ALGEBRA AND GEOMETRY
IN ELEMENTARY AND SECONDARY SCHOOL

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Abstracts

František Kuřina: Geometry in Secondary School
Geometry as a part of mathematics generally appears as a finished area of science, well organized with axioms, definitions, theorems and proofs — as to be seen in EUCLID's *Elements* or HILBERT's *Grundlagen der Geometrie*. As a subject of school–mathematics, however, geometry should be treated in its *nascend state* — as mathematics coming to life. In our workshop we present and comment an ample collection of problems serving that purpose. In particular we consider the relation between geometry and algebra where computations with numbers, vectors etc. provide a natural method for solving geometric problems with the intent to shift that way from "Mathematics for experts" to "Mathematics for all". The participants of the workshop will have the opportunity to work exemplarily in their own ways to get a feeling what is necessary for teaching geometry.

Christian Siebeneicher: Algebra in elementary school — In Search of a Lost Art
Commonly in mathematics–education the *mathematics* part of elementary school arithmetic seems to be a well understood and finished subject. For two particular (different) points of view see chapter 22 on *algebra* in RICHARD P. FEYNMAN, *The Feynman Lectures on Physics*, vol. 1, 1963, and LIPING MA, *Knowing and Teaching Elementary Mathematics*, 1999. In that situation I came to know of the 5th European Summer University (ESU 5) planned for summer 2007 in Prague. To find out what I as a mathematician could contribute to THE HISTORY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION I started last year a Google search for '*algebra*' and '*elementary school*'. Google provided more than *one million* items in 0,15 seconds and shows that intensive research in math–education is directed to the following subject–matters:

algebraic thinking	*algebraic skills*	*algebraic–symbolic notation*
algebraic concepts	*algebraic understanding*	*patterns and algebraic thinking*
algebraic reasoning	*algebraic problem solving*	*algebraic relations and notations.*

Correponding research papers suggest that these concepts are considered ready to be implemented into elementary school; my ESU 5 workshop is devoted to the question: Is the *mathematics* component of elementary arithmetic really as well understood as it seems?

Preliminary remark: The three hours of the workshop have been divided into two seemingly independent parts. The first two hours were directed to the question: *What is the mathematics component of elementary arithmetic*? The problems presented in the second part provide material to answer the question: *What is Mathematics for all*? The two parts are intimately interlinked with each other by a question of crucial importance for teaching mathematics: *What is learning Mathematics*?

Algebra in Elementary School

In Search of a Lost Art[1]

to the memory of
Karl Peter Grotemeyer

1 Leonhard Euler on Arithmetic and Algebra

What, then, actually constitutes the mathematics component of elementary arithmetic? Modern text–books on elementary arithmetic enrich the subject with todays common lingo — thereby making it difficult to identify what is what. Therefore I consider the *Einleitung zur Rechenkunst* (Introduction to the Art of Reckoning) of Leonhard Euler (1707–1783) which he had written 1738 for Russian schools. A reader of that book will notice at once that the symbols $+$ $-$ $\cdot$ and $\div$ (which today are considered as indispensable constituents of elementary school arithmetic and it's teaching[2]) are not present. The equal sign — which is taught to German elementary school kids within their first weeks in school — is missing, too, and one may wonder how in the time of Euler arithmetic could be done at all.

"*Lisez Euler*", Pierre–Simon Laplace recommended to his students,"*c'est notre maître à tous*"[3,4] and consequently we read what Leonhard Euler has to say in the preliminary report (Vorbericht) on elementary arithmetic:

> *Since learning the art of reckoning without some basis in reason is neither sufficient for treating all possible cases nor apt to sharpen the mind — as should be our special intent — so we have striven, in the present guide, to expound and explain the reasons for all rules and operations in such a way that even persons who are not yet skilled in thourough discussion can see and understand them; nonetheless, the rules and shortcuts appropriate to calculation were described in detail and extensively clarified by examples.*
>
> *By this device, we hope that young people, besides acquiring an adequate proficiency in calculation, will always be aware of the true reason behind every operation, and in this way gradually become accustomed to thorough reflection. For, when they thus not only grasp the rules, but also clearly see*

[1]by Christian Siebeneicher.

