Noncommutative algebra

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1 Basics of rings and modules

1.1 Rings

We consider rings R which are *unital*, so there is $1 \in R$ with $r1 = 1r = r$ for all $r \in R$. Examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, $R[x]$ of ring of polynomials in an indeterminate x with coefficients in a ring R, $M_n(R)$ the ring of $n \times n$ matrices with entries in a ring R.

A *subring* of a ring is a subset $S \subseteq R$ which is ring under the same operations, with the same unity as R. A *ring homomorphism* is a mapping $\theta : R \to S$ preserving addition and multiplication and such that $\theta(1) = 1$.

A *(two-sided) ideal* in a ring R is a subgroup $I \subseteq R$ such that $rx \in I$ and $xr \in I$ for all $r \in R$ and $x \in I$. The ideal *generated by* a subset $S \subseteq R$ is

$$(S) = \{\sum_{i=1}^{n} r_i s_i r'_i : n \geq 0, r_i, r'_i \in R, s_i \in S\}.$$

If I is an ideal in R, then R/I is a ring.
Examples: $\mathbb{F}_p = \mathbb{Z}/(p) = \mathbb{Z}/p\mathbb{Z}$, $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$.

The isomorphism theorems (see for example, P.M.Cohn, Algebra, vol. 1).
(1) If $\theta : R \to S$ then $\operatorname{Im}\theta \cong R/\operatorname{Ker}\theta$.
(2) If I is an ideal in R and S is a subring of R then $S/(S \cap I) \cong (S + I)/I$.
(3) If I is an ideal in R, then the ideals in R/I are of the form J/I with J an ideal in R containing I, and $(R/I)/(J/I) \cong R/J$.

The opposite ring R^{op} is obtained from R by using the multiplication $\cdot$, where $r \cdot s = sr$. The transpose defines an isomorphism $M_n(R)^{op} \to M_n(R^{op})$.

1.2 Modules

Let R be a ring. A *(left) R-module* consists of an additive group M equipped with a mapping $R \times M \to M$ which is an *action*, meaning
- $(rr')m = r(r'm)$ for $r, r' \in R$ and $m \in M$,
- it is distributive over addition, and
- it is unital: $1m = m$ for all m.

Dually there is the notion of a *right R-module* with an action $M \times R \to R$. Apart from notation, it is the same thing as a left R^{op}-module. If R is commutative, the notions coincide.

If R and S are rings, then an *R-S-bimodule* is given by left R-module and right S-module structures on the same additive group M, satisfying $r(ms) = (rm)s$ for $r \in R$, $s \in S$ and $m \in M$.

If $\theta : R \to S$ is a ring homomorphism, any S-module ${}_SM$ becomes an R-module denoted ${}_RM$ or ${}_\theta M$ by *restriction*: $r.m = \theta(r)m$.

An R-module *homomorphism* $\theta : M \to N$ is a map of additive groups with $\theta(rm) = r\theta(m)$ for $r \in R$ and $m \in M$. A *submodule* of a R-module M is a subgroup $N \subseteq M$ with $rn \in N$ for all $r \in R$, $n \in N$. Given a submodule N of M one gets a quotient module M/N. Similarly for right modules and bimodules.

A ring R is naturally an R-R-bimodule. A (two-sided) ideal of R is a sub-bimodule of R. A *left* or *right ideal* of R is a submodule of R as a left or right module.

The isomorphism theorems for R-modules (see for example P.M.Cohn, Algebra, vol. 2).
(1) If $\theta : M \to N$ then $\operatorname{Im}\theta \cong M/\operatorname{Ker}\theta$.
(2) If L and N are submodules of a module M, then $L/(L\cap N) \cong (L+N)/N$.
(3) If N is a submodule of M, then the submodules of M/N are of the form L/N where L is a submodule of M containing N, and $(M/N)/(L/N) \cong M/L$.

1.3 Exact sequences

A sequence of modules and homomorphisms

$$\cdots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is said to be *exact* at M if $\operatorname{Im} f = \operatorname{Ker} g$. It is *exact* if it is exact at every module. A *short exact sequence* is one of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

so f is injective, g is surjective and $\operatorname{Im} f = \operatorname{Ker} g$.

Any map $f : M \to N$ gives an exact sequence

$$0 \to \operatorname{Ker} f \to M \to N \to \operatorname{Coker} f \to 0$$

where $\operatorname{Coker} f = M/\operatorname{Im} f$, and short exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0, \quad 0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0.$$

Snake Lemma. Given a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(0 & \longrightarrow &)L & \xrightarrow{\theta} & M & \xrightarrow{\phi} & N & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & L' & \xrightarrow{\theta'} & M' & \xrightarrow{\phi'} & N'(& \longrightarrow & 0)
\end{array}$$

there is an induced exact sequence

$$(0 \to) \operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{c} \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h (\to 0).$$

The connecting homomorphism c is given by diagram chasing.

1.4 Algebras

Fix a commutative ring K (usually a field). An *(associative) algebra* over K, or K*-algebra* consists of a ring which is at the same time a K-module, with the same addition, and such that multiplication is a K-module homomorphism in each variable.

To turn a ring R into a K-algebra is the same as giving a homomophism from K to the centre of R, $Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}$. Given the K-module structure on R, we have the map $K \to Z(R)$, $\lambda \mapsto \lambda 1$. Given a map $f : K \to Z(R)$ we have the K-module structure $\lambda.m = f(\lambda)m$.

A ring is the same thing as a $\mathbb{Z}$-algebra.

Any module for a K-algebra R becomes naturally a K-module via $\lambda.m = (\lambda 1)m$. It can also be considered as a R-K-bimodule.

If R and S are K-algebras, then unless otherwise stated, one only considers R-S-bimodules for which the left and right actions of K are the same.

A *K-algebra homomorphism* is a ring homomorphism which is also a K-module homomorphism, or equivalenty a ring homomorphism which is compatible with the ring homomorphisms from K.

1.5 Hom spaces

Let R be a K-algebra (including the case of a ring, with $K = \mathbb{Z}$). The set of all R-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_R(M, N)$ and it is a K-module. The set of endomorphisms $\mathrm{End}_R(M)$ is a K-algebra.

Bimodule structures on M or N give module structures on $\mathrm{Hom}_R(M, N)$. For example if M is an R-S-bimodule and N is an R-T-bimodule then $\mathrm{Hom}_R(M, N)$ becomes an S-T-bimodule via $(s\theta t)(m) = \theta(ms)t$.

Lemma. If M is an R-module, there is an R-module isomorphism

$$\mathrm{Hom}_R(R, M) \cong M, \quad \theta \mapsto \theta(1), \quad (r \mapsto rm) \leftarrow m.$$

Taking $M = R$, this gives an isomorphism of rings

$$\mathrm{End}_R(R) \cong R^{op}.$$

(If we used right modules, we wouldn't need the opposite here.)

1.6 Products and sums

The *(Cartesian) product* of sets X_i $(i \in I)$ is $\prod_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i\}$. $\prod_{i \in I} X$ is the set of functions $I \to X$, also denoted X^I.

The axiom of choice says that if all $X_i \neq \emptyset$ then $\prod_{i \in I} X_i \neq \emptyset$. We shall freely use the axiom of choice. In particular:

Zorn's lemma. If in a non-empty partially ordered set, every chain (=totally ordered subset) has an upper bound, then the set has a maximal element.

Well-ordering Theorem. Any set can be well ordered, that is, given a total order with the property that any non-empty subset has a least element.

A product of rings $\prod_{i\in I} R_i$ is naturally a ring, e.g. $\mathbb{Z}^n = \mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}$.

A product of R-modules $\prod_{i\in I} X_i$ is naturally an R-module. There is also the *(external) direct sum* or *coproduct* of modules:

$$\bigoplus_{i\in I} X_i \left(\text{or } \coprod_{i\in I} X_i\right) = \left\{(x_i)_{i\in I}\in\prod_{i\in I} X_i \colon x_i = 0 \text{ for all but finitely many } i\right\}.$$

One writes $X^{(I)} = \bigoplus_{i\in I} X$.

If the X_i ($i\in I$) are submodules of an R-module X, then addition gives a homomorphism

$$\bigoplus_{i\in I} X_i \to X,\quad (x_i)_{i\in I}\mapsto \sum_{i\in I} x_i.$$

The image is the *sum* of the X_i, denoted $\sum_{i\in I} X_i$. If this homomorphism is an isomorphism, then the sum is called an *(internal) direct sum*, and also denoted $\bigoplus_{i\in I} X_i$.

Lemma. There are natural isomorphisms

$$\mathrm{Hom}_R(X,\prod_i Y_i)\cong\prod_i \mathrm{Hom}_R(X,Y_i),$$

$$\mathrm{Hom}_R(\bigoplus_i X_i,Y)\cong\prod_i \mathrm{Hom}_R(X_i,Y).$$

In particular $\mathrm{End}_R(X^n)\cong M_n(\mathrm{End}_R(X))$ and more generally

$$\mathrm{End}_R(X_1\oplus\cdots\oplus X_n)\cong\begin{pmatrix}\mathrm{Hom}(X_1,X_1) & \mathrm{Hom}(X_2,X_1) & \dots & \mathrm{Hom}(X_n,X_1)\\ \mathrm{Hom}(X_1,X_2) & \mathrm{Hom}(X_2,X_2) & \dots & \mathrm{Hom}(X_n,X_2)\\ \dots & & & \\ \mathrm{Hom}(X_1,X_n) & \mathrm{Hom}(X_2,X_n) & \dots & \mathrm{Hom}(X_n,X_n)\end{pmatrix}.$$

1.7 Generators and relations

If $(m_i)_{i\in I}$ is family of elements of an R-module M, the submodule *generated by* (m_i) is

$$\sum_{i\in I} Rm_i = \{\sum_{i\in I} r_i m_i : r_i\in R,\text{ all but finitely many zero}\},$$

or equivalently the image of the map $R^{(I)} \to M$, $(r_i) \mapsto \sum_{i \in I} r_i m_i$. Any element of the kernel gives a relation of the form

$$\sum_{i \in I} r_i m_i = 0$$

(with all but finitely many $r_i = 0$). The family (m_i) is *linearly independent* if the map $R^{(I)} \to M$ is injective, or equivalently there are no non-trivial relations; it is an *(R-)basis* for M if it is linearly independent and generates M.

The following are equivalent (in which case M is said to be a *free module*).
(i) M has an R-basis
(ii) $M \cong R^{(I)}$ for some set I.

The module generated by a family of indeterminates $(m_i)_{i \in I}$ subject to a set of relations of the form

$$\sum_{i \in I} r_i m_i = 0$$

is $M = R^{(I)}/L$ where the m_i are identified with the canonical basis of $R^{(I)}$ and L is the submodule of $R^{(I)}$ generated by the elements $\sum_{i \in I} r_i m_i$.

Every module M has a generating set (eg M itself), so there is always a map from a free module onto M.

1.8 Finitely generated modules

A module M is *finitely generated* if it has a finite generating set. Equivalently if there is a map from R^n onto M for some $n \in \mathbb{N}$.

Properties. (i) Any quotient of of a finitely generated module is finitely generated.

(ii) $\bigoplus M_i$ is finitely generated if and only if the M_i arefinitely generated, and all but finitely many are zero.

(iii) $\mathrm{Hom}_R(M, \bigoplus Y_i) \cong \bigoplus \mathrm{Hom}_R(M, Y_i)$ for M finitely generated.

(iv) Any proper submodule of a finitely generated module is contained in a maximal proper submodule. (Apply Zorn's Lemma to the set of proper submodules containing the submodule. Finite generation ensures that the union of a chain of proper submodules is a proper submodule.)

