Faculty of Mathematics,
Bielefeld University

Bielefeld,
April 2015

Optimization and Dynamics

Summer semester 2015

Exercise sheet 0

Due 12pm, 10.04.2015

1. (a) Define what it means for a sequence $(a_n)_{n\geq 1}$ to converge.

   (b) For each of the following sequences $(a_n)_{n\geq 1}$, either find the limit or briefly state why the limit does not exist.

$$a_n := \frac{(-1)^n}{n^2} \quad ; \quad a_n := \frac{n(1-3n)}{n^2-5} \quad ; \quad a_n := \frac{n}{\ln(n+1)}$$

2. Which of the following functions are linear?

   (a) $f : \mathbb{R} \to \mathbb{R};\ f(x) := 17x - \pi$

   (b) $f : \mathbb{R}^2 \to \mathbb{R};\ f(x,y) := xy$

   (c) $f : \mathbb{R} \to \mathbb{R};\ f(x) := e^x$

3. Consider the matrices

$$A := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

   (a) Show that A and B do not commute.

   (b) Does A have an inverse? If so, find A^{-1}.

4. Let

$$A := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

   and let

$$x_0 := A\,; \quad x_n := A^{n+1} \text{ for } n \in \mathbb{N}\,.$$

   Find A^2, A^3 and A^4 and hence express x_n in the form $x_n = f(n)\,x_0$ for $n \in \mathbb{N}$.

5. Find the eigenvalues and eigenvectors of the matrix

$$M := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Write down the diagonalisation of M and hence (or otherwise) find M^n for $n \in \mathbb{N}$.

6. Express the following complex numbers in the form $x + iy$.

   (a) $\frac{i}{1+i}$

   (b) $\left(\frac{i}{1+i}\right)^{15}$ (*Hint: Use polar coordinates.*)

7. Solve the differential equations

   (a) $x' = (1 + e^t)x^2$,

   (b) $x' - x \cos t = \cos t$.

8. Solve the initial value problem

$$\dot{x} = t \tan x, \qquad x(0) = \frac{\pi}{6}.$$

9. Consider the function $f(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$. Show that $x = 0$ is a fixed point of the Newton-Raphson iteration and that for any starting value $x_0 \neq 0$, the Newton-Raphson method will fail to converge to the root $x = 0$.

10. Consider the function $f(x) = x^3 - 2x + 2$. Show that f has at least one real root. Then, taking as initial value $x_0 = 0$, apply the Newton-Raphson method. What happens?