Faculty of Mathematics,
Bielefeld University

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Optimization and Dynamics

Summer semester 2015

Exercise sheet 4

Due 12pm, 08.05.2015

1. Consider the discrete dynamical systems defined by the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = ax + x^2 \,,$$

for $a \in \mathbb{R}$. Find the fixed points and discuss how their properties depend on the value of a.

2. (a) Show that $x = 0$ is an attracting fixed point of the dynamical system defined by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x^2)$.

   (b) Consider $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos x$. By plotting $y = g(x)$ and $y = x$ on the same set of axes, show that g has one fixed point, $\bar{x} \in (0, \frac{\pi}{2})$. Without attempting to calulate the value of $\bar{x}$, prove that it is an attracting fixed point.

   (c) Show that for all $b \in (0, 1)$, $g(x) = \cos(bx)$ has an attracting fixed point in $(0, \frac{\pi}{2})$. What can happen for $b > 1$?

   (d) Now consider $g(x) = \cos(\pi x)$. Show that g has an attracting fixed point.

3. Consider the dynamical system defined by $x_{n+1} = a\,x_n + b$. Use the principal of mathematical induction to show that for all $n \in \mathbb{N}$,

$$x_n = \begin{cases} x_0 + nb \,, & \text{if } a = 1 \\ a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} \,, & \text{if } a \neq 1 \,. \end{cases}$$