Faculty of Mathematics,
Bielefeld University

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Optimization and Dynamics

Summer semester 2015

Exercise sheet 7

Due 12pm, 29.05.2015

1. Consider the following family of dynamical systems
$$f(x) = ax + x^3.$$
Discuss the bifurcation that occurs at $a = 1$ and sketch the corresponding diagram.

2. Consider the family of dynamical systems defined by
$$f_a(x) = x - x(x^2 - a)(x^2 - 4a)\,,$$
as discussed in Example 4.15. Show that for $a = 0$, the fixed point $x = 0$ is attracting and stable.

3. Consider the family of dynamical systems defined by
$$f_a(x) = x - (x^2 - a)(x^2 - 4a)\,,$$
as discussed in Example 4.14.
   (a) Show that for $a = 0$, the fixed point $x = 0$ is neither attracting nor repelling and is hence unstable.
   (b) For which values of a do you expect another bifurcation?

4. Let A be a real 2×2 matrix with a complex eigenvalue λ and corresponding eigenvalue v. Set $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ and $v = x + iy$, $x, y \in \mathbb{R}^2$.
   (a) Show $Ax = \alpha x - \beta y$ and $Ay = \alpha y + \beta x$.
   (b) Hence show that $\bar{v} = x - iy$ is an eigenvector of A corresponding to the eigenvalue $\bar{\lambda}$.
   (c) Prove that x and y are linearly independent.
   (d) Let S be the matrix whose columns are the vectors x and y, that is, $S = (x, y)$. Show that
$$S^{-1}AS = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$
Hint: Note that $S^{-1}x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $S^{-1}y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.