For later use, we consider the following rather specialized idea that I saw in R. Wisbauer, Foundations of module and ring theory. If M is a module which can be written as a direct sum of finitely generated modules, say $M = \bigoplus M_\lambda$, then we define

$$\widehat{\mathrm{Hom}}_R(M,Y) = \{\theta \in \mathrm{Hom}(M,Y) : \theta(M_\lambda) = 0 \text{ for all but finitely many } \lambda\}.$$

Contrary to what it seems to say in §51, p485 of the book by Wisbauer, this definition depends on the given decomposition of M. (For example if $M = R^{(\mathbb{N})}$ is the free module with natural basis $e_0, e_1, e_2, \dots$, then the projection onto the 0th summand $\pi : M \to R$ is in $\widehat{\mathrm{Hom}}_R(M,R)$ with respect to the natural decomposition of M as a direct sum of copies of R, but not with respect to the decomposition coming from the basis $e_0, e_0+e_1, e_0+e_2, \dots$.) Clearly

$$\widehat{\mathrm{Hom}}_R(M,Y) \cong \bigoplus \mathrm{Hom}_R(M_\lambda, Y), \text{ and}$$

$$\widehat{\mathrm{Hom}}_R(M, \bigoplus Y_i) \cong \bigoplus_i \widehat{\mathrm{Hom}}_R(M, Y_i).$$

1.9 Simple and semisimple modules

A module S is *simple* (or irreducible) if it has exactly two submodules, namely $\{0\}$ and S. It is equivalent that S is non-zero and any non-zero element is a generator.

Schur's Lemma. Any homomorphism between simple modules must either be zero or an isomorphism, so if S is simple, $\mathrm{End}_R(S)$ is a division ring, that is, all non-zero elements are invertible.

Theorem. If M is a module, the following are equivalent, in which case M is said to be *semisimple* (or completely reducible).
(a) M is isomorphic to a direct sum of simple modules.
(b) M is a sum of simple modules.
(c) Any submodule of M is a direct summand.

Sketch. For fuller details see P.M.Cohn, Algebra 2, §4.2.

(a) implies (b) is trivial. Assuming (b), say $M = \sum_{i\in I} S_i$ and that N is a submodule of M, one shows by Zorn's lemma that $M = N \oplus \bigoplus_{i\in J} S_i$ for some subset J or I. This gives (a) and (c).

The property (c) is inherited by submodules $N \subseteq M$, for if $L \subseteq N$ and $M = L \oplus C$ then $N = L \oplus (N \cap C)$. Let N be the sum of all simple submodules. It has complement C, and if non-zero, then C has a non-zero finitely generated submodule F. Then F has a maximal proper submodule P. Then P has a complement D in F, and $D \cong F/P$, so it is simple, so $D \subseteq N$. But $D \subseteq C$, so its intersection with N is zero.

Corollary. Any submodule or quotient of a semisimple module is semisimple.

Proof. We showed above that condition (c) passes to submodules. Now if M is semisimple and M/N is a quotient, then N has a complement C in M, and $M/N \cong C$, so it is semisimple.

The theory of vector spaces:

Theorem. If R is a field, or more generally a division ring, every R-module is free and semisimple.

Proof. R is a simple R-module, and it is the only simple module up to isomorphism, since if S is a simple module and $0 \neq s \in S$ then the map $R \to S$, $r \mapsto rs$ must be an isomorphism. Thus free = semisimple. The result follows.

Artin-Wedderburn Theorem. For a ring R, the following are equivalent:
(i) R is a left artinian and its Jacobson radical is zero. (We didn't discuss these conditions yet.)
(ii) R is semisimple as an R-module.
(iii) Every R-module is semisimple.
(iv) R is isomorphic to a finite direct product of matrix rings over division rings.

Sketch (excluding (i)). (ii) implies that every free module is semisimple, and since any module is a quotient of a free module, (iii) follows. If (ii) holds then, since R is finitely generated as a module (by 1), it must be a finite direct sum of simples. Collecting terms we can write

$$R = S_1 \oplus \cdots \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_2 \oplus \cdots \oplus S_n \oplus \cdots \oplus S_n$$

where $S_1, \dots, S_n$ are non-isomorphic simples, and there are n_i copies of each S_i. Then by by Schur's lemma and the last statement in §1.6, one gets $\mathrm{End}_R(R) \cong \prod_{i=1}^n M_{n_i}(D_i)$ with $D_i = \mathrm{End}_R(S_i)$. Now take the opposite ring to get R, giving (iv).

Conversely, if (iv) holds, say $R \cong \prod_{i=1}^n M_{n_i}(D_i)$ then $R = \bigoplus_{i=1}^n \bigoplus_{j=1}^{n_i} I_{ij}$

where I_{ij} is the left ideal in $M_{n_i}(D_i)$ consisting of matrices which are zero outside the jth column. This is isomorphic to the module consisting of column vectors $D_i^{n_i}$, and for D_i a division algebra, this is a simple module, giving (ii).

1.10 An exotic example

If M is a free R-module, say $M \cong R^{(I)}$, then

$$\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_R(R^{(I)}, R) \cong (\mathrm{Hom}_R(R, R))^I \cong R^I.$$

This is either finitely generated (if $R = 0$ or I is finite) or uncountable (if $R \neq 0$ and I is infinite). The following result thus shows that $\mathbb{Z}^{\mathbb{N}}$ cannot be free.

Theorem (Specker, 1950). $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is a free $\mathbb{Z}$-module with basis $(\pi_i)_{i\in\mathbb{N}}$ where $\pi_i(a) = a_i$.

Proof. (cf. Scheja and Storch, Lehrbuch der Algebra, Teil 1, 2nd edition, Satz III.C.4, p230) It is clear that the π_i are linearly independent. Let (e_i) be the standard basis of $\mathbb{Z}^{(\mathbb{N})} \subset \mathbb{Z}^{\mathbb{N}}$. Let $h : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, and let $b_i = h(e_i)$. Let (c_n) be a sequence of positive integers such that c_{n+1} is a multiple of c_n and

$$c_{n+1} \geq n + 1 + \sum_{i=1}^{n} |c_i b_i|.$$

Let $c = h((c_n))$.

For each $m \in \mathbb{N}$ there is $y_m \in \mathbb{Z}^{\mathbb{N}}$ with

$$(c_n) = \sum_{i=0}^{m} c_i e_i + c_{m+1} y_m.$$

Applying h gives

$$c = \sum_{i=0}^{m} c_i b_i + c_{m+1} h(y_m),$$

so

$$|c - \sum_{i=0}^{m} c_i b_i| = c_{m+1} |h(y_m)|$$

is either 0 or $\geq c_{m+1}$. But if $m \geq |c|$, then

$$|c - \sum_{i=0}^{m} c_i b_i| \leq |c| + \sum_{i=0}^{m} |c_i b_i| < c_{m+1}.$$

Thus $c = \sum_{i=0}^{m} c_i b_i$ for all $m \geq |c|$. But this implies $b_i = 0$ for all $i > |c|$. Then the linear form $h - \sum_{i=0}^{|c|} b_i \pi_i$ vanishes on all of the standard basis elements e_i.

It remains to show that if $g \in \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ vanishes on all the e_i, then it is zero. Suppose given $(c_i) \in \mathbb{Z}^{\mathbb{N}}$. Expanding $c_i = c_i(3-2)^{2i}$, we can write $c_i = v_i 2^i + w_i 3^i$ for some $v_i, w_i \in \mathbb{Z}$. Then $g((c_i)) = g((v_i 2^i)) + g((w_i 3^i))$. Now for any m, $(v_i 2^i) = \sum_{i=0}^{m-1} v_i 2^i e_i + 2^m z_m$ for some $z_m \in \mathbb{Z}^{\mathbb{N}}$. Thus $g((v_i 2^i)) \in 2^m \mathbb{Z}$. Thus $g((v_i 2^i)) = 0$. Similarly for w. Thus $g((c_n)) = 0$.

See also the example of Rickard on mathoverflow.net (in answer to question 218113), the $\mathbb{Z}$-module A of bounded sequences of elements of $\mathbb{Z}[\sqrt{2}]$ satisfies $A \cong A \oplus \mathbb{Z}^2 \not\cong A \oplus \mathbb{Z}$.

1.11 Tensor products

If X is a right R-module and Y is a left R-module, the tensor product $X \otimes_R Y$ is defined to be the additive group generated by symbols $x \otimes y$ $(x \in X, y \in Y)$ subject to the relations:
- $(x + x') \otimes y = x \otimes y + x' \otimes y$,
- $x \otimes (y + y') = x \otimes y + x \otimes y'$,
- $(xr) \otimes y = x \otimes (ry)$ for $r \in R$.

Properties. (1) Assume that R is a K-algebra (including the case $K = \mathbb{Z}$, if R is a ring), then $X \otimes_R Y$ is a K-module via $\lambda(x \otimes y) = x\lambda \otimes y$. If Z is a K-module, a map $\phi : X \times Y \to Z$ is *K-bilinear* if it is K-linear in each argument, and *R-balanced* if $\phi(xr, y) = \phi(x, ry)$ for all x, y, r. The set $B_{K,R}(X, Y, Z)$ of K-bilinear R-balanced maps is naturally a K-module, and there is an isomorphism

$$\mathrm{Hom}_K(X \otimes_R Y, Z) \cong B_{K,R}(X, Y, Z)$$

sending θ to the map $(x, y) \mapsto \theta(x \otimes y)$ and ϕ to the map sending $\sum x_i \otimes y_i$ to $\sum \phi(x_i, y_i)$.

(2) If X is an S-R-bimodule, then $X \otimes_R Y$ becomes an S-module via $s(x \otimes y) = (sx) \otimes y$, and for a left S-module Z, there is a natural isomorphism

$$\mathrm{Hom}_S(X \otimes Y, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z)).$$

Both sides correspond to the subset of $B_{K,R}(X, Y, Z)$ consisting of bilinear balanced maps which are also S-linear in the first argument.

Dually, if Y is an R-T-bimodule, then $X \otimes_R Y$ is a right T-module and

$$\mathrm{Hom}_T(X \otimes Y, Z) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_T(Y, Z)).$$

(3) There are natural isomorphisms $X \otimes_R R \cong X$, $x \otimes r \mapsto xr$ and $R \otimes_R Y \cong Y$, $r \otimes y \mapsto ry$. There are natural isomorphisms

$$\left(\bigoplus_{i \in I} X_i\right) \otimes_R Y \cong \bigoplus_{i \in I} (X_i \otimes_R Y), \ X \otimes_R \left(\bigoplus_{i \in I} Y_i\right) \cong \bigoplus_{i \in I} (X \otimes_R Y_i).$$

Thus $X \otimes_R {}_R R^{(J)} \cong X^{(J)}$ and $R_R^{(I)} \otimes_R Y \cong Y^{(I)}$, so $R_R^{(I)} \otimes_R {}_R R^{(J)} \cong R^{(I \times J)}$.

(4) If $\theta : X \to X'$ is a map of right R-modules and $\phi : Y \to Y'$ is a map is left R-modules, then there is a map

$$\theta \otimes \phi : X \otimes_R Y \to X' \otimes_R Y', \ x \otimes y \mapsto \theta(x) \otimes \phi(y).$$

If θ is a map of S-R-bimodules, then this is a map of S-modules, etc.

(5) If $X' \subseteq X$ is an R-submodule of X then $(X/X') \otimes_R Y$ is isomorphic to quotient of $X \otimes_R Y$ by the subgroup generated by all elements of the form $x' \otimes y$ with $x' \in X'$, $y \in Y$ (so the cokernel of the map $X' \otimes_R Y \to X \otimes_R Y$). Similarly for $X \otimes_R (Y/Y')$ if Y' is a submodule of Y.

Thus if I is a right ideal in R,

$$(R/I) \otimes_R Y \cong (R \otimes_R Y)/\operatorname{Im}(I \otimes_R Y \to R \otimes_R Y) \cong Y/IY.$$

Similarly if J is a left ideal in R then $X \otimes_R (R/J) \cong X/XJ$.

Thus $(R/I) \otimes_R (R/J) \cong R/(I + J)$. eg. $(\mathbb{Z}/2\mathbb{Z}) \otimes_Z (\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/\mathbb{Z} = 0$.

(6) If X is a right S-module, Y a S-R-bimodule and Z a left R-module, then there is a natural isomorphism

$$X \otimes_S (Y \otimes_R Z) \cong (X \otimes_S Y) \otimes_R Z.$$

(7) Tensor product of algebras. If R and S are K-algebras, then $R \otimes_K S$ becomes a K-algebra. For example $M_n(K) \otimes_K S \cong M_n(S)$. An R-S-bimodule (for which the two actions of K agree) is the same thing as a left $R \otimes_K S^{op}$-module.

