Optimization and Dynamics

Summer semester 2015

Exercise sheet 8

Due 12pm, 05.06.2015

1. Let $\lambda \in \mathbb{R}$ and consider the Jordan block $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

   (a) Prove that $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

   (b) Under what condition(s) is J invertible? In this case, calculate J^{-1} and J^{-n}.

   (c) Show that if $\lambda > 1$ then $\lim_{n\to\infty} ||J^n x|| = \infty$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

   (d) Show that if $\lambda < 1$ then $\lim_{n\to\infty} ||J^n x|| = 0$ for all $x \in \mathbb{R}^2$.

2. Consider the two dimensional linear dynamical system $x_{n+1} = Ax_n$ given by the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

   (a) Find the eigenvalues of A. Does the system have any fixed points other than $x = 0$?

   (b) By writing A in the form $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, determine the periodic points of the system.

   (c) Hence or otherwise, show that $x = 0$ is a stable but not attracting fixed point of the system.

3. Consider the two dimensional linear dynamical system $x_{n+1} = Ax_n$ given by the matrix

$$A = \begin{pmatrix} 4 & -3 \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix}.$$

   Find the eigenvalues and eigenvectors of A and hence the stable and unstable subspaces of the dynamical system.