Optimization and Dynamics
Summer semester 2015

Exercise sheet 14

Solutions will be discussed in the tutorial on 15.07.15.

1. Find a Lyapunov function for the system

$$\begin{cases} x' &= -2xy - 2y^2 \\ y' &= x^2 - y^3 + xy \end{cases}$$

   and hence show that the origin is a stable fixed point of the system. Is it asymptotically stable?

2. Find a Lyapunov function for the system

$$\begin{cases} x' &= -3x^3 - y \\ y' &= x^5 - 2y^3 \end{cases}$$

   and hence show that the origin is an asymptotically stable fixed point.

   Hint: Look for a function of the form $E(x, y) = Ax^p + By^q$, for $A, B > 0$ and $p, q \in \mathbb{N}$ even.

3. Consider the system $x' = f(x)$ in $\mathbb{R}^d$, where f is a Lipschitz function on $\mathbb{R}^d$ such that $f(0) = 0$ and

$$\sum_{k=1}^{d} x_k f_k(x) < 0 \qquad \text{if } x \neq 0 \ .$$

   Prove that $x(t) \to 0$ when $t \to \infty$ for all solutions $x(t)$ of the system, independently of the starting value $x(0)$.

   Hint: Prove that $E(x) = \|x\|^2 = x_1^2 + \ldots + x_n^2$ is a Lyapunov function for the system.