(8) Base change. If S is a commutative K-algebra then $R \otimes_K S$ is naturally an S-algebra.

1.12 Idempotents

An *idempotent* in a ring R is an element $e \in R$ with $e^2 = e$.

Properties. (1) If M is an R-module, then $eM = \{m \in M : em = m\}$, for if $em = m$ then $m \in eM$, while if $m \in eM$ then $m = em' = e^2 m' = e(em') = em$.

(2) The map $\mathrm{Hom}_R(Re, M) \to eM, \theta \mapsto \theta(e)$ is an isomorphism of additive groups. The inverse sends m to the map $r \mapsto rm$.

(3) $eRe \cong \mathrm{End}_R(Re)^{op}$ is a ring. It is not a subring of R since the identity element is different.

A family of idempotents (e_i) is *orthogonal* if $e_i e_j = 0$ for $i \neq j$. For a family of orthogonal idempotents, the following are equivalent, in which case it is called a *complete* family of orthogonal idempotents.
(i) I is finite and $\sum_{i \in I} e_i = 1$;
(ii) $R = \bigoplus_{i,j \in I} e_i R e_j$ (*Peirce decomposition*);
(iii) For any left module M one has $M = \bigoplus_{i \in I} e_i M$ and for any right module N one has $N = \bigoplus_{i \in I} N e_i$.

Given any idempotent e, the pair $e, 1 - e$ is a complete set of orthogonal idempotents.

Lemma. For an R-module M, there is a bijection

$$\{\text{idempotents in } \mathrm{End}_R(M)\} \to \{R\text{-module decompositions } M = X \oplus Y\}.$$

Proof. An idempotent endomorphism e gives $M = \mathrm{Im}\, e \oplus \mathrm{Ker}\, e$. A decomposition $M = X \oplus Y$ gives $e =$ projection onto X.

Similarly, complete families of n orthogonal idempotents in $End_R(M)$ correspond to decompositions $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

Definitions. (i) A module M is *indecomposable* if it is non-zero and in any decomposition $M = X \oplus Y$, either $X = 0$ or $Y = 0$. Thus M is indecomposable if and only if $\mathrm{End}_R(M)$ has no non-trivial idempotents (other than 0 and 1).

(ii) An idempotent $e \in R$ is *primitive* if Re is indecomposable, or equivalently if eRe contains no idempotents other than 0 and e. Equivalently if e can't be written in a non-trivial way as a sum of two orthogonal idempotents.

(iii) Idempotents $e, f \in R$ are *equivalent* if $Re \cong Rf$. Equivalently if there are elements $p \in eRf$ and $q \in fRe$ with $pq = e$ and $qp = f$.

1.13 Non-unital rings

For fun, I thought it would be nice to extend definitions and theorems to the following non-unital rings.

A *ring with enough idempotents* is an additive group with an associative multiplication which is distributive over addition, and which has a complete set of orthogonal idempotents, that is, a set of orthogonal idempotents $(e_i)_{i \in I}$ satisfying

$$R = \bigoplus_{i,j \in I} e_i R e_j.$$

By a left module for such a ring, one means an additive group M equipped with an action $R \times M \to M$ which satisfies $r(r'm) = (rr')m$, is distributive over addition, and is unital in the sense that

$$M = \bigoplus_{i \in I} e_i M$$

(equivalently $RM := \{\sum_{j=1}^n r_j m_j : n \geq 0, r_j \in R, m_j \in M\}$ is equal to M.)

Examples.
(a) A direct sum of rings $\bigoplus_{i \in I} R_i$ (unital or with enough idempotents) is a ring with enough idempotents.
(b) If I is a set and R a ring (unital or with enough idempotents), write $R^{(I \times I)}$ for the set of matrices with entries in R, with rows and columns indexed by I, and only finitely many non-zero entries. It is a ring with enough idempotents.
(c) If M is a direct sum of finitely generated R-modules, then $\widehat{\mathrm{End}}_R(M)$ is a ring with enough idempotents, and for any set I,

$$\widehat{\mathrm{End}}_R(M^{(I)}) \cong \widehat{\mathrm{End}}_R(M)^{(I \times I)}.$$

(d) If R has enough idempotents then $R = \bigoplus Re_i$ so ${}_RR$ is a direct sum of finitely generated modules, and $R = \bigoplus e_iR \cong \bigoplus \mathrm{Hom}_R(Re_i, R) \cong \widehat{\mathrm{End}}_R(R)^{op}$.

2 Constructions of algebras

We consider K-algebras, where K is a commutative ring. Maybe $K = \mathbb{Z}$, so we consider rings.

2.1 Tensor algebras and free algebras

If V is an R-R-bimodule, one defines the *tensor powers* by

$$T_R^n V = \underbrace{V \otimes_R V \otimes_R \cdots \otimes_R V}_{n},$$

with the convention that $T_R^0 V = R$. The *tensor algebra* is

$$T_R V = \bigoplus_{n \in N} T_R^n V = R \oplus V \oplus (V \otimes_R V) \oplus (V \otimes_R V \otimes_R V) \oplus \dots$$

with its natural K-algebra structure.

A K-algebra homomorphism $\phi : T_R V \to C$ gives rise to a K-algebra homomorphism $\theta : R \to C$ and an R-R-bimodule map $\psi : V \to {}_\theta C_\theta$. Conversely any such θ and ψ uniquely determine a K-algebra map $\phi : T_R V \to C$, $\phi(r) = \theta(r)$, $\phi(v_1 \otimes \cdots \otimes v_n) = \psi(v_1) \dots \psi(v_n)$.

Similarly, a module for $T_R V$ is the same thing as an R-module M and an R-module map $V \otimes_R M \to M$.

In case $R = K$ and V is the free K-module with basis X, we can identify $T_K V$ with the *free (associative) algebra* $K\langle X \rangle$. This is the free module K-module on the set of all words in the letters of X. It becomes a K-algebra by concatenation of words. For example for $X = \{x, y\}$ we write $K\langle x, y \rangle$, and it has basis

$$1, x, y, xx, xy, yx, yy, xxx, xxy, \dots$$

In case $X = \{x\}$ one recovers the polynomial ring $K[x]$.

If C is any K-algebra, there is a bijection

$$\mathrm{Hom}_{K\text{-algebra}}(K\langle X \rangle, C) \to \mathrm{Hom}_{\mathrm{set}}(X, C).$$

Any algebra can be written as a quotient of a free algebra by an ideal, $K\langle X \rangle / I$. For example take X to be a basis of a free K-module V mapping onto the algebra.

2.2 Free products of algebras

The *free product* (or, better, *coproduct*) of two K-algebras A and B is an algebra, denoted $A*B$ (or $A*_K B$), equipped with K-algebra maps $A \to A*B$ and $B \to A*B$, which has the universal property that for each K-algebra C and pair of K-algebra maps $A \to C$ and $B \to C$ there is a unique K-algebra map $A*B \to C$ whose composition with the maps from A and B is the given maps. Thus there is a bijection

$$\mathrm{Hom}_{K\text{-alg}}(A*B, C) \cong \mathrm{Hom}_{K\text{-alg}}(A, C) \times \mathrm{Hom}_{K\text{-alg}}(B, C).$$

By the universal property, if the free product exists, then it is unique up to a unique isomorphism. The free product exists, for if we write $A = K\langle X \rangle / I$ and $B = K\langle Y \rangle / J$ then one can take $A * B = K\langle X \cup Y \rangle / (I \cup J)$.

This is not the same as the tensor product $A \otimes_K B$, as the elements from A and B need not commute in $A * B$. In fact $A \otimes_K B \cong (A * B)/(ab - ba : a \in A, b \in B)$.

Example. The free product of two copies of $K \times K$ is the free product of two copies of $K[e]/(e^2 - e)$, so it is $K\langle e, f \rangle / (e^2 - e, f^2 - f)$. It has basis the alternating words in e and f (either one can show this by hand, or it is a very trivial case of the Diamond Lemma, coming later.) The tensor product is $K^2 \otimes_K K^2 \cong K[e, f]/(e^2 - e, f^2 - f) \cong K \times K \times K \times K$.

More generally one can take free products of any family of algebras, and also free products with amalgamation: given $R \to A$ and $R \to B$ there is

$$\begin{array}{ccc} R & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow \\ B & \longrightarrow & A *_R B. \end{array}$$

where one can take $A *_R B = A *_K B/(\text{image of } r \text{ in } A \text{ - image of } r \text{ in } B : r \in R)$.

2.3 Matrix algebras and nth roots

Given a matrix ring $M_n(R)$, we write e^{ij} for the matrix which is 1 in position (i, j) and 0 elsewhere.

Proposition. The following are equivalent for a ring S and integer n:
(i) S is isomorphic to $M_n(R)$ for some ring R;
(ii) S contains elements e^{ij} $(1 \le i, j \le n)$ satisfying

$$e^{ij}e^{pq} = \delta_{jp}e^{iq}, \quad e^{11} + \cdots + e^{nn} = 1.$$

(*matrix units*)
(iii) S contains a complete set of n pairwise equivalent orthogonal idempotents;
(iv) ${}_SS \cong M^n$ for some S-module M.

Proof. (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) are trivial. If (iv) holds then $S \cong \mathrm{End}_S(M^n)^{op} \cong M_n(\mathrm{End}_S(M))^{op} \cong M_n(\mathrm{End}_S(M)^{op})$.

Observe that if (ii) holds, and we let $R = e^{11}Se^{11}$, then we get an isomorphism $M_n(R) \to S$, $(r_{ij}) \mapsto \sum_{i,j} e^{i1}r_{ij}e^{1j}$ so it sends the e^{ij} in $M_n(R)$ to e^{ij} in S.

Lemma. If R and S are rings, there is a bijection

$$\{\text{ring homs } R \to S\} \to \{\text{ring homs } M_n(R) \to M_n(S) \text{ sending } e^{ij} \text{ to } e^{ij}\}.$$

Proof. Given $\theta : R \to S$ we define $\Theta : M_n(R) \to M_n(S)$ by $\Theta(A)_{ij} = \theta(A_{ij})$. Conversely given Θ with $\Theta(e^{ij}) = e^{ij}$ we define θ by $\theta(r) = \Theta(re^{11})_{11}$. For example if you start with Θ, construct θ and then construct Θ', then

$$\Theta'(A)_{ij} = \theta(A_{ij}) = \Theta(A_{ij}e^{11})_{11} = \Theta(e^{1i}Ae^{j1})_{11} = (e^{1i}\Theta(A)e^{j1})_{11} = \Theta(A)_{ij}$$

so $\Theta' = \Theta$.

If A is a K-algebra, we can form $A * M_n(K)$. This is a ring which contains a set of matrix units e^{ij}, so it is of the form $M_n(\sqrt[n]{A})$ where we define

$$\sqrt[n]{A} = e^{11}(A * M_n(K))e^{11}.$$

(Bergman, Coproducts and some universal ring constructions, Transactions of the AMS 1974. Notation from L. Le Bruyn and G. Van de Weyer, Formal structures and representation spaces, J. Algebra 2002.)

Proposition. For any K-algebra C, there is a bijection

$$\mathrm{Hom}_{K\text{-alg}}(\sqrt[n]{A}, C) \to \mathrm{Hom}_{K\text{-alg}}(A, M_n(C)).$$

Proof. The space $\mathrm{Hom}_{K\text{-alg}}(\sqrt[n]{A}, C)$ is isomorphic to the set of morphisms in $\mathrm{Hom}_{K\text{-alg}}(M_n(\sqrt[n]{A}), M_n(C))$ sending the e^{ij} to the e^{ij}. This space is $\mathrm{Hom}_{K\text{-alg}}(A, M_n(C)) \times \mathrm{Hom}_{K\text{-alg}}(M_n(K), M_n(C))$, and the condition on the e^{ij} exactly fixes the element of $\mathrm{Hom}_{K\text{-alg}}(M_n(K), M_n(C))$.