[2]*Specific for mathematical work* — declare guidelines for mathematics in German elementary schools — *is the use of particular symbols and chains of symbols. In elementary school, symbolism is mainly restricted to digits and chains of digits for numerals, the arithmetic operators* $+, -, \cdot, :$, *the signs for relations* $>, =, <$ *and to variables which are denoted by geometric figures* □, △, ○, ... *or letters* $a, b, x, \ldots$. *From the first year on children shall be accustomed to the use of variables — without making variables to a subject of discussion. [...] As early as possible computing has to be extended with respect to the following aspects: [...] the sign* $=$ *must not be interpreted only in the sense of* "yields"*, but increasingly also as a symbol for equal value on both sides; [...]* — Rund Erlass des Kultusministers vom 2.4.1985, Auszug aus dem Gemeinsamen Amtsblatt des Kultusministeriums und des Ministeriums für Wissenschaft und Forschung des Landes Nordrhein–Westfalen 5/85, S. 282, Grundschule — *Richtlinien und Lehrpläne, Mathematik*, Ausgabe 2003.

[3]Read Euler. He is the master of us all.

[4]and Carl Friedrich Gauss adds: *Studying Eulers works is the best school for the different parts of mathematics and cannot be replaced by anything else*, letter to P.H. von Fuss, September 16, 1849.

their basis and origin, they will in some measure be enabled to invent new rules of their own, and, by means of these, solve problems for which the ordinary rules are insufficient.

When working through the *Rechenkunst* one realizes that in this exceptionally clear and readable exposition of the subject Euler uses algebra right from the beginning. Hence it may come as a surprise to the modern reader that *algebraic–symbolic notation* is not present: instead of symbols Euler uses words of everyday language.

To understand the role of symbols it is helpful to compare the *Rechenkunst* with Euler's more advanced *Vollständige Anleitung zur Algebra* (Complete Initiation to Algebra) of 1769 — addressed to devotees of higher arithmetic. After a short introduction, Euler defines the signs $+$ $-$ and $\cdot$ (a sign for division is missing!) and these signs are used throughout as *tools* in computing. This means: with these symbols as shortcut for the operations of arithmetic, one is able to reckon not only with numbers, but also with sums of numbers, differences, products, powers and roots, then with sums of these, differences, products, powers and roots and so on. Such composed expressions — made up of the digits and the symbols of arithmetic (and also parantheses to fix the order of the operations) — do not change their value if one moves around and changes their constituent parts while respecting the *laws of arithmetic*. That way computations can be done *algebraically* and in numerical computations it is almost always advantageous *not* to figure out as fast as possible what could be figured out. An exercise will demonstrate what is meant: Determine $47 \cdot 47$ *algebraically*, i.e. not using the well known algorithm from elementary school.

In the first nineteen chapters of Euler's *Algebra* the equal sign is still not present. Only in number 206 of Chapter 20 — *Of the different Methods of Calculation, and of their mutual Connections* — it is introduced and in order to come to a deeper understanding of the relevance of *algebraic–symbolic notation* it is again advisable to read Euler:

Hitherto we have explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the raising into powers, and the extraction of roots.

It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connections among them in order to see whether or not other operations of the same kind are possible.

To this end we need a new character, which may replace the expression that has been so often repeated, "is as much as"*. This sign is = and it is read* "is equal to"*: thus, when I write $a = b$, this means that a is equal to b : so, for example,* $3 \times 5 = 15$.

Once that '*new character*' has been introduced to relate different forms of one and the same number it becomes an extraordinarily powerful tool in the hands of someone who in chapter 20 has already achieved a mastery[5] in reckoning which is unattainable

[5] To experience the genuine scope of $=$ employ its companion $\equiv$ in computing. Carl Friedrich Gauss introduced $\equiv$ on the first page of his *Disquisitiones Arithmeticæ* into arithmetic and commented, "*I choose that sign because of the great analogy which takes place between equality and congruence*". Exercise: With congruence in mind determine by use of a hand–held calculator first the recurring decimal of 1/17 and then that of 2/119. As a warm–up use the notion of equality to determine the number of hours in a year — algebraically!

for elementary school kids as well as for most elementary school teachers. A moment's reflection might suggest that = in elementary school is about as "*apt to sharpen the mind*" of young children ("*as should be our special intent* !"), as a razor blade in their hands is to foster fine–motoric coordination.

Since abstract algebraic–symbols are shortcuts of common speech used in counting and reckoning the question arises quite naturally for what reason the regime of math–education is so eager to obtrude upon young children abstract algebraic symbolism as early as possible. An answer will be useful for teaching elementary arithmetic.

