Proposition 2.28. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with an attracting fixed point $\bar{x}$. Then $\bar{x}$ is a stable fixed point of the system $x_{n+1} = f(x_n)$.*

Proof. Let $I := (a, b)$ be the maximal interval containing $\bar{x}$ such that $\lim_{k\to\infty} f^k(x) = \bar{x}$ for all $x \in I$ (note that it is possible that $a = -\infty$ and/or $b = \infty$). Then I can contain no other fixed point and it must be true that for any $x \in I \setminus \{\bar{x}\}$, exactly one of the inequalities

$$f(x) < x \qquad \text{or} \qquad f(x) > x$$

holds.

Let $J := f(I)$, which is an interval by the continuity of f, and must contain $\bar{x}$, as $\bar{x}$ is a fixed point. That is, $J \cap I \neq \emptyset$, so the intervals overlap. But for all $x \in J$, $f^n(x) \to \bar{x}$, so by the maximality of I, $J \subseteq I$. This shows that the orbit of each point in I is contained in I.

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f(x) - x$. Again by the continuity of f, g is continuous, and

$$g(x) = 0 \;\Leftrightarrow\; f(x) = x\,,$$

so g has only one root in the interval I, namely $\bar{x}$. Thus g has the same sign on the interval $I_r := (\bar{x}, b)$ (and on the interval $I_\ell := (a, \bar{x})$).

Suppose now that $g > 0$ on I_r, that is, that $f(x) > x$ for all $x \in I_r$ and let $x \in I_r$. We have $f^{n+1}(x) = f(f^n(x)) > f^n(x) > x$ for all $n \in \mathbb{N}_0$, so $(f^n(x))$ is an increasing sequence in I_r and thus cannot have limit $\bar{x}$. This is a contradiction, so we must have that $g < 0$ on I_r, that is, that $f(x) < x$ for all $x \in I_r$. A similar argument shows that $f(x) > x$ for all $x \in I_\ell$.

Similarly, for each $n \in \mathbb{N}$ we may define $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = f^n(x) - x$ and, observing that there can be no periodic points in I (other than the fixed point $\bar{x}$), exactly the same argument shows that

$$f^n(x) < x \quad \text{for all } \; x \in I_r\,, n \in \mathbb{N}\,; \tag{1}$$

$$f^n(x) > x \quad \text{for all } \; x \in I_\ell\,, n \in \mathbb{N}\,. \tag{2}$$

Let $\varepsilon > 0$. As f is continuous, we may choose $\delta > 0$ such that

$$|f(x) - \bar{x}| < \varepsilon \qquad \text{whenever} \qquad |x - \bar{x}| < \delta\,; \tag{3}$$

in particular, we may choose δ such that $\delta \leq \varepsilon$ (and such that $B_\delta(\bar{x}) \subseteq (a, b)$). Then let $x \in I_r$ such that $|x - \bar{x}| < \delta$.

The sequence

$$x > f(x) > f^2(x) > \ldots$$

is monotone decreasing as long as $f^i(x) \geq \bar{x}$. If $f^i(x) \geq \bar{x}$ for all i we have that $x \searrow \bar{x}$ and we are done. Assume therefore that $f^i(x) < \bar{x}$ for some i and let j be the smallest value such that

$$f^{j+1}(x) < \bar{x} < f^j(x) < f^{j-1}(x) < \ldots < x\,.$$

Then for all $n = 1, 2, \ldots, j$, we have

$$|f^n(x) - \bar{x}| < |x - \bar{x}| < \delta \leq \varepsilon\,.$$

In particular, $|f^j(x) - \bar{x}| < \delta$, and then by (3),

$$|f^{j+1}(x) - \bar{x}| < \varepsilon\,.$$

As $f^j(x) \in I_r$ and $f^{j+1}(x) \in I_\ell$, we must have (from (1) and (2)) that

$$f^{j+1}(x) < f^{j+1+k}(x) < f^j(x) < x$$

for all $k \in \mathbb{N}$. Then as both $f^{j+1}(x), f^j(x) \in B_\varepsilon(\bar{x})$, we have for all $k \in \mathbb{N}$ that

$$|f^{j+1+k}(x) - \bar{x}| < \varepsilon\,.$$

Thus, for all $n \in \mathbb{N}$, $|f^n(x) - \bar{x}| < \varepsilon$ and this holds for all $x \in I_r$ such that $|x - \bar{x}| < \delta$. Similarly, this holds for all $x \in I_\ell$ such that $|x - \bar{x}| < \delta$, and thus for all $x \in \mathbb{R}$ such that $|x - \bar{x}| < \delta$, and we see that $\bar{x}$ is an attracting fixed point. □

Example 2.33. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined, in polar co-ordinates, by

$$f(r,t) := (\sqrt{r}, \sqrt{2\pi t})\,,\ r > 0\,, t \in [0, 2\pi)\,.$$

(Recall that polar co-ordinates are related to the usual cartesian co-ordinates by $x = r\cos t$, $y = r\sin t$. In this case, the polar form of f is much simpler than the cartesian form, and easier to analyse.)

The function f is continuous and has fixed points at $(r,\theta) = (0,0)$ and $(r,\theta) = (1,0)$. For an initial point (r_0, t_0), we have

$$r_1 = r_0^{\frac{1}{2}}\,,\ r_2 = \left(r_0^{\frac{1}{2}}\right)^{\frac{1}{2}} = r_0^{\frac{1}{4}}\,, \ldots\,,\ r_n = r_0^{\frac{1}{2^n}}$$

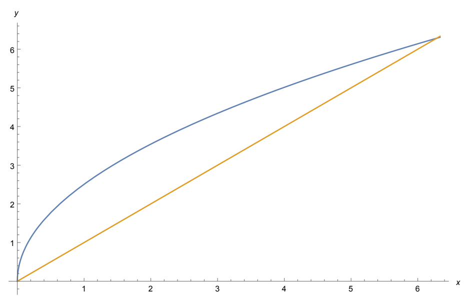
and

$$t_1 = (2\pi t)^{\frac{1}{2}}\,,\ t_2 = \left(2\pi(2\pi t)^{\frac{1}{2}}\right)^{\frac{1}{2}} = (2\pi)^{\frac{3}{4}} t^{\frac{1}{4}}\,, \ldots\,,\ t_n = (2\pi)^{1-\frac{1}{2^n}} t^{\frac{1}{2^n}}\,.$$

Thus as $n \to \infty$ we have $r_n \to 1$ and $t_n \to 2\pi$, and we see that the orbit of every (r,t) converges to $(1,0)$. Thus the fixed point $(1,0)$ is attracting.

In fact, this fixed point is also unstable. If we take any point in the first quadrant of the plane (that is, with $0 < t < \frac{\pi}{2}$), the orbit of the point takes an anticlockwise path around the origin before approaching $(1,0)$ from the fourth quadrant. We can see this by examining the behaviour of $f_1 : \mathbb{R} \to \mathbb{R}$, $f(t) = \sqrt{2\pi t}$ on the interval $(0, 2\pi)$. For all $x \in (0, 2\pi)$, f_1 is increasing, with $f_1(x) > x$ and $0 < f_1(x) < 2\pi$.

The graph below plots $y = f_1(x)$ and $y = x$.



The picture below shows the trajectories of the points

$$(r, t) = (1.1, 0.01), (1.01, 0.02), (0.75, 0.03), (0.5, 0.22), (0.45, \tfrac{\pi}{5})\,.$$