2.4 Path algebras

A *quiver* is a quadruple $Q = (Q_0, Q_1, h, t)$ where Q_0 and Q_1 are sets, called the sets of vertices and arrows, and $h, t : Q_1 \to Q_0$ are mappings, specifying the head and tail vertices of each arrow,

$$\overset{t(a)}{\bullet} \overset{a}{\to} \overset{h(a)}{\bullet}.$$

A *path* in Q of length $n > 0$ is a sequence $p = a_1 a_2 \dots a_n$ of arrows satisfying $t(a_i) = h(a_{i+1})$ for all $1 \le i < n$,

$$\bullet \overset{a_1}{\leftarrow} \bullet \overset{a_2}{\leftarrow} \bullet \cdots \bullet \overset{a_n}{\leftarrow} \bullet.$$

The head and tail of p are $h(a_1)$ and $t(a_n)$. For each vertex $i \in Q_0$ there is also a *trivial path* e_i of length zero with head and tail i.

Assume that Q has only finitely many vertices. The *path algebra* KQ is the free K-module on the paths. It becomes an algebra with the product of two paths given by $p \cdot q = 0$ if the tail of p is not equal to the head of q, and otherwise $p \cdot q = pq$, the concatenation of p and q.

Examples. (i) $1 \overset{a}{\to} 2 \overset{b}{\to} 3$. Then KQ has K-basis e_1, e_2, e_3, a, b, ba.

(ii) If only one vertex, then KQ is the free algebra $K\langle Q_1 \rangle$.

(iii) Suppose there is at most one path between any two vertices. Label the vertices $1, \dots, n$. Then KQ is isomorphic to the subalgebra

$$\{C \in M_n(K) \colon C_{ij} = 0 \text{ if there is no path from } j \text{ to } i\}$$

of $M_n(K)$. Under this isomorphism, the matrix e^{ij} which is 1 in position (i, j) corresponds to the path from j to i.

Properties. (1) The trivial paths are a complete set of orthogonal idempotents: $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$ and $\sum_{i \in Q_0} e_i = 1$.

(2) The spaces KQe_i, $e_j KQe_i$ and $e_j KQ$ have as K-bases the paths with tail at i and/or head at j.

(3) If K is a domain, $0 \neq f \in KQe_i$ and $0 \neq g \in e_i KQ$ then $fg \neq 0$. Explicitly p and q are paths of maximal length involved in f and g, then the coefficient of pq in fg must be non-zero.

(4) If K is a domain, then the e_i are primitive idempotents, for $e_i KQ e_i$ contains no zero-divisors.

(5) If $i \neq j$ then the idempotents e_i and e_j are inequivalent. Otherwise, reducing modulo a maximal ideal of K we may assume that K is a field. Then there are $f \in e_j KQ e_i$ and $g \in e_i KQ e_j$ with $fg = e_j$ and $gf = e_i$. But by the argument in (3), f and g can only involve trivial paths.

Notes. (a) If Q has infinitely many vertices then KQ still makes sense as a ring with enough idempotents.

(b) $KQ \cong T_R V$ where $R = K^{(Q_0)}$ and $V = \bigoplus_{a \in Q_1} {}_{\pi_{h(a)}} K_{\pi_{t(a)}}$, where $\pi_i : R \to K$ is the projection onto the ith summand.

2.5 Representations of quivers

A *(K-)representation* V of Q consists of a K-module V_i for each vertex i and a K-module map $V_a : V_i \to V_j$ for each arrow $a : i \to j$ in Q. If there is no risk of confusion, we write $a : V_i \to V_j$ instead of V_a.

If V and W are representations of Q, a homomorphism $\theta : V \to W$ consists of a K-module map $\theta_i : V_i \to W_i$ for each vertex i satisfying $W_a \theta_i = \theta_j V_a$ for all arrows $a : i \to j$.

A *subrepresentation* of a representation V is given by a K-submodule W_i of V_i for each i, such that $V_a(W_i) \subseteq W_j$ for all arrows $a : i \to j$. It becomes a representation by taking W_a to be the restriction of V_a to W_i. The *quotient representation* V/W is given by K-modules $(V/W)_i = V_i/W_i$ and the induced K-module maps $V_i/W_i \to V_j/W_j$ for $a : i \to j$. Given a family of representations V^λ $(\lambda \in \Lambda)$, the *direct sum* $\bigoplus_\lambda V^\lambda$ is given by the K-modules $\bigoplus_\lambda V_i^\lambda$ and maps $\bigoplus_\lambda V_i^\lambda \to \bigoplus_\lambda W_i^\lambda$ for $a : i \to j$ sending (v_λ) to $(V_a^\lambda(v))$.

Construction. Any representation V of Q determines a KQ-module ${}^\oplus V$ via ${}^\oplus V = \bigoplus_{i \in Q_0} V_i$ with the action given as follows:
- For $v = (v_i)_{i \in Q_0}$ we have $e_i v = v_i \in V_i \subseteq {}^\oplus V$. That is, the trivial path e_i acts on ${}^\oplus V$ as the projection onto V_i.
- and $a_1 a_2 \dots a_n v = V_{a_1}(V_{a_2}(\dots(V_{a_n}(v_{t(a_n)}))\dots)) \in V_{h(a_1)} \subseteq {}^\oplus V$.

Properties. (1) Any KQ-module is isomorphic to one of the form ${}^\oplus V$. This is essentially the fact that an internal direct sum is isomorphic to the external direct sum.

(2) Any homomorphism $\theta : V \to W$ of representations defines a homomorphism ${}^{\oplus}\theta : {}^{\oplus}V \to {}^{\oplus}W$ defined by ${}^{\oplus}\theta((v_i)) = (\theta_i(v_i))$. This defines a bijection $\mathrm{Hom}_{\mathrm{reps}}(V, W) \to \mathrm{Hom}_{KQ}({}^{\oplus}V, {}^{\oplus}W)$.

(3) Any subrepresentation W of V defines a submodule ${}^{\oplus}W = \bigoplus_{i \in Q_0} W_i$ of ${}^{\oplus}V$. Any submodule of ${}^{\oplus}V$ is of this form, and the quotient module can be identified with ${}^{\oplus}(V/W)$. Also direct sums of families of representations and modules correspond.

Examples for $\mathbb{Z}Q$ where $Q = 1 \xrightarrow{a} 2$. For example $\mathbb{Z}^2 \xrightarrow{(4\ 6)} \mathbb{Z}$ is isomorphic to the direct sum of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ and $\mathbb{Z} \to 0$. Smith normal form. Also, does $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ have a subrepresentation isomorphic to $\mathbb{Z} \xrightarrow{3} \mathbb{Z}$?

2.6 Submodules of free modules for path algebras

Let K be a field. In this case every submodule of a free module for a path algebra KQ is isomorphic to a direct sum of left ideals generated by trivial paths KQe_i.

This is known by G.M.Bergman, Modules over coproducts of rings, 1974. For simplicity we prove that every left ideal in a free algebra is a free module. We use a Gröbner basis type argument. Gröbner bases are really for ideals in polynomial rings. They have been adapted to ideals in free algebras, e.g. by T. Mora, An introduction to commutative and noncommutative Gröbner bases, 1994. The generalization to left ideals in path algebras is straightforward. The generalization from left ideals to submodules of free modules can be incorporated by another refinement of the Gröbner basis argument, using a well-ordering on the free basis of the module. Alternatively use a theorem of Kaplansky, see Theorem I.5.3 in Cartan and Eilenberg, Homological algebra, 1956.

Let $R = K\langle X \rangle$ and let W be the set of words involving letters in X. Choose a well-ordering on X, and give W the *length-lexicographic ordering*, so $x_1 \dots x_n < x'_1 \dots x'_m$ if
- $n < m$, or
- $n = m$ and there is $1 \le k \le n$ with $x_i = x'_i$ for $i < k$ and $x_k < x'_k$.

Lemma 1. This is a well-ordering on W and if $u \le u'$ and $w \le w'$ then $uw \le u'w'$, with equality only if $u = u'$ and $w = w'$.

Proof. Straightforward.

Definition. If $0 \neq r \in R$, we define $\mathrm{tip}(r)$ to be the maximal word involved in r with a non-zero coefficient.

Lemma 2. If $0 \neq r, q \in R$, then $rq \neq 0$, and $\mathrm{tip}(rq) = \mathrm{tip}(r)\,\mathrm{tip}(q)$.

Proof. The product rq is a linear combination of products uw with u involved in r and w involved in q. By Lemma 1, the maximal such product is given by taking u and w maximal, and nothing else can cancel with it, so $rq \neq 0$.

Theorem. Any left ideal I in R is a free left R-module. Explicitly, let

$$T = \{\mathrm{tip}(q) : 0 \neq q \in I\} \text{ and } S = T \setminus \{wt : w \text{ non-trivial word}, t \in T\},$$

and for each $s \in S$ choose $0 \neq q_s \in I$ with $\mathrm{tip}(q_s) = s$. Then the map $\theta : R^{(S)} \to I$, $\theta((r_s)) = \sum_s r_s q_s$ is an isomorphism.

Proof. (Injective) Say $\sum_s r_s q_s = 0$ with not all $r_s = 0$. Then not all $r_s q_s = 0$ by Lemma 2. In order for the sum to be zero, there must be distinct $s, s' \in S$ with $\mathrm{tip}(r_s q_s) = \mathrm{tip}(r_{s'} q_{s'})$. Thus by Lemma 2, $\mathrm{tip}(r_s)\,\mathrm{tip}(q_s) = \mathrm{tip}(r_{s'})\,\mathrm{tip}(q_{s'})$. But then, swapping s and s' if necessary, to ensure that $\mathrm{length}\,\mathrm{tip}(r_s) \leq \mathrm{length}\,\mathrm{tip}(r_{s'})$, we must have $\mathrm{tip}(r_{s'}) = w\,\mathrm{tip}(r_s)$ and $\mathrm{tip}(q_s) = w\,\mathrm{tip}(q_{s'})$ for some word w. Then w must be a trivial path since $\mathrm{tip}(q_s) = s \in S$. But then $s = s'$, a contradiction.

(Surjective) If not, by the well-ordering we can choose $a \in I \setminus \mathrm{Im}\,\theta$ with $t = \mathrm{tip}(a)$ minimal. Amongst decompositions $t = ws$ of t as a product of words with $s \in T$ (for example $t = 1t$), the one with s of minimal length must have $s \in S$. Then a and wq_s both have tip t, so there is a non-zero scalar $\lambda \in K$ such that $a' = a - \lambda w q_s$ is either zero or has tip smaller than t. But since $\lambda w q_s \in \mathrm{Im}(\theta)$ we have $a' \in I \setminus \mathrm{Im}\,\theta$, so it can't be zero, and it can't have tip smaller than t.

Example. The ideal (x) in the free algebra $R = K\langle x, y\rangle$ has as K-basis the words which involve x at least once, and any such word can be written uniquely as uxy^n for some word u and some n. It follows that

$$(x) = \bigoplus_{n=0}^{\infty} Rxy^n.$$

Thus (x) is isomorphic as a left R-module to the free module $R^{(\mathbb{N})}$. In this case T consists of all words involving x at least once and $S = \{xy^n : n \in \mathbb{N}\}$.

2.7 Power series

The formal power series path algebra $K\langle\langle Q\rangle\rangle$ of a quiver Q (say finite) is the algebra whose elements are formal sums

$$\sum_{p \text{ path}} a_p p$$

with $a_p \in K$, but with no requirement that only finitely many are non-zero. Multiplication makes sense because any path p can be obtained as a product qq' in only finitely many ways.

In the special case of a loop one gets the formal power series algebra $K[[x]]$. The element $1+x$ is invertible in $K[[x]]$ since it has inverse $1-x+x^2-x^3+\dots$.

Lemma. An element of $K\langle\langle Q\rangle\rangle$ is invertible if and only if the coefficient of each trivial path e_i is invertible in K.

Proof. If the condition holds one can multiply first by a linear combination of trivial paths to get it in the form $1+x$ with x only involving paths of length ≥ 1. Then the expression $1-x+x^2-x^3+\dots$ makes sense in $K\langle\langle Q\rangle\rangle$, and is an inverse.

