Attractors for ergodic and monotone random dynamical systems

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In this talk, we relate ergodicity, monotonicity and attractors of a random dynamical system (rds). The main result is that under suitable conditions, ergodicity and monotonicity together imply the existence of a weak random attractor which consists of a single random point only. We also show that ergodicity alone does not suffice to guarantee the existence of a weak attractor – not even when the one-point motion is a diffusion on $\mathbf{R}^d$ for $d \geq 2$. The results can be applied to a large class of systems, eg to certain SPDEs. The talk is largely based on joint work with Igor Chueshov (Kharkov), see [2].

Our motivation for studying this problem is to provide a rather general sufficient condition for a random attractor. Indeed several authors have proved the existence of random attractors for particular systems using ad hoc methods and in many cases the systems were in fact monotone and ergodic. When an rds is monotone, then this property is usually very easy to prove. Ergodicity is not always easy to prove – but in most cases still considerably easier than to prove the existence of an attractor. The result below provides a stronger conclusion than most other results on attractors in that we show, that the attractor is trivial (i.e. consists of a single point). On the other hand, due to the generality of the set-up, we can only show that the attractor attracts all deterministic compact sets in probability while some authors prove almost sure attraction for a larger class of sets.

We now state the assumptions and the main result. Let (X, d) be a complete, separable metric space and let $\mathbf{T}$ be either $\mathbf{R}$ or $\mathbf{Z}$. Further $\mathbf{T}_+$ denotes the set of nonnegative elements from $\mathbf{T}$. We denote by $\mathcal{B}(X)$ the Borel σ -algebra of subsets of X . By definition, a *random dynamical system* with time $\mathbf{T}_+$ and state space X is a pair (ϑ, φ) consisting of the following

two objects:

- A *metric dynamical system* (mds) $\vartheta \equiv (\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta(t), t \in \mathbf{T}\})$, i.e. a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a family of measure preserving transformations $\vartheta \equiv \{\vartheta(t) : \Omega \rightarrow \Omega, t \in \mathbf{T}\}$ such that (a) $\vartheta(0) = \text{id}$, $\vartheta(t) \circ \vartheta(s) = \vartheta(t+s)$ for all $t, s \in \mathbf{T}$; (b) the map $(t, \omega) \mapsto \vartheta(t)\omega$ is measurable and (c) $\vartheta(t)\mathbf{P} = \mathbf{P}$ for all $t \in \mathbf{T}$.
- A (perfect) *cocycle* φ over ϑ of continuous mappings of X with one-sided time $\mathbf{T}_+$, i.e. a measurable mapping

$$\varphi : \mathbf{T}_+ \times \Omega \times X \rightarrow X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x$$

such that the mapping $x \mapsto \varphi(t, \omega)x$ is continuous for every $t \geq 0$ and $\omega \in \Omega$ and it satisfies the cocycle property:

$$\varphi(0, \omega) = \text{id}, \quad \varphi(t+s, \omega) = \varphi(t, \vartheta(s)\omega) \circ \varphi(s, \omega)$$

for all $t, s \geq 0$ and $\omega \in \Omega$.

We call an rds *ergodic* in case there exists a probability measure π on X such that for each $x \in X$, the law of $\varphi(t, \omega)x$ converges weakly to π . Note that we do not assume that the one-point motion is Markovian.

Next, we introduce the concept of a *random attractor*.

Definition 0.1 Let $\mathcal{C}$ be the family of compact subsets of X . The mapping $\mathcal{A} : \Omega \rightarrow \mathcal{C}$ is called an *invariant random compact set* if

- (i) $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for each $x \in X$, where $d(x, \mathcal{A}) = \inf_{y \in \mathcal{A}} d(x, y)$.
- (ii) There exists a set $\tilde{\Omega} \in \mathcal{F}$ of full measure which is invariant under $\vartheta(t)$ for each $t \in \mathbf{T}$ such that $\varphi(t, \omega)(\mathcal{A}(\omega)) = \mathcal{A}(\vartheta(t)\omega)$ for all $\omega \in \tilde{\Omega}$ and $t \in \mathbf{T}_+$.

An invariant random compact set is called a *pullback attractor* if for each compact set $B \subseteq X$

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(\varphi(t, \vartheta(-t)\omega, x), \mathcal{A}(\omega)) = 0 \quad \text{a. s.},$$

and a *weak attractor* if for each compact set $B \subseteq X$

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(\varphi(t, \vartheta(-t)\omega, x), \mathcal{A}(\omega)) = 0 \quad \text{in probability.}$$

Since almost sure convergence implies convergence in probability, every pullback attractor is also a weak attractor. The converse is not true (see, e.g., [5] for examples). The concept of a pullback attractor was proposed independently in [1] and [6]. Weak attractors were introduced in [4]. If an attractor (weak or pullback) exists, then it is unique up to sets of measure zero.

In the talk, we will provide examples, namely isotropic Brownian flows with drift on $\mathbf{R}^d$, $d \geq 2$ showing that an ergodic rds does not necessarily admit a weak attractor.

To introduce a *monotone* rds, we need a partially ordered state space X . We will assume that (X, d) is a suitable subset of an ordered Banach space V :

Let V be a real separable Banach space with a cone $V_+ \subset V$. By definition, V_+ is a closed convex set in V such that $\lambda v \in V_+$ for all $\lambda \geq 0$, $v \in V_+$ and $V_+ \cap (-V_+) = \{0\}$. The cone V_+ defines a partial order relation on V via $x \leq y$ iff $y - x \in V_+$ which is compatible with the vector space structure of V . If V_+ has nonempty interior $\text{int } V_+$, we say that the cone V_+ is *solid*. For elements a and b in V such that $a \leq b$ we define the (conic) closed *interval* $[a, b]$ as the set of the form

$$[a, b] = \{x \in V : a \leq x \leq b\} .$$

If the cone V_+ is solid, then any bounded set $B \subset V$ is contained in some interval. A cone V_+ is said to be *normal* if every interval $[a, b]$ is bounded.

An rds (ϑ, φ) taking values in a subset X of V is called *monotone* or *order preserving*, if (possibly up to a universal set of measure zero) $x \leq y$ implies $\varphi(t, \omega)x \leq \varphi(t, \omega)y$ for all $t \in \mathbf{T}_+$ and $\omega \in \Omega$.

Theorem 0.2 *Let (ϑ, φ) be an ergodic and monotone rds taking values in a separable Banach space V with a solid and normal cone V_+ . Then the rds has a weak attractor which consists of a single (random) point.*

The theorem remains true, if V is replaced by a suitable subset $X \subseteq V$ (see [2]), but it does not hold for arbitrary subsets X of V (we will provide an example in the talk). The assumptions of the theorem do not guarantee that the rds has a pullback attractor. We point out, that another paper dealing with attractors for an ergodic rds (with independent increments) is [3].

Our result can be applied to one-dimensional rds, to multi-dimensional rds with *cooperative* drift, to certain stochastic delay differential equations,

certain parabolic spde's and some (monotone) interacting particle systems like the (dynamical) Ising model.

We will conclude the talk with some remarks about the relationship between the existence of a singleton attractor and the convergence of the *Propp-Wilson algorithm* – an algorithm for the perfect simulation of the invariant measure of a finite discrete time Markov chain via *coupling from the past*.

References

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