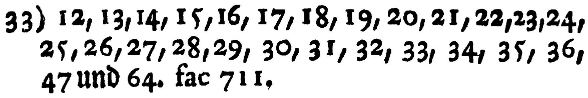
2 The purpose of computing is insight, not numbers

To get a somehow clearer conception of the role algebra can play for school children it is appropriate to take a model–computation which is easy enough to be done by a kid, as for example

Problem 33 : Compute $1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$

from the *Algebra* (p. 13) of Israel M. Gelfand and Alexander Shen.
The authors comment: *A legend says that as a schoolboy Karl Gauss (later a great German mathematician) shocked his school teacher by solving this problem instantly (as the teacher was planning to relax while the children were busy adding the hundred numbers).*

Since meanwhile the Gauss anecdote is common property I opened on page 64 the second edition of Christian Stephan Remer's *Arithmetica theoretico–practica* of 1737. There I spotted in the chapter on *addition* the 270 year old companion


33) 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24,
25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
47 und 64. fac 711.

of the modern problem 33. Carl Friedrich Gauss (1777–1855) had Remer's *Arithmetica*[6] at the age of eight — a treasure chest to entertain the mind of a child[7]; together with Leonhard Euler's *Einleitung zur Rechenkunst* it encompasses the legacy of *The Enlightenment* in elementary arithmetic — in everyday language and waiting by now another 270 years to be rediscovered for elementary school.

As todays school mathematics has not yet detected the emancipating character of mathematics[8], *expert knowledge* decrees what elementary arithmetic is about. That way

[6]See the articles by Ludwig Schlesinger and Philipp Maeennchen in Gauss's Werke, X_2, as well as Philipp Maennchen, *Methodik des mathematischen Unterrichts*. Schlesinger reports that Remer's book — which at his time was in the Gauss–Bibliothek in Göttingen — carried the inscription "Johann Friedrich Karl Gauss, Braunschweig, 16. December Anno 1785" and Maennchen complemented that Gauss wrote to the inside of the book–cover "Liebes Büchlein" (dear little book). In *Carl Friedrich Gauss und seine Welt der Bücher*, Göttingen, 1979, Martha Küssner states that Remer's *Arithmetica* is no more present in the Gauss–Bibliothek, and since Schlesingers reports that amongst the text there were computations by the hand of the young Gauss, the reader would like to know if in §§. 80–84 on pages 259–262 (*Beschreibung einer Geometrischen Progression*) of Remer's *Arithmetica* there are traces from Gauss's hand.

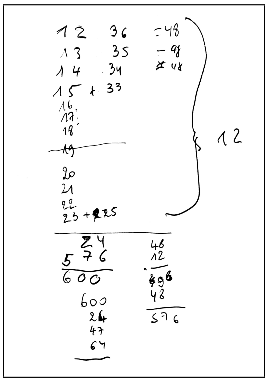
[7]There is another great book to entertain the mind of a child: According to Emil Fellman (*Leonhard Euler*, rororo, 1995, p. 11), in a short biographical note from 1767, Leonhard Euler tells: *... since my father was one of the students of the world–famous* Jacob Bernoulli *he strove betimes to teach me the fundamentals of mathematics. To this end, he used the* Coss *of* Christoph Rudolph, *with the annotations of* Michael Stiefel *which I studied with all diligence for several years.*

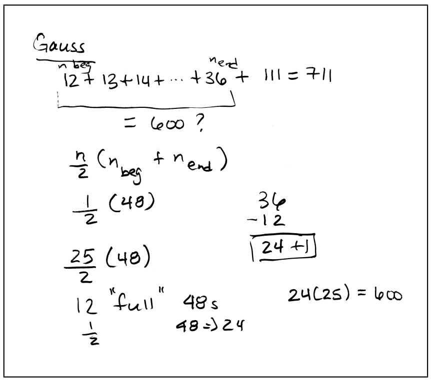
[8]*Das Wesen der Mathematik liegt gerade in ihrer Freiheit* — The essence of mathematics resides precisely in its freedom, Georg Cantor.

the subject is put in the straitjacket of dogmatism and a precious cultural heritage is constantly impeded to be handed over to the next generation of children.

3 The Workshop

To give an idea of how the "*freedom inherent in mathematics*" can inspire teaching of elementary arithmetic I put example 33 of Remer's Arithmetica to an overhead projector and asked the participants of my ESU 5 workshop to write down own answers to an empty transparency lying on a second projector. After some moments of hesitation we had the following two slides:





Since there were no further questions, the workshop could have finished when these two slides were completed. So I pointed to the left slide and asked: is it possible to sum the arithmetic progression even more *algebraically*?