Now suppose K is a field. We say that a KQ-module M is *nilpotent* if there is some N such that $pM = 0$ for any path p of length $> N$.

Proposition. If K is a field, then finite-dimensional $K\langle\langle Q\rangle\rangle$-module correspond exactly to finite dimensional nilpotent modules for KQ.

Proof. We consider restriction via the homomorphism $KQ \to K\langle\langle Q\rangle\rangle$. Clearly any nilpotent KQ-module is the restriction of a $K\langle\langle Q\rangle\rangle$-module. Conversely suppose that M is a finite-dimensional $K\langle\langle Q\rangle\rangle$-module whose restriction to KQ is not nilpotent. By finite-dimensionality, there is some $m \in M$ such that $pm \neq 0$ for arbitrarily long paths p. By König's Lemma (using that Q is finite) we can find an infinite sequence of arrows $a_1, a_2, \dots$ such that $m_n = a_n \dots a_2 a_1 m \neq 0$ for all n. By finite-dimensionality there is a linear relation among the m_n. Thus we can write some m_i in terms of m_j with $j > i$. Then $m_i = x m_i$ where x is a linear combination of paths of length > 1. But then $1 - x$ is invertible, so $m_i = 0$, a contradiction.

2.8 Algebras given by quivers with relations

We say that an algebra is given by a *quiver with relations* if it is given as $A = KQ/(S)$ where $S \subseteq \bigcup_{i,j} e_j KQe_i$. Any ideal can be generated in this way, for if $x \in KQ$ then $x = \sum_{i,j} e_j xe_i$.

Example $K(\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet)/(ba)$.

Given a representation V of Q and an element $s \in e_j KQe_i$ one gets a map $V_i \to V_j$. We say that V *satisfies the relation* s if this map is zero. If $A = KQ/(S)$ then A-modules correspond to representations of Q satisfying all the relations in S.

Given a quiver Q, we denote by $\overline{Q}$ the double of Q, obtained by adjoining an inverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q.

If Q has finitely many vertices, the *preprojective algebra* for Q is

$$\Pi(Q) = K\overline{Q}/(c)$$

where $c = \sum_{a \in Q}(aa^* - a^*a)$.

Observe that $e_i ce_j = 0$ if $i \neq j$, so $\Pi(Q)$ is given by the relations

$$c_i = e_i ce_i = \sum_{a \in Q, h(a)=i} aa^* - \sum_{a \in Q, t(a)=i} a^*a$$

$(i \in Q_0)$.

Examples. For $Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ the relations are

$$aa^* = 0, bb^* = a^*a, b^*b = 0.$$

If Q is a loop x, then $\Pi(Q) = K\langle x, x^* \rangle/(xx^* - x^*x) \cong K[x, x^*]$, a polynomial ring in two variables.

Given $\lambda \in K^{Q_0}$ there is also the *deformed preprojective algebra*

$$\Pi^\lambda(Q) = K\overline{Q}/(c - \sum_{i \in Q_0} \lambda_i e_i).$$

Theorem (A. Mellit, Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation, Algebra Discrete Math. 2004.)

If $P_1(x), \ldots, P_k(x) \in K[x]$ are monic polynomials of degree $d_i \geq 2$ which are products of linear factors, then

$$K\langle f_1, \ldots, f_k\rangle/(f_1 + \cdots + f_k, P_1(f_1), \ldots, P_k(f_k)) \cong e_0\Pi^\lambda(Q)e_0$$

for some λ, where Q is star-shaped with central vertex 0 and arms $1, \ldots, k$, with vertices $(i, 1), (i, 2), \ldots, (i, d_i - 1)$ going outwards on arm i and arrows $a_{i,1}, \ldots, a_{i,d_i-1}$ pointing inwards.

Proof. We do the special case $P_i(x) = x^{d_i}$, in which case $\lambda = 0$. Let the algebra on the left be A and the one on the right be $B = e_0\Pi(Q)e_0$. Now B is spanned by the paths in $\overline{Q}$ which start and end at vertex 0. If vertex (i, j) is the furthest out that a path reaches on arm i, then it must involve $a_{ij}a^*_{ij}$, and if $j > 1$, the relation

$$a_{ij}a^*_{ij} = a^*_{i,j-1}a_{i,j-1}$$

shows that this path is equal in B to a linear combination of paths which only reach $(i, j - 1)$. Repeating, we see that B is spanned by paths which only reach out to vertices $(i, 1)$. Thus we get a surjective map

$$K\langle f_1, \ldots f_k\rangle \to B$$

sending each f_i to $a_{i1}a^*_{i1}$. It descends to a surjective map $\theta : A \to B$ since it sends $f_1 + \cdots + f_k$ to 0 and $f_i^{d_i}$ is sent to

$$\begin{aligned}
(a_{i1}a^*_{i1})^{d_i} &= a_{i1}(a^*_{i1}a_{i1})^{d_i-1}a_{i1} \\
&= a_{i1}(a_{i2}a^*_{i2})^{d_i-1}a^*_{i1} \\
&= a_{i1}a_{i2}(a^*_{i2}a_{i2})^{d_i-2}a^*_{i2}a^*i1 \\
&= \cdots = \\
&= a_{i1}a_{i2}\ldots a_{i,d_i-1}(a^*_{i,d_i-1}a_{i,d_i-1})a^*_{i,d_i-1}\ldots a^*_{i1} = 0
\end{aligned}$$

since $a^*_{i,d_i-1}a_{i,d_i-1} = 0$.

To show that θ is an isomorphism it suffices to show that any A-module can be obtained by restriction from a B-module, for if $a \in \operatorname{Ker}\theta$ and $M = {}_\theta N$, then $aM = \theta(a)N = 0$. Thus if A can be obtained from a B-module by restriction, then $aA = 0$, so $a = 0$.

Thus take an A-module M. We construct a representation of $\overline{Q}$ by defining $V_0 = M$ and $V_{(i,j)} = f_i^j M$. with a_{ij} the inclusion map, and a^*_{ij} multiplication by f_i. This is easily seen to satisfy the preprojective relations, so it becomes a module for $\Pi(Q)$. Then $e_0V = M$ becomes a module for $e_0\Pi(Q)e_0 = B$. Clearly its restriction via θ is the original A-module M.

2.9 Diamond lemma

This is: G.M.Bergman, The diamond lemma for ring theory, Advances in Mathematics 1978. Notes. (1) Is it useful to generalize it from free algebras to path algebras? (2) It looks like Gröbner basis theory.

We consider the algebra $A = K\langle X\rangle/(S)$ generated by a set X of indeterminates, subject to a set S of relations of the form

$$w_j = s_j \quad (j \in J)$$

where w_j is a word in the indeterminates and $s_j \in K\langle X\rangle$.

We assume that there is well-ordering on the set W of words in the indeterminates with the following properties. (More generally one can use a partial ordering with the descending chain condition.)

(i) (semigroup ordering) if $w < w'$ then $uwv < uw'v$ for all $w, w', u, v \in W$.

(ii) (compatibility with the relations) each s_j only involves words w with $w < w_j$.

In practice we normally use the length-lexicographic ordering coming from a well-ordering on the set X. Thus $w < w'$ if
- length $w <$ length w', or
- length $w =$ length w' and $w < w'$ in the lexicographic ordering.

Example. Consider the algebra A generated by x, y subject to

$$x^2 = x, \quad y^2 = 1, \quad yx = 1 - xy$$

(so $w_1 = xx$, $s_1 = x$, etc.) One has compatibility for the length-lexicographic ordering with the usual alphabet ordering of x, y.

Definition. The *reduction* with respect to words u, v and relation $w_j = s_j$, is the K-linear map $K\langle X\rangle \to K\langle X\rangle$ sending uw_jv to us_jv and sending any other word to itself. We write $f \rightsquigarrow g$ to indicate that g is obtained from f by applying reduction with respect to some u, v and $w_j = s_j$.

Example. For the algebra with relations $x^2 = x$, $y^2 = 1$, $yx = 1 - xy$.
- The element $x^2 + xy^2$ can be reduced to $x^2 + x$ and then to $x + x = 2x$, or to $x + xy^2$ and then to $2x$ as well.
- The element yx^2 can be reduced to yx and then to $1 - xy$, or alternatively to $(1 - xy)x = x - xyx$, and then to $x - x(1 - xy) = x^2y$, and then to xy.

Lemma 1. If $f \rightsquigarrow g$ and u', v' are words, then $u'fv' \rightsquigarrow u'gv'$.

Proof. If g is the reduction of f with respect to u, v and the relations $w_j = s_j$, then $u'gv'$ is the reduction of $u'fv'$ with respect to $u'u$, vv' and the relation $w_j = s_j$.

Definition. We say that f is *irreducible* if $f \rightsquigarrow g$ implies $g = f$. It is equivalent that no word involved in f can be written as a product uw_jv.

Lemma 2. Any $f \in K\langle X\rangle$ can be reduced by a finite sequence of reductions to an irreducible element.

Proof. Any $f \in K\langle X\rangle$ which is not irreducible involves words of the form uw_jv. Amongst all words of this form involved in f, let $\mathrm{tip}(f)$ be the maximal one. Consider the set of tips of elements which cannot be reduced to an irreducible element. For a contradiction assume this set is non-empty. Then by well-ordering it contains a minimal element. Say it is $\mathrm{tip}(f) = w = uw_jv$. Writing $f = \lambda uw_jv + f'$ where $\lambda \in K$ and f' only involving words different from uw_jv, we have $f \rightsquigarrow g$ where $g = \lambda us_jv + f'$. By the properties of the ordering, us_jv only involves words which are less than $uw_jv = w$, so $\mathrm{tip}(g) < w$. Thus by minimality, g can be reduced to an irreducible element, hence so can f. Contradiction.

Definition. We say that f is *reduction-unique* if there is a unique irreducible element which can be obtained from f by a sequence of reductions. If so, the irreducible element is denoted $r(f)$.

Lemma 3. The set of reduction-unique elements is a subspace of $K\langle X\rangle$, and the assignment $f \mapsto r(f)$ is a K-linear map.

Proof. Consider a linear combination $\lambda f + \mu g$ where f, g are reduction-unique and $\lambda, \mu \in K$. Suppose there is a sequence of reductions (labelled (1))

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h$$

with h irreducible. Let a be the element obtained by applying the same reductions to f. By the Lemma 2, a can be reduced by some sequence of reductions (labelled (2)) to an irreducible element. Since f is reduction-unique, this irreducible element must be $r(f)$.

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f).$$

Applying all these reductions to g we obtain elements b and c, and after applying more reductions (labelled (3)) we obtain an irreducible element, which must be $r(g)$.

$$g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} b \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} c \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(g).$$

But $h, r(f)$ are irreducible, so these extra reductions don't change them:

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} h,$$

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f) \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(f).$$

Now the reductions are linear maps, hence so is a composition of reductions, so $h = \lambda r(f) + \mu r(g)$. This shows that $\lambda f + \mu g$ is reduction-unique and that $r(\lambda f + \mu g) = \lambda r(f) + \mu r(g)$.

Definition. We say that two reductions of f, say $f \rightsquigarrow g$ and $f \rightsquigarrow h$, satisfy the *diamond condition* if there exist sequences of reductions starting with g and h, which lead to the same element, $g \rightsquigarrow \cdots \rightsquigarrow k$, $h \rightsquigarrow \cdots \rightsquigarrow k$. (You can draw this as a diamond.)

In particular we are interested in this in the following two cases:

An *overlap ambiguity* is a word which can be written as $w_i v$ and also as uw_j for some i, j and some words $u, v \neq 1$, so that w_i and w_j overlap. There are reductions $f \rightsquigarrow s_i v$ and $f \rightsquigarrow us_j$.

An *inclusion ambiguity* is a word which can be written as w_i and as $uw_j v$ for some $i \neq j$ and some u, v. There are reductions $f \rightsquigarrow s_i$ and $f \rightsquigarrow us_j w$.

