Faculty of Mathematics, Bielefeld University — Bielefeld, May 2015

Optimization and Dynamics

Summer semester 2015

Exercise sheet 5

Due 12pm, 15.05.2015

1. Consider the dynamical systems defined by the following functions. In each case, show that $x = 0$ is a fixed point of the dynamical system and discuss its stability properties.
   (a) $f(x) = -x - x^3$
   (b) $f(x) = -x + x^3$
   (c) $f(x) = -x + x^2$
   (d) $f(x) = -x - x^2$

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := -x^2 + 5x - 4$.
   (a) Find the unique fixed point, $\bar{x}$, of f.
   (b) Show that for all $x \in B_1(\bar{x}) = (\bar{x} - 1, \bar{x} + 1)$, the inequality
   $$|f(x)| \leq |x|$$
   holds, with equality if and only if $x = \bar{x}$.
   (c) Show that $x = \bar{x}$ is neither an attracting nor a repelling fixed point.
   (d) Explain the difference between this situation and that of Proposition 3.8.

3. Consider the function $f : [0, 1] \to [0, 1]$ defined by
   $$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2^{k+1}} & \text{for } \frac{1}{2^k} < x \leq \frac{1}{2^{k-1}},\ k \in \mathbb{N}, \\ 0 & \text{for } x = 0\,. \end{cases}$$
   (a) Show that for $k \in \mathbb{N}$, $x \in \left(\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]$,
   $$\lim_{n\to\infty} f^n(x) = \frac{1}{2^k}\,.$$
   Hint: Problem 3 from Exercise sheet 4 may be useful!
   (b) Show that the fixed point $x = 0$ of f is stable.

4. Consider the dynamical system given by $f : \mathbb{R} \to \mathbb{R}$, $f(x) := 2|x| - 1$. Show that for every $m \in \mathbb{N}$, there exists a periodic point of f with minimal period m.