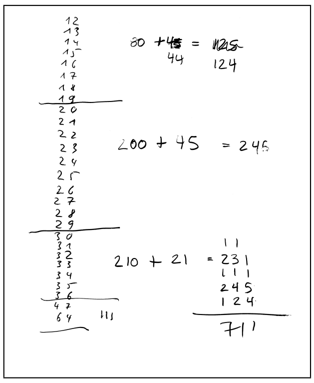
But what was obvious for me was not so obvious for my co–workers. Only when I suggested to consider the twelve 48's as part of the game, someone had the idea to decompose the lone 24 between the last entries 23 and 25 (in the two columns respectively) into the product $2 \cdot 12$. Then $48 \cdot 12$ and $2 \cdot 12$ fit together and distributivity provides $50 \cdot 12$ — easily calculated as 600.

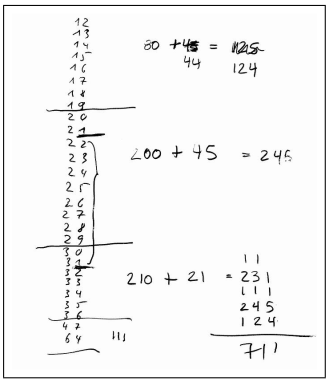
"Can one do even better?" I asked.

"Of course, multiply double of 50 with half of 12, hence 100 with 6: that pushes 6 two places to the left and no calculation is needed at all!"

Then I asked for a computation with an elementary school kid in mind, i.e. a computation on the base of common sense — without any prior knowledge in patterns, sum formulae and all that.

That was quickly done (left slide).



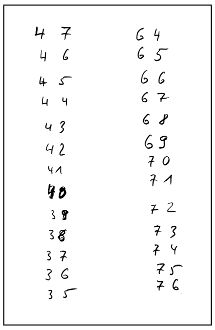


When the computation was finished I added two small marks under 21 and 31 and a brace bracketing these.

"Aha", was the prompt reaction, "now the digits 2,3,4,5,6,7,8,9,0,1 are somehow a shorter arithmetic progression, and that progression occurs even twice!"

All that is part of the well known game of algebra which some children already play by themselves intuitively[9] before schooling starts.

"What if now one made the two extra numbers 47 and 64 part of the same game", I proposed, "this time with constant sum 111?!" That lead to write down the further numbers shown to the right. Question:
"Is there something interesting in that pattern?"
Silence!
I insisted: "Maybe there is an interesting pair of numbers?!"
After some time of reflection one of my co–workers stated that the pair 37 74 is interesting: 74 is the double of 37!



So in the end the concept of arithmetic progression implicit in example 33 can be applied to the two extra numbers 47 and 64 as well — with the amazing consequence that $3 \times 37 = 111$ and demonstrating that the mental operations required in reckoning are quite different from those which a student of language employs in declining and conjugating his nouns and verbs.

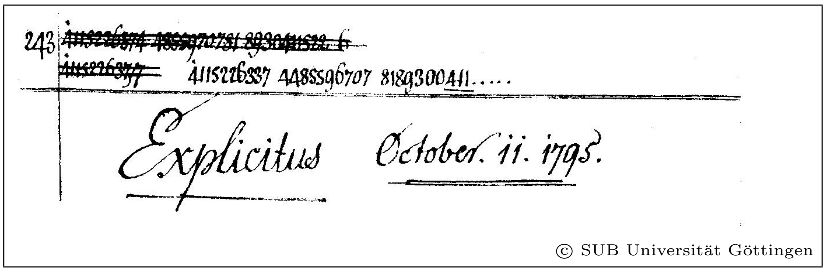
The equation $3 \times 37 = 111$ — coming from nowhere ! — tells us that the two numbers 3×37 and 111 are equal and using that fact one may deduce *algebraically* the number of hours in a year[10].

After these examples of algebraic reckoning[11] I came to the basis of reckoning:

[9] *Sapere Aude ! — Have courage to use your own understanding !*

[10] But of course there are other ways to determine that number — think for example of the binomial identity $(a + b)(a - b) = a^2 - b^2$; and still many further ways come into mind once one had started to *play* with the problem.