Examples. (1) For the relations $x^2 = x$, $y^2 = 1$, $yx = 1 - xy$ the ambiguities are:

$$(xx)x = x(xx), \quad (yy)y = y(yy), \quad (yy)x = y(yx), \quad (yx)x = y(xx).$$

The diamond condition fails for the ambiguity $(yx)x = y(xx)$.

(2) For the relations $x^2 = x$, $y^2 = 1$, $yx = y - xy$ the ambiguities are:

$$(xx)x = x(xx), \quad (yy)y = y(yy), \quad (yy)x = y(yx), \quad (yx)x = y(xx).$$

Does the diamond condition hold for these?
$(xx)x \rightsquigarrow xx \rightsquigarrow x$ and $x(xx) \rightsquigarrow xx \rightsquigarrow x$. Yes.

$(yy)y \rightsquigarrow 1y = y$ and $y(yy) \rightsquigarrow y1 = y$. Yes.
$(yy)x \rightsquigarrow 1x = x$ and $y(yx) \rightsquigarrow y(y - xy) = y^2 - yxy = y^2 - (yx)y \rightsquigarrow y^2 - (y - xy)y = xyy = x(yy) \rightsquigarrow x1 = x$. Yes.
$(yx)x \rightsquigarrow (y - xy)x = yx - xyx \rightsquigarrow yx - x(y - xy) = yx - xy + xxy \rightsquigarrow yx - xy + xy = yx \rightsquigarrow \dots$ and $y(xx) \rightsquigarrow yx \rightsquigarrow \dots$. Yes.

Diamond Lemma. The following conditions are equivalent:
(a) The diamond condition holds for all overlap and inclusion ambiguities.
(b) Every element of $K\langle X\rangle$ is reduction-unique.
In this case the algebra $A = K\langle X\rangle/(S)$ has K-basis given by the irreducible words, with multiplication given by $f.g = r(fg)$.

Example. Consider our example of A generated by x, y subject to $x^2 = x$, $y^2 = 1$, $yx = y - xy$. The irreducible words $1, x, y, xy$ form a K-basis of A with multiplication table

	1	x	y	xy
1	1	x	y	xy
x	x	x	xy	xy
y	y	$y - xy$	1	$1 - x$
xy	xy	0	x	0

For example $y(xy) = (yx)y \rightsquigarrow (y-xy)y = yy - xyy \rightsquigarrow 1 - xyy \rightsquigarrow 1 - x$, and $(xy)(xy) = x(yx)y \rightsquigarrow x(y - xy)y = xyy - xxyy \rightsquigarrow x - xxyy \rightsquigarrow x - xyy \rightsquigarrow x - x = 0$.

Example. (P. Shaw, Appendix A, Generalisations of Preprojective algebras, Ph. D. thesis, Leeds, 2005. Available from homepage of WCB.) The algebra with generators b, c and relations $b^3 = 0$, $c^2 = 0$ and $cbcb = cb^2c - bcbc$ fails the diamond condition for the overlap $cbc(b^3) = (cbcb)b^2$. But this calculation shows that the equation $cb^2cb^2 = bcb^2cb - b^2cb^2c$ holds in the algebra, and if you add this as a relation, the diamond condition holds.

Stupid example. Suppose R is a K-algebra which is free as a K-module, with basis $\{1\} \cup \{r_i : i \in I\}$. Then $r_ir_j = \sum_k \lambda_{ijk}r_k + \mu_{ij}1$ for some $\lambda_{ijk}, \mu_{ij} \in K$. We get a map

$$K\langle x_i : i \in I\rangle/(x_ix_j - \sum_k \lambda_{ijk}x_k - \mu_{ij}1) \to R$$

sending each x_i to r_i which is onto. The associative law ensures that the ambiguities $(x_ix_j)x_k = x_i(x_jx_k)$ satisfy the diamond condition. Thus by the Diamond Lemma the left hand side is the free K-module with basis $\{1\} \cup \{x_i : i \in I\}$. Thus the map is an isomorphism.

Example. Suppose R and S are K-algebras, free as K-modules with bases $\{1\} \cup \{r_i\}$ and $\{1\} \cup \{s_j\}$. Then $R * S$ is free as a K-module with basis the alternating products of the r_i and s_j.

$$1, r_i, s_j, r_i s_j, s_j r_i, r_i s_j r_{i'}, \ldots.$$

Namely $R*S$ is generated by x_i, y_j subject to relations coming from products $r_i r_{i'}$ and $s_j s_{j'}$, and there are no ambiguities involving both the r_i and the s_j.

Proof of Diamond Lemma. (b)$\Rightarrow$(a) is trivial, so we prove (a)$\Rightarrow$(b). Since the reduction-unique elements form a subspace, it suffices to show that every *word* is reduction-unique. For a contradiction, suppose not. Then there is a minimal word w which is not reduction-unique. Let $f = w$. Suppose that f reduces under some sequence of reductions to g, and under another sequence of reductions to h, with g, h irreducible. We want to prove that $g = h$, giving a contradiction.

Let the elements obtained in each case by applying one reduction be f_1 and g_1. Thus

$$f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g, \qquad f \rightsquigarrow h_1 \rightsquigarrow \cdots \rightsquigarrow h.$$

By the properties of the ordering, g_1 and h_1 are linear combinations of words which are less than w, so by minimality they are reduction-unique. Thus $g = r(g_1)$ and $h = r(h_1)$.

It suffices to prove that the reductions $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$ satisfy the diamond condition, for if there are sequences of reductions $g_1 \rightsquigarrow \cdots \rightsquigarrow k$ and $h_1 \rightsquigarrow \cdots \rightsquigarrow k$, combining them with a sequence of reductions $k \rightsquigarrow \cdots \rightsquigarrow r(k)$, we have $g = r(g_1) = r(k) = r(h_1) = h$.

Thus we need to check the diamond condition for $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$. Recall that $f = w$, so these reductions are given by subwords of w of the form w_i and w_j. There are two cases:

(i) If these words overlap, or one contains the other, the diamond condition follows from the corresponding overlap or inclusion ambiguity. For example w might be of the form $u'w_i v v' = u' u w_j v'$ where $w_i v = u w_j$ is an overlap ambiguity and u', v' are words. Now condition (a) says that the reductions $w_i v \rightsquigarrow s_i v$ and $u w_j \rightsquigarrow u s_j$ can be completed to a diamond, say by sequences of reductions $s_i v \rightsquigarrow \cdots \rightsquigarrow k$ and $u s_j \rightsquigarrow \cdots \rightsquigarrow k$. Then Lemma 1 shows that the two reductions of w, which are $w = u' w_i v v' \rightsquigarrow u' s_i v v'$ and $w = u' u w_j v' \rightsquigarrow u' v s_j v'$, can be completed to a diamond by reductions leading to $u' k v'$.

(ii) Otherwise w is of the form uw_ivw_jz for some words u, v, z, and $g_1 = us_ivw_jz$ and $h_1 = uw_ivs_jz$ (or vice versa). Writing s_i as a linear combination of words, $s_i = \lambda t + \lambda' t' + \dots$, we have

$$r(g_1) = r(us_ivw_jz) = \lambda r(utvw_jz) + \lambda' r(ut'vw_jz) + \dots \ .$$

Reducing each word on the right hand side using the relation $w_j = s_j$, we have $utvw_jz \rightsquigarrow utvs_jz$, and $ut'vw_jz \rightsquigarrow ut'vs_jz$, and so on, so

$$r(g_1) = \lambda r(utvs_jz) + \lambda' r(ut'vs_jz) + \dots \ .$$

Collecting terms, this gives $r(g_1) = r(us_ivs_jz)$. Similarly, writing s_j as a linear combination of words, we have $r(h_1) = r(us_ivs_jz)$. Thus $r(h_1) = r(g_1)$, so the diamond condition holds.

For the last part we show that $r(f) = 0$ if and only if $f \in (S)$. If $f \rightsquigarrow g$ then $f - g \in (S)$, so $f - r(f) \in (S)$ giving one direction. For the other, (S) is spanned by words of the form $u(w_j - s_j)v$, and $uw_jv \rightsquigarrow us_jv$ so $r(uw_jv) = r(us_jv)$, so $r(u(w_j - s_j)v) = 0$.

Thus r defines a K-module isomorphism from $A/(S)$ to the K-span of the irreducible words.

Noetherian property (add to section 1.8)

Definition. A module M is *noetherian* if it satisfies the following equivalent conditions
(i) Any ascending chain of submodules $M_1 \subseteq M_2 \subseteq \dots$ becomes stationary, that is, for some n one has $M_n = M_{n+1} = \dots$.
(ii) Any non-empty set of submodules of M has a maximal element.
(iii) Any submodule of M is finitely generated.

Proof of equivalence. (i) $\implies$ (ii) because otherwise we choose M_1 to be any of the submodules, and iteratively, since M_i isn't maximal, we can choose $M_i < M_{i+1}$. This gives an ascending chain which doesn't become stationary.

(ii) $\implies$ (iii). Let N be a submodule, let L be a maximal element of the set of finitely generated submodules of N, and $n \in N$. Then $L + Rn$ is also a finitely generated submodule of N, so equal to L by maximality. Thus $n \in L$, so $N = L$, so it is finitely generated.

(iii) $\implies$ (i) Choose a finite set of generators for $N = \bigcup_i M_i$. Some M_i must contain each of these generators, so be equal to N. Thus $M_i = M_{i+1} = \dots$.

Lemma. If L is a submodule of M then M is noetherian if and only if L and M/L are noetherian. If $M = L + N$ and L and N are noetherian, then so is M.

Proof. If M is noetherian then clearly L and M/L are noetherian. Now suppose $M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of submodules of M. If L and M/L are noetherian, then $L \cap M_i = L \cap M_{i+1} = \dots$ and $(L + M_i)/L = (L + M_{i+1})/L = \dots$ for some i, so $L + M_i = L + M_{i+1} = \dots$. Now if $m \in M_{i+1}$, then $m = \ell + m'$ with $\ell \in L$ and $m' \in M_i$. Then $\ell = m - m' \in L \cap M_{i+1} = L \cap M_i$, so $m \in M_i$. Thus $M_i = M_{i+1} = \dots$. For the last part, use that $(L + N)/L \cong N/(L \cong N)$.

Definition. A ring R is *left noetherian* if it satisfies the following equivalent conditions
(a) ${}_R R$ is noetherian (so R is has the ascending chain condition on left ideals, or any left ideal in R is finitely generated).
(b) Any finitely generated left R-module is noetherian.

Proof of equivalence. For (a) $\implies$ (b), any finitely generated module is a quotient of a finite direct sum of copies of R.

Definition. A ring is *noetherian* if it is left noetherian and right noetherian (i.e. noetherian for right modules, or equivalently R^{op} is left noetherian).

Remarks. (1) Division rings and principal ideal domains such as $\mathbb{Z}$ are noetherian. Hilbert's Basis Theorem says that if K is noetherian, then so is $K[x]$.

(2) If $R \to S$ is a ring homomorphism and M is an S-module such that ${}_R M$ is noetherian, then M is noetherian. Thus if ${}_R S$ is a finitely generated R-module, and R is left noetherian, then so is S. Thus, for example, if R is noetherian, so is $M_n(R)$.

(3) If K is noetherian and R is a finitely generated K-algebra, then R is noetherian, as it is a quotient of a polynomial ring $K[x_1, \dots, x_n]$. This is not true for R non-commutative.

Artin-Tate Lemma. Let A be a finitely generated K-algebra with K noetherian, and let Z be a K-subalgebra of $Z(A)$. If A is finitely generated as a Z-module, then Z is finitely generated as a K-algebra, hence Z and A are noetherian rings.

Proof. Let $A = K\langle a_1, \dots, a_n\rangle = Zb_1 + \dots + Zb_m$. Let $a_i = \sum_j z_{ij} b_j$ and $b_i b_j = \sum_k z_{ijk} b_k$. Let $Z' = K[z_{ij}, z_{ijk}]$. This is noetherian, and $A = Z'b_1 + \dots + Z'b_m$, so it is a finitely generated Z'-module. Then $Z \subseteq A$ is a finitely generated Z'-module. In particular it is finitely generated as a Z'-algebra, and hence also as a K-algebra.

