Faculty of Mathematics,
Bielefeld University

Bielefeld,
April 2015

Optimization and Dynamics

Summer semester 2015

Exercise sheet 3

Due 12pm, 04.05.2015[1]

1. Let $X \subseteq \mathbb{R}$ and define $f : X \to \mathbb{R}$ by $f(x) := \frac{1}{4}(x^2 + 3)$.
   (a) Take $X = \left[0, \frac{3}{2}\right]$. Show that $f(X) \subseteq X$ and that in this case $f : X \to X$ is a contraction.
   (b) Hence, or otherwise, find the fixed point(s) of f in $\left[0, \frac{3}{2}\right]$ and describe their properties.
   (c) Now take $X = \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is not a contraction. Does f have any fixed points in $\mathbb{R} \setminus \left[0, \frac{3}{2}\right]$? Do they have the same properties as those of (b)?

2. Let $f : I \to I$ be a continuous and strictly increasing function on the closed and bounded interval $I \subset \mathbb{R}$ and consider the time discrete system given by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}_0$.
   (a) Prove that there are no eventually periodic points in I.
   (b) Prove that any orbit under f is either constant (that is we have a fixed point) or strictly monotonous.
   (c) Prove that any non-constant orbit converges to a fixed point.
   (d) Does the fixed point from (c) have to be unique, or may different orbits converge to different fixed points?

3. Consider the dynamical system defined by $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} 1 & \text{for } \frac{1}{4} < x < 1\,, \\ \frac{1}{2} & \text{for } x = \frac{1}{4}, 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   (a) Sketch the graph and the phase portrait.
   (b) Find the fixed points of f and describe their properties.
   (c) Find the periodic orbits of f and describe their properties.

[1] Note later due date as Friday 1st May is a public holiday.