Faculty of Mathematics, Bielefeld University — Bielefeld, April 2015

Optimization and Dynamics

Summer semester 2015

Exercise sheet 1

Due 12pm, 17.04.2015

1. Consider the difference equation

$$x_{n+1} = -x_n\,,\; n \in \mathbb{N}; \quad x_0 \in \mathbb{R}\,.$$

Find an expression for x_n in terms of x_0. What are the fixed points? Are there periodic points?

2. Consider the difference equation

$$x_{n+1} = 4x_n(1 - x_n).$$

Let $x_0 = (\sin\phi_0)^2$. Describe the sequence $(x_0, x_1, x_2, x_3, \ldots)$ (ie, the *orbit* of x_0) for each of the initial values $\phi_0 = \frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{\pi}{5}$.
Hint: Use the identity $(\sin(\theta))^2 = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$.

3. Consider the difference equation

$$x_{n+1} = ax_n(1 - x_n)\,.$$

   (a) Let $0 < a < 1$.
      (i) Describe the behaviour of the system for intial states $x_0 = 0$ and $x_0 = 1$.
      (ii) Show that for $0 < x_0 < 1$, $(x_n)_{n=0}^{\infty}$ is a strictly monotone decreasing sequence that is bounded below, and hence that the sequence converges to 0.
   (b) Now let $a \geq 1$.
      (i) Show that if $x_0 < 0$ then $x_n < 0$ for all $n \in \mathbb{N}$ and hence that the sequence is strictly monotone decreasing. Does it converge?
      (ii) What happens if $x_0 > 1$?

4. Solve the differential equation

$$x'(t) = (x(t))^2$$

for the initial value $x(0) = x_0 > 0$ in the time interval $0 \leq t < \frac{1}{x_0}$. Sketch the graph. What happens at $t = \frac{1}{x_0}$? Does the same problem arise if x_0 is negative and $t \geq 0$?