2.10 Skew polynomial rings

If R is a K-algebra and M is an R-R-bimodule, a *(K-)derivation* $d : R \to M$ is a mapping of K-modules which satisfies $d(rr') = rd(r') + d(r)r'$ for all $r, r' \in R$.

Observe that for $d(1) = d(1) + d(1)$ so $d(1) = 0$. Also, for $\lambda \in K$, $d(\lambda 1) = \lambda d(1) = 0$ by linearity.

We write $\mathrm{Der}_K(R, M)$ for the set of derivations. It is naturally a K-module.

Examples. (i) For any $m \in M$ the map $r \mapsto rm - mr$ is a derivation, called an *inner derivation.*

(ii) The map $\frac{d}{dx} : K[x] \to K[x]$,

$$\frac{d}{dx}(\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n) = \lambda_1 + 2\lambda_2 x + \cdots + n\lambda_n x^{n-1}.$$

is a derivation. More generally $\partial/\partial x_i : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$.

Definition. If R is a K-algebra and $\sigma, \delta : R \to R$, we write $R[x; \sigma, \delta]$ for a K-algebra containing R as a subalgebra, which consists of all polynomials

$$r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n$$

with $r_i \in R$, with the natural addition and a multiplication satisfying

$$xr = \sigma(r)x + \delta(r)$$

for $r \in R$. If such a ring exists, the multiplication is uniquely determined. It is called a *skew polynomial ring* or *Ore extension of R.*

Theorem. $R[x; \sigma, \delta]$ exists if and only if σ is a K-algebra endomorphism of R and $\delta \in \mathrm{Der}_K(R, {}_\sigma R)$. [One says δ is a σ-derivation of R.]

Proof. If such an algebra exists, then clearly $\sigma, \delta \in \operatorname{End}_K(R)$ and

$$\begin{aligned}\sigma(rr')x + \delta(rr') &= x(rr') \\ &= (xr)r' \\ &= (\sigma(r)x + \delta(r))r' \\ &= \sigma(r)(xr') + \delta(r)r' \\ &= \sigma(r)(\sigma(r')x + \delta(r')) + \delta(r)r'.\end{aligned}$$

Thus $\sigma(rr') = \sigma(r)\sigma(r')$ and $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$. For the converse, identify R with the subalgebra of $E = \operatorname{End}_K(R^{\mathbb{N}})$, with $r \in R$ corresponding to left multiplication by r. Define $X \in E$ by

$$(Xs)_i = \sigma(s_{i-1}) + \delta(s_i)$$

for $s = (s_0, s_1, \dots) \in R^{\mathbb{N}}$, where $s_{-1} = 0$. Then

$$\begin{aligned}(X(rs))_i &= \sigma(rs_{i-1}) + \delta(rs_i) \\ &= \sigma(r)\sigma(s_{i-1}) + \sigma(r)\delta(s_i) + \delta(r)s_i \\ &= \sigma(r)X(s)_i + \delta(r)s_i.\end{aligned}$$

Thus $X(rs) = \sigma(r)X(s) + \delta(r)s$, so $Xr = \sigma(r)X + \delta(r)$. Observe also that the coefficients of a polynomial $f = \sum r_i X$ are uniquely determined since $X(e_i) = e_{i+1}$ so $f(e_0) = (r_0, r_1, \dots)$. It follows that the subalgebra of E generated by the $\hat{r}$ and X is a suitable algebra.

Special cases. If $\delta = 0$ the skew polynomial ring is isomorphic to $T_R(R_\sigma)$ and we denote it $R[x;\sigma]$. If $\sigma = 1$ denote it $R[x;\delta]$.

Properties. Let $S = R[x;\sigma,\delta]$.

(1) $x^n r = \sigma^n(r)x^n +$ lower degree terms. Proof by induction on n.

(2) If R is a domain (no zero-divisors) and σ is injective then the degree of a product of two polynomials is equal to the sum of their degrees. In particular S is a domain. Proof. $(r_0 + \cdots + r_n x^n)(s_0 + \cdots + s_m x^m) = r_n \sigma^n(s_m)x^{n+m} +$ lower degree terms.

(3) If R is a division ring then σ is automatically injective and S is a principal left ideal domain. Proof. Suppose I is a non-zero left ideal. It contains a non-zero polynomial $f(x)$ of least degree d, which we may suppose to be monic. If $g(x)$ is a polynomial with leading term rx^{d+n}, then $g(x) - rx^n f(x)$ has strictly smaller degree. An induction then shows that $I = Sf(x)$.

(4) If σ is an automorphism then $rx = x\sigma^{-1}(r) - \delta(\sigma^{-1}(r))$, so $S^{op} = R^{op}[x; \sigma^{-1}, -\delta\sigma^{-1}]$.

Hilbert's Basis Theorem. Assume σ is an automorphism. If R is left (respectively right) noetherian, then so is $R[x; \sigma, \delta]$.

Proof. By the observation above, it suffices to prove this for right noetherian. Let J be a right ideal in S which is not finitely generated, and take a polynomial f_1 of least degree in J. By induction, if we have found $f_1, \ldots, f_k \in J$, then since J is not finitely generated $J \setminus \sum_{i=1}^k f_i S \neq \emptyset$, and we take f_{k+1} of least possible degree. We obtain an infinite sequence of polynomials $f_1, f_2, \dots \in J$. Let f_i have leading term $r_i x^{n_i}$. By construction $n_1 \leq n_2 \leq \ldots$. The chain

$$r_1 R \subseteq r_1 R + r_2 R \subseteq \ldots$$

must become stationary, so some $r_{k+1} = \sum_{i=1}^k r_i r_i'$ with $r_i' \in R$. Then

$$f_{k+1} - \sum_{i=1}^{k} f_i \sigma^{-n_i}(r_i') x^{n_{k+1} - n_i} \in J \setminus \sum_{i=1}^{k} f_i S$$

and it has degree $< n_{k+1}$, contradicting the choice of f_{k+1}.

2.11 Weyl algebra

Definition. The *nth Weyl algebra* $A_n(K)$ is the K-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the relations

$$y_i x_j - x_j y_i = \delta_{ij}, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i$$

By the Diamond Lemma it has K-basis the elements $x^\alpha y^\beta$ where $\alpha, \beta \in \mathbb{N}^n$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $y^\beta = y_1^{\beta_1} \ldots y_n^{\beta_n}$.

Lemma 1. The derivations $\partial/\partial x_j$ and $\partial/\partial y_j$ can be extended to inner derivations of $A_n(K)$ by defining

$$(\partial/\partial x_j)(a) = y_j a - a y_j, \quad (\partial/\partial y_j)(a) = a x_j - x_j a.$$

They satisfy

$$\frac{\partial}{\partial x_j}\left(x^\alpha y^\beta\right) = \alpha_j x^{\alpha - e_j} y^\beta, \quad \frac{\partial}{\partial y_j}\left(x^\alpha y^\beta\right) = \beta_j x^\alpha y^{\beta - e_j}.$$

Proof. Use that $(\partial/\partial x_j)(y_i) = 0$ and $(\partial/\partial x_j)(x_i) = \delta_{ij}$ and similarly for $\partial/\partial y_j$.

Proposition. $A_n(K)$ is isomorphic to the iterated skew polynomial algebra

$$K[x_1, \dots, x_n][y_1; \partial/\partial x_1][y_2; \partial/\partial x_2] \dots [y_n; \partial/\partial x_n].$$

In particular, if K is a noetherian domain, so is $A_n(K)$. Moreover, if K is a field of characteristic 0 then $A_n(K)$ is a simple ring (no non-trivial ideals).

Proof. Observe that $\partial/\partial x_j$ defines a derivation of the subalgebra

$$Q_j = K\langle x_1, \dots, x_n, y_1, \dots, y_{j-1}\rangle \subseteq A_n(K),$$

and $Q_{j+1} \cong Q_j[y_j; \partial/\partial x_j]$.

Suppose I is a non-zero ideal. Choose $0 \neq c \in I$. Choosing an element $x^\alpha y^\beta$ involved in c with non-zero coefficient λ, with $|\alpha| + |\beta|$ maximal (where $|\alpha| = \alpha_1 + \dots + \alpha_n$), and applying $(\partial/\partial x)^\alpha (\partial/\partial y)^\beta$, we get

$$\lambda \prod \alpha_i! \prod \beta_i! \in I.$$

If K is a field of characteristic zero, then $1 \in I$, so $I = A_n(K)$.

Definition. Let R be a commutative K-algebra. We define the set of differential operators of order $\leq n$ to be

$$D_{\leq n}(R) = \{P \in \mathrm{End}_K(R) : [r, P] \in D_{\leq n-1}(R) \text{ for all } r \in R\}$$

where by convention $D_{\leq -1}(R) = 0$. Here we identify $r \in R$ with the multiplication operator and the commutator is $[a, b] = ab - ba$. The *ring of differential operators on* R is

$$D(R) = \bigcup_n D_{\leq n}(R).$$

Note that the commutator satisfies $[a, a] = 0$, $[a, b] = -[b, a]$ and the *Jacobi identity*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

Lemma 2. If $P \in D_{\leq m}(R)$ and $Q \in D_{\leq n}(R)$ then $PQ \in D_{\leq m+n}(R)$, so $D(R)$ is a subalgebra of $\mathrm{End}_K(R)$.

Proof. It is clear if $m = -1$ or $n = -1$. In general we use

$$[r, PQ] = [r, P]Q + P[r, Q]$$

and by induction the two terms on the right hand side are in $D_{\leq m+n-1}(R)$.

Proposition. $D_{\leq 0}(R) = R$ and $D_{\leq 1}(R) = R \oplus \mathrm{Der}_K(R, R)$.

Proof. $D_{\leq 0}(R) = \{P : [r, P] = \forall r\} = \mathrm{End}_R(R) = R$, with the identification above.

If $d \in \mathrm{Der}_K(R, R)$ then $(rd - dr)(s) = rd(s) - d(rs) = -d(r)s$, so $rd - dr = -d(r) \in D_{\leq 0}(R)$, so $d \in D_{\leq 1}(R)$.

Clearly $R \cap \mathrm{Der}_K(R, R) = 0$ since any derivation sends 1 to 0.

If $P \in D_{\leq 1}(R)$. Then for any $a \in R$ we have $[a, P] = r$ for some r. Moreover $r = r1 = aP(1) - P(a)$. Letting $d = P - P(1)$, we have $d(ab) = P(ab) - abP(1) = (Pa)(b) - (aP)(b) + aP(b) - abP(1) = -[a, P](b) + aP(b) - abP(1) = -rb + aP(b) - abP(1) = (P(a) - aP(1))b + a(P(b) - bP(1)) = d(a)b + ad(b)$, so d is a derivation, so $P \in R + \mathrm{Der}_K(R, R)$.

Theorem. If K is a field of characteristic 0, then there is an isomorphism $A_n(K) \to D(K[x_1, \ldots, x_n])$ sending x_i to x_i and y_i to $\partial/\partial x_i$. Moreover $D_{\leq k}(K[x_1, \ldots, x_n])$ has K-basis the operators $x^\alpha(\partial/\partial x)^\beta$ where $|\beta| \leq k$.

Proof. Let C_k be the K-span of the operators $x^\alpha\partial^\beta$ where $|\beta| \leq k$. It is easy to see that $C_k \subseteq D_{\leq k}$. Suppose by induction that $C_{k-1} = D_{\leq k-1}$.

