomorphisms

$$X \xrightarrow{\sim} \lim X/I^p \quad , \quad \pi_1(X) \xleftarrow{\sim} \operatorname{colim} \pi_1(X/I^p) \quad \text{etc.}$$

If X is weakly zigzag-finite, one wants a *reducing filtration* that moreover satisfies:
— X/I^0 has no essential contour;
— I^p/I^{p+1} is a reducing ideal of X/I^{p+1}, $\forall p \geq 0$.
There is a technical problem in the construction that is explained and overcome in [Fi], 2.3; see also [Ge], 14.6; this should be discussed with more detail. In any case, the freeness of $\pi_1(X)$ is immediate provided X admits a reducing filtration ([Fi], 2.4, 3.1).

5) *Roiter's vanishing theorem*

The cohomology group $H^2(X, G)$ allows an alternative description as "G-valued contour functions modulo exact contour functions" ([BGRS], 8.2) which one could just state and accept. Then by an inductive argument given in [Fi], 3.2, $H^2(X, G)$ vanishes provided X admits a reducing filtration; see also [Ge], 14.7.

6) *Interval-finiteness: preparation*

The interval-finiteness of a zigzag-finite *simply connected* X rests on [Bo1], 2.3, which one should accept, or rather on two corollaries whose proofs should at least be sketched in order to avoid misunderstandings:
— simple connectedness of convex subcategories ([BrG], 2.8);
— separation criterion of Bautista-Larrión ([BrG], 2.9; [Bo1], 2.3c)).
Then [Fi], 4.2, 4.3, 4.4 should be proved completely.

7) *Interval-finiteness: conclusion*

The strategy can be explained as in [Fi], 5.1, and then [Fi], 5.2 should be proved completely.

References

[BGRS] Bautista, Gabriel, Roiter, Salmerón: Representation-finite algebras and multiplicative bases
[Bo1] Bongartz: A criterion for finite representation type
[Bo2] Bongartz: Indecomposables live in all smaller lengths
[BrG] Bretscher, Gabriel: The standard form of a representation-finite algebra
[Fi] Fischbacher: Zur Kombinatorik der Algebren mit endlich vielen Idealen
[Ge] Geiss: Darstellungsendliche Algebren und multiplikative Basen (Diplomarbeit)