[11] The illustration below shows the recurring decimal for 1/243 as written down by Carl Friedrich Gauss. To determine the quotient digits by the ordinary pencil–and–paper method involves a certain amount of guesswork and ingenuity on the part of the person doing the division. Guesswork and ingenuity become unnecessary to a great extent if a table with the first nine multiples of 243 is present. By October 11, 1795 Gauss had finished a table which allows to read off the recurring decimal for every proper fraction with denominator a power of a prime number below 1000. According to G. Waldo Dunnington, in *Carl Friedrich Gauss, Titan of Science*, Gauss left Brunswick on October 11, 1795 to register on October 15, 1795 in Göttingen as a university student.

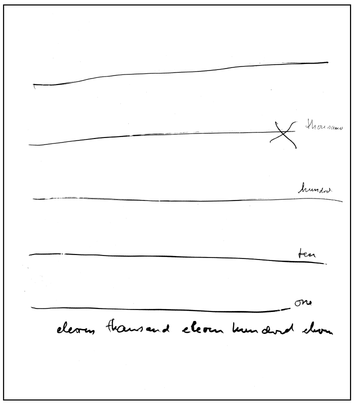


Gauss's table is the basis for a method he devised to divide by numbers which can be expressed as a product of primes and powers of primes below 1000: the result of such a division can be determined without long division to as many places as one wishes! The details are in section 6 of his *Disquisitiones Arithmeticæ*.

Since the result of long division does not depend on a factorization of the divisor, it is clear that any possible factorization must lead to one and the same recurring decimal.

Hence a question of general interest suggests itself: Is it possible to factorize a number in more than one way? Gauss answers that question within the first seven pages of his *Disquisitiones* by stating and proving the *Fundamental Theorem of Elementary Arithmetic*, with the notion of *congruence* as the numbers written with the ten digits 1234567890.

Since we are so accustomed to decimal reckoning with paper and pencil it is difficult to imagine that one can do without digits.



To show how this can be done I drew lines on a transparency, following thereby a suggestion of Adam Riese[12]: *Draw lines: the first and nethermost, means one, the other above, ten, the third, hundred, the fourth, thousand. Likewise, going further, the next line above, always ten times as much as the previous one thereunder.* Then I layed out[13] with cent pieces on the lines eleven thousand eleven hundred eleven from Chapter 1 of Euler's Rechenkunst — however, after two hours my time was over and the workshop on algebra in elementary school ended.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ *The End of my Workshop* ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

But since digits are at the heart of the art of reckoning I carry on and consider a conception from *new math* which Richard Feynman comments in *"Surely You're Joking, Mr. Feynman!"*[14] as follows:

principal tool. Since in his proof Gauss does not use Euclid's algorithm the remark at the end of § 14 (p 7) is of particular interest. According to Harold Davenport, *The Higher Arithmetic*, 1952, p. 19, this seems to be the first clear statement and proof of a fact, which is certainly not a '*law of thought*'.

Felix Klein in *Development of mathematics in the 19th century* reports in the section '*prehiestoric period*' that Gauss *calculates endlessly, with stunning diligence and indefatigable endurance* and that he determined *decimal fractions to unbelievably many places.* Klein did not mention that the latter are an essential ingredient of a new method Gauss invented for dividing by large numbers. At the age of eighteen he had added a new chapter to the apparently closed history of the four operations of arithmetic! Has this breakthrough in reckoning been overlooked?

Part of the conceptual framework of Section 6 has entered school mathematics: as expert knowledge in the form of professional sounding jargon — ready for teaching, ready for learning.

[12] Adam Riese, 1522, *Rechenbuch auff Linien und Ziphren* — reckoning on lines and with digits.

[13] There are many ways to lay out a given number with pennies on lines; if the same number had to be written with digits that liberty vanishes since, *eventually it has to be noticed, that at no time more than nine of a sort can be written since 10 pieces of a sort constitute one piece of the next sort and consequently belong there* (Leonhard Euler, *Einleitung zur Rechenkunst*, Chapter 1). Instead of applying that rule to digits it can, of course, also be used for pennies on lines — leading to a unique representation of the given number by pennies. But in contrast to digits, pennies on lines do not require the application of that rule, and the freedom to fiddle with pennies on lines — using thereby words of everyday language — provides a first hand intuitive understanding of the functionality of the decimal place value system and conveys right from start *meaning* to that game with numbers which is called elementary arithmetic.

Exercises: Write down with paper and pencil eleven thousand eleven hundred eleven. Lay out that number in at least two different ways with pennies on lines.

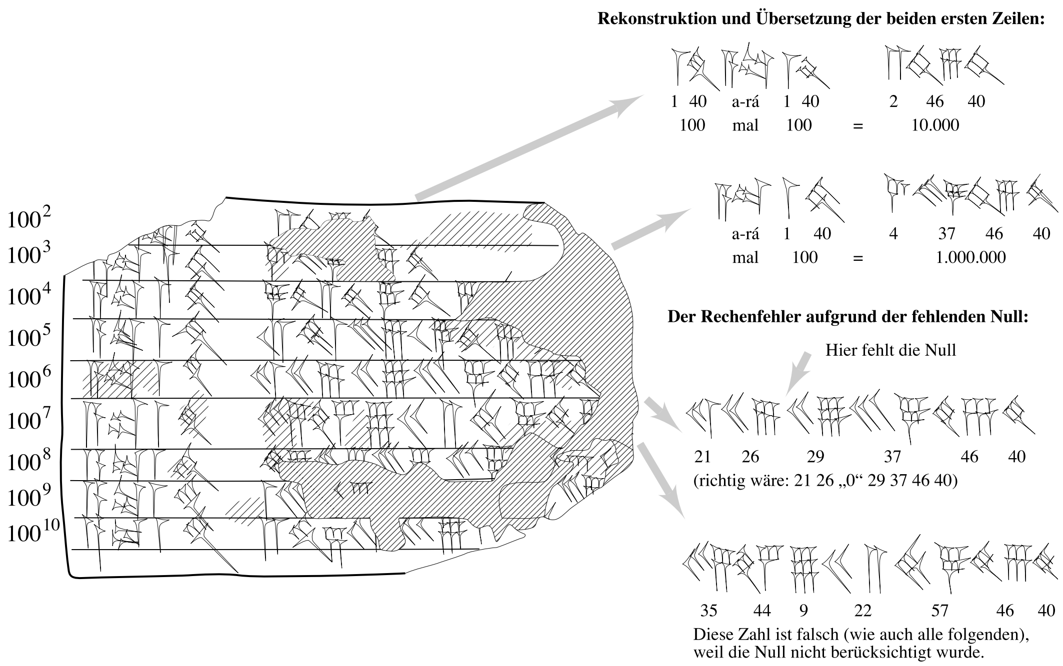
Remark: When pennies on lines are used for dividing *one* by *seven* it leaps to the eye that the succession of residues in the division process coincides with the sequence of powers of the first residue 3. Remer emphasizes this amazing fact on pages 260/261 of his *Arithmetica*, and it finds a detailed exposition, leaving no open question, in the third Section — *On Power Residues* — of the *Disquisitiones Arithmeticæ*.

[14] Vintage, 1985, p. 293.

They would talk about different bases of numbers — five, six, and so on — to show the possibilities. That would be interesting for a kid who could understand base ten — something to entertain his mind. But what they turned it into, in these books, was that every child had to learn another base! And then the usual horror would come: "Translate these numbers, which are written in base seven, to base five." Translating from one base to another is an utterly useless thing. If you can do it, maybe it's entertaining; if you can't do it, forget it. There's no point to it.

To entertain the mind of those who understand base ten we look at a Babylonian clay tablet dating from the end of the third millenium B.C.

Peter Damerow[15] detected on it a computation containing a phenomenon which is constitutive for reckoning with digits in a place value system. His legend is worth to be read — word by word, sign by sign:



The tablet shows the powers of Babylonian 1 40 — one hour and fourty minutes (or one hundred minutes in decimal language) denoted in the Babylonian (base 60) number system[16]. An arrow points to the location of the relevant phenomenon: It is the blank between Babylonian 26 and 29 in the row of 100^6.

At the time when the tablet was prepared people had nothing to denote "*nothing*" on an abacus by "*something*" to be impressed into the soft clay; hence that place remained untouched. According to Otto Neugebauer[17], not until a millenium later an imprint occured on Babylonian clay tablets corresponding to nothing on the abacus[18].

[15] H.J. Nissen, P. Damerow and R.K. Englund, *Frühe Schrift und Techniken der Wirtschaftsverwaltung im alten Orient — Informationsspeicherung und –verarbeitung vor 5000 Jahren*, Franzbecker, 1991, p. 195.

[16] Exercise: To experience the feelings of a kid unsure with reckoning in base ten compute some of the powers of Babylonian 1 40.

[17] *The exact sciences in antiquity*, Princeton University Press, 1952.

[18] For more on base 60 see Donald E. Knuth, *The Art of Computer Programming*, Vol. 2, third

Donald E. Knuth, in his *Art of Computer Programming*, also comments the use[19] of an abacus:

> *Since handwriting was not always a common skill, and since abacus users need not memorize addition and multiplication tables* — making really easy to reckon with an abacus — people at that time probably felt *it would be silly even to suggest that computing could be done better on "scratch paper"*.

Hence Romans were able to reckon without having first to memorize tables[20] — and, moreover, using their numerals to write down numbers they not even needed something to denote the empty place on the abacus.

It is worthwhile to review the time when both abacus and written decimal numbers were in use for reckoning.

The German *Rechenmeister* Adam Riese (1492–1559) lived at that time of transition, and in his first *Rechenbuch* he gives instructions for reckoning with pennies on lines and with written decimal numbers as well. He starts with the arithmetic operations on an abacus with lines and only after that he turns to the operations with written decimal numbers — not without reason. In his more detailed *Rechnung nach der lenge auff den Linihen vnd Feder* from 1550 he explains why: *When teaching arithmetic to young children, always those who started with the lines of an abacus came to a better understanding than those who started straight away with written decimal numbers. With the lines they became current and fluent in counting and reckoning and after that had been accomplished they had no trouble to switch to arithmetic with written numbers.*

In todays *formalistic*[21,22] conception of school mathematics also that lesson from history did not enter elementary school — unfortunately[23].

If my ESU 5 contribution can draw attention to the *mathematics* component of elementary school arithmetic and thereby open school mathematics[24] both to *The Enlightenment's* sapere aude and to Georg Cantor's famous motto, future generations of school children might benefit[25].

4 Final remark

This is not all what a deeper insight into the *mathematics* component of elementary arithmetic has to offer — in particular to those who want to teach mathematics to children. In the chapter on *algebra* in his *Lectures on Physics*, (Addison–Wesley,

edition, Addison–Wesley, 1998, p. 196.

[19]In my paraphrase of the original text on p. 196/197 I use roman typefaces.

[20]Memorizing as the basis of German elementary school arithmetic starts, when during the first weeks in school, work–sheets arrive in the classroom: i.e. forms which the kids have to fill in with the silent understanding: *The quicker the better*.

[21]*Specific for mathematical work is the use of particular symbols and chains of symbols . . . etc. etc.* Guidelines for mathematics, Nordrhein–Westfalen.

[22]*Non ex notationibus sed ex notionibus*, Carl Friedrich Gauss. *Elementa doctrinæ Residuorum*, 71, manuscript, Berlin–Brandenburgische Akademie der Wissenschaften, Nachlass Dirichlet.

[23]For '*unfortunately*' see Hans Freudenthal, *Didactical Phenomenology of Mathematical Structures*, p. 92.

[24]*The principal obstacle against the progress of science is the belief to know already what is not yet known*, Georg Christoph Lichtenberg (1742–1799).

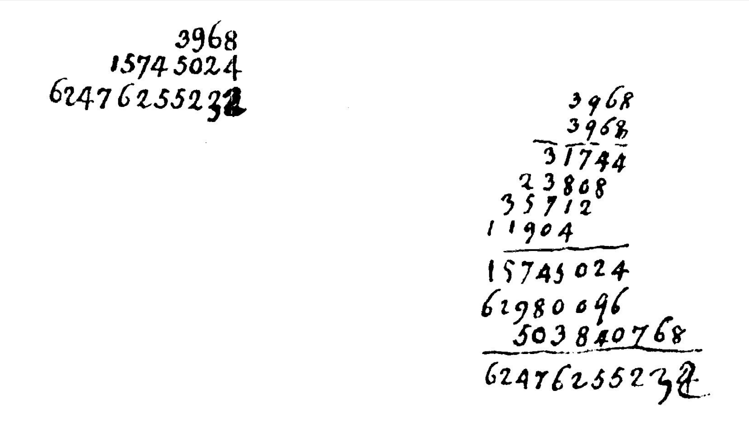
[25]To facilitate teenagers reading by themselves the books mentioned in my ESU 5 contribution I put PDF versions of these to my *web–page* http://www.math.uni–bielefeld.de/˜sieben/Rechnen.html.

1963, p. 22–1) Richard P. Feynman asks, *“What is mathematics doing in a physics lecture?”* and answers:

> *We have several possible excuses: first, of course, mathematics is an important tool, but that would only excuse us for giving the formula [the most remarkable, almost astounding, formulas in all of mathematics] in two minutes. On the other hand, in theoretical physics we discover that all our laws can be written in mathematical form; and that this has a certain simplicity and beauty about it. So, ultimately, in order to understand nature it may be necessary to have a deeper understanding of mathematical relationships. But the real reason is that the subject is enjoyable, and although we humans cut nature up in different ways, and we have different courses in different departments, such compartmentalization is really artificial, and we should take our intellectual pleasures where we find them.*

A final challenge:

Find the *algebra* in the pattern of digits[26] which Carl Friedrich Gauss composed[27] more than two centuries ago!



Truly, it is not knowing but learning[28]*,*
not possessing but acquiring,
not being there but getting there,
which yields the greatest enjoyment.

Carl Friedrich Gauss

Letter to Wolfgang Bolyai, September 2, 1808

[26]or with other words: Make sense of the pattern! Provide meaning to it!

[27]detected in Christian Leiste, *Die Arithmetik und Algebra zum Gebrauch bey dem Unterrichte*, Wolfenbüttel, 1790, handwritten addendum by Carl Friedrich Gauss, copy of the Gauss Bibliothek in Göttingen.

[28]*Please forget what you have learned in school; you haven’t learned it*! Edmund Landau, 1929, *Foundations of Analysis — Preface for the Beginner.*

Geometry in Secondary School[1]

1 Historical problems

1.1 *Heron*: Find the formula for the area of the triangle with given sides a, b, c.

Heron's original solution is published in MORITZ CANTOR'S *Vorlesungen über die Geschichte der Mathematik* (1894); we arrive at the known solution by means of trigonometry and algebra.

1.2 *Apollonius* (I): Find the set of all points in the plane which have given ratio of distances from two given points.

1.3 *Apollonius* (II): Construct all circles which are tangent with three given circles.

The natural solution of problems 1.2 and 1.3 is analytical. We translate the conditions of the problems into the language of algebra and calculate.

1.4 *Euclid*: Prove: From all rectangles with given perimeter the square has maximal area.

The historical solution is known from EUCLID's *Elements*, today it is possible to solve the problem by means of differential calculus.

2 School problems

2.1 Prove: If the straight lines AX, BX are perpendicular, then X is a point on the circle with diameter AB (Thales).

2.2 Prove: The altitudes of a triangle meet in a point (Gauss).

2.3 Prove: The medians of a triangle meet in a point which is two–thirds of the distances from any vertex to the midpoint of the opposite side.

3 Problems for participants

3.1 Given three non–collinear points A, B, C. Construct the circles with centres in these points such that every two of these circles are externally tangent.

3.2 Prove: If two rectangles have equal areas and equal perimeters, then they are congruent.

3.3 Prove: The bisector of the angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

3.4 ABC is an isosceles triangle with the base AC. Find a point X on the side AB and a point Y on the side BC such that $|AX| = |XY| = |YC|$.

3.5 $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are parallelograms in space, A, B, C, D are centres of segments $A_1A_2, B_1B_2, C_1C_2, D_1D_2$. Prove that $ABCD$ is a parallelogram.

3.6 $KABH, BHGC, CGEF$ are squares. Find the sum of angles ABK, ACK, AFK.

3.7 Find the area of the regular dodecagon inscribed in the circle with radius r.

3.8 AB is a segment with centre S, k is the circle with diameter AB, m, n are circles with diameters AS, SB. Construct the circle which is tangent with k, m, n.

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3.9 In circle k with center S, AB and CD are mutually perpendicular diameters, n is the circle with diameter CS. Find all circles which are tangent with k, n and AB.

3.10 Find all right–angled triangles ABC with hypotenuse AB, midpoints M, N of sides AC, BC and centroid T with this propery: quadrilateral $MTNC$ is circumscribed.

3.11 Prove: If a, b, c, d are sides, e, f diagonales of the inscribed quadrilateral, then $ac + bd = ef$ (Ptolemaios).

3.12 In the triangle ABC are CO median, CP altitude. Prove: If CP is part of the angle ACB and the angles ACO, BCP are congruent, then ABC is right angled triangle with hypotenuse AB.

Geometry–teachers are invited to work through carefully as many of these problems as possible. Geometry–students will appreciate the effort of their teachers.