We show that if $P_1, \ldots, P_n \in C_{k-1}$ satisfy $[P_i, x_j] = [P_j, x_i]$ for all i, j, then there is $Q \in C_k$ with $P_i = [Q, x_i]$. Suppose by induction on j we have $Q' \in C_k$ with $[Q', x_i] = P_i$ for all $i < j$. Then $[[Q', x_i], x_j] = [P_j, x_i]$ by the Jacobi identity, and this equals $[P_i, x_j]$, so letting $G = [Q', x_j] - P_j \in C_{k-1}$ we have $[G, x_i] = 0$ for $i < j$. Now

$$[x^\alpha\partial^\beta, x_i] = \beta_i x^\alpha(\partial/\partial x)^{\beta - e_i}$$

from which it follows that G can be written as a linear combination

$$G = \sum \lambda_{\alpha\beta} x^\alpha(\partial/\partial x)^\beta$$

where the sum is over all α and β with $|\beta| = k - 1$ and $\beta_i = 0$ for $i < j$. Let

$$Q'' = \sum \frac{\lambda_{\alpha\beta}}{\beta_j + 1} x^\alpha(\partial/\partial x)^{\beta + e_j} \in C_k.$$

Then $[Q'', x_i] = 0$ for $i < j$ by construction, and $[Q'', x_j] = G$, so $[Q' - Q'', x_i] = P_i$ for $i \leq j$, giving the induction.

Now if $P \in D_{\leq k}$, then $P_i = [P, x_i] \in D_{\leq k-1} = C_{k-1}$, and $[P_i, x_j] = [P_j, x_i]$ by the Jacobi identity. Thus there is $Q \in C_k$ with $[Q, x_i] = P_i = [P, x_i]$. Thus $[P - Q, x_i] = 0$ for all i. This $P - Q \in \mathrm{End}_{K[x_1,\dots,x_n]}(K[x_1, \dots, x_n]) = K[x_1, \dots, x_n]$. Thus $P \in C_k$. Thus $D_{\leq k} = C_k$.

Now we get a map $A_n(K) \to D(K[x_1, \dots, x_n])$ sending x_i to x_i and y_i to $\partial/\partial x_i$ Namely, for any $f \in K[x_1, \dots, x_n]$ we have $(\partial/\partial x_j)(x_i f) = \delta_{ij} f + x_j(\partial/\partial x_j)(f)$. This map is onto, and since $A_n(K)$ is simple, it is injective. Thus there are no relations between the monomials $x^\alpha \partial^\beta$.

Remark. For $R = K[x_1, \dots, x_n]$ this shows that $D(R)$ is the subalgebra of $\mathrm{End}_K(R)$ generated by R and $\mathrm{Der}_K(R, R)$. This is not true for all commutative K-algebras R.

Remark (Connection with differential equations). Let $A = D(K[x_1, \dots, x_n])$. Various rings of functions become A-modules, for example polynomial functions $K[x_1, \dots, x_n]$, or the smooth functions $C^\infty(U)$ on an open subset of $\mathbb{R}^n$ (if $K = \mathbb{R}$) or the holomorphic functions $\mathcal{O}(U)$ on an open subset of $\mathbb{C}^n$ (if $K = \mathbb{C}$). Let F be one of these A-modules of functions. Given $P = (P_{ij}) \in M_{m\times n}(A)$ we consider the system of differential equations

$$P \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = 0$$

with $f_i \in F$. The set of solutions is identified with $\mathrm{Hom}_A(M, F)$ where M is the cokernel of the map $A^m \to A^n$ given by right multiplication by P.

2.12 Group algebras

If G is a group, written multiplicatively, the *group algebra* KG is the free K-module with basis the elements of G, and with multiplication given by $g \cdot h = gh$ for $g, h \in G$. Thus a typical element of KG can be written as $\sum_{g\in G} a_g g$ with $a_g \in K$, almost all zero, and

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_h h) = \sum_{k\in G}(\sum_{g\in G} a_g b_{g^{-1}k})k.$$

If V is a KG-module, then one gets a group homomorphism $\rho : G \to \mathrm{GL}(V) = \mathrm{Aut}_K(V)$ via $\rho(g)(v) = gv$. Conversely if V is a K-module and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism, one gets a KG-module.

If G is finite, and $1/|G| \in K$ we define $e_G = \frac{1}{|G|}\sum_{g\in G} g$. It is an idempotent in KG. We have $he_G = e_G h = e_G$ for any $h \in G$, so $KGe_G = e_G KG = e_G KGe_G = Ke_G$.

Maschke's Theorem. If G is a finite group and K is a field of characteristic 0, then KG is semisimple. Thus if K is algebraically closed, $KG \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K)$.

Proof. Given KG-modules $N \subseteq M$, there exists a K-vector space complement $M = N \oplus U$, and let π be the projection onto N. Define $\theta : M \to M$ by $\theta(m) = (1/|G|)\sum_{g\in G} g\pi(g^{-1}m)$. Then θ is a KG-module map, $\operatorname{Im}\theta \subseteq N$ and $\theta(n) = n$ for $n \in N$. Thus $\theta^2 = \theta$ and $M = \operatorname{Im}\theta \oplus \operatorname{Ker}\theta = N \oplus \operatorname{Ker}\theta$.

Example. If G is cyclic of order n and K is a field containing $1/n$ and a primitive nth root of 1, then $KG \cong K \times \cdots \times K$. We have

$$\sum_{j=0}^{n-1} \epsilon^{ij} = \begin{cases} n & \text{(if } n \text{ divides } i\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

as in the second case its product with $\epsilon^i - 1$ is $\epsilon^{in} - 1 = 0$. Letting $G = \langle\sigma\rangle$ with $\sigma^n = 1$, it follows that the elements

$$e_i = \frac{1}{n}\sum_{j=0}^{n-1} \epsilon^{ij}\sigma^j \in KG \quad (0 \le i < n)$$

are orthogonal idempotents. They must be linearly independent, so a basis for KG.

2.13 Invariant rings

An action of a group G on an algebra R is given by a group homomorphism $\rho : G \to \operatorname{Aut}_{K\text{-alg}}(R)$. We write ${}^g r$ for $\rho(g)(r)$. Thus
- ${}^g(\lambda r + \mu s) = \lambda {}^g r + \mu {}^g s$,
- ${}^g(rs) = ({}^g r)({}^g s)$,
- ${}^g 1 = 1$,
- ${}^1 r = r$,
- ${}^{gh} r = {}^g({}^h r)$.

Example. Any subgroup $G \subseteq \mathrm{GL}_n(K)$ acts on $K[x_1, \dots, x_n]$ (or $K\langle x_1, \dots, x_n\rangle$

or $K[[x_1,\dots,x_n]]$ or $K\langle\langle x_1,\dots,x_n\rangle\rangle$) via

$$^{g}x_j = \sum_i g_{ij} x_i.$$

For example

$$\begin{aligned} {}^{g}({}^{h}x_j) &= {}^{g}(\sum_i h_{ij} x_i) = \sum_i h_{ij} {}^{g}x_i = \sum_i h_{ij} \sum_k g_{ki} x_k \\ &= \sum_{i,k} g_{ki} h_{ij} x_k = \sum_k (gh)_{kj} x_k = {}^{(gh)}x_j. \end{aligned}$$

Definition. The *invariants* are $R^G = \{r \in R : {}^{g}r = r \text{ for all } g \in G\}$. This is a subalgebra of R.

Definition. The *Kleinian singularities* are the rings $K[x,y]^G$ with G a finite sugroup of $\mathrm{SL}_2(K)$ and K an algebraically closed field of characteristic 0.

Example (Cyclic Kleinian singularity). Suppose ϵ is a primitive nth root of 1 and $\sigma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \in \mathrm{SL}_2(K)$. Then $G = \langle \sigma \rangle \cong C_n$ acts on $K[x,y]$ via ${}^{\sigma}x = \epsilon x$, ${}^{\sigma}y = \epsilon^{-1} y$, so ${}^{\sigma}(x^i y^j) = \epsilon^{i-j} x^i y^j$. Thus $K[x,y]^G$ is spanned by the monomials $x^i y^j$ with $i - j$ divisible by n. Thus it has basis $u^i v^j w^k$ where $u = x^n$, $v = y^n$, $w = xy$ and $k < n$. Now the map from

$$K[u,v,w]/(w^n - uv) = K\langle u,v,w : vu = uv, wu = uw, wv = vw, w^n = uv\rangle$$

to $K[x,y]^G$ is an isomorphism by the Diamond Lemma.

Theorem (Hilbert-Noether) If a finite group G acts on a finitely generated commutative K-algebra R, and K is noetherian, then R^G is a finitely generated K-algebra and R is a finitely generated R^G-module.

Proof. For $r \in R$ we have $p_r(r) = 0$, where

$$p_r(x) = \prod_{g \in G} (x - {}^{g}r) \in R[x].$$

This is a monic polynomial in $R[x]$, but it is unchanged by the action of G, so it is in $R^G[x]$. If $r_1,\dots,r_k$ are K-algebra generators of R, then there is a surjective map $R^G[x_1,\dots,x_k]/(p_{r_i}(x_i)) \to R$, and the left hand side generated as an R^G-module by the monomials $x_1^{i_1} \dots x_k^{i_k}$ with all $i_j < n$. Thus R is a finitely generated R^G-module. Now use the Artin-Tate Lemma.

Given an action, R becomes a KG-module via $g \cdot r = {}^g r$.

Lemma. If G is finite and $1/|G| \in K$, then the assigment

$$\rho(r) = e_G r = \frac{1}{|G|} \sum_{g \in G} {}^g r$$

defines an R^G-module map $R \to R^G$ with $\rho(r) = r$ for $r \in R^G$. [It is called the Reynolds operator.]

2.14 Skew group algebras

If G acts on R one can form the *skew group algebra* $R\#G$, which has elements $\sum_{g\in G} a_g \# g$ with $a_g \in R$, all but finitely many zero, and multiplication

$$(a\#g)(b\#h) = a({}^g b)\#gh.$$

Example. If G is cyclic of order n, acting by cyclically permuting the factors in $R = K^n$, then $R\#G \cong M_n(K)$. Namely, suppose that $G = \langle \sigma \rangle$ and ${}^\sigma e_i = e_{i+1}$ with indices modulo n. Then $R\#G$ has basis the elements $e^{ij} = e_i \# \sigma^{i-j}$,

$$e^{ij} e^{st} = (e_i \# \sigma^{i-j})(e_s \# \sigma^{s-t}) = e_i e_{s+i-j} \# \sigma^{i-j+s-t} = \delta_{js} e_i \# \sigma^{i-t} = \delta_{js} e^{it}$$

and $\sum_i e^{ii} = \sum e_i \# 1 = 1\#1 = 1$.

Lemma. We can consider R as an $R\#G$-module via $(a\#g)r = a \cdot {}^g r$, and the map $(R^G)^{op} \to \mathrm{End}_{R\#G}(R)$, $x \mapsto (r \mapsto rx)$ is an isomorphism.

Proof. We need to prove that any $\phi \in \mathrm{End}_{R\#G}(R)$ is given by right multiplication by an element of R^G. For $g \in G$ we have

$${}^g\phi(1) = (1\#g)\phi(1) = \phi((1\#g)1) = \phi({}^g 1) = \phi(1).$$

Thus $\phi(1) \in R^G$. Now for any $r \in R$,

$$\phi(r) = \phi((r\#1)1) = (r\#1)\phi(1) = r\phi(1).$$

Lemma. Suppose G is a finite group and $1/|G| \in K$, and consider e_G as the idempotent $(1/|G|)\sum_{g\in G} 1\#g$ in $R\#G$. Then $R \cong (R\#G)e_G$, and so also $R^G \cong e_G(R\#G)e_G$.

Proof. The maps

$$\theta : R \to (R\#G)e_G, \quad r \mapsto (r\#1)e_G = \frac{1}{|G|}\sum_{g\in G} r\#g$$

and

$$\phi : (R\#G)e_G \to R, \quad \sum_{h\in G} a_h\#h \mapsto \sum_{h\in G} a_h$$

are $R\#G$-module maps, $\phi(\theta(r)) = r$ and for $x = \sum_{h\in G} a_h\#h \in (R\#G)e_G$ we have

$$x = xe_G = \frac{1}{|G|}\sum_{g,h\in G} a_h\#hg = \frac{1}{|G|}\sum_{k,h\in G} a_h\#k = \theta(\sum_{h\in G} a_h) = \theta(\phi(x)).$$

These kinds of constructions appear in the theory of 'Symplectic reflection allgebras'. We shall consider a simpler notion.

Definition (CB and M.P.Holland). Let K be an algebraiclly closed field of characteristic 0. If G is a finite subgroup of $\mathrm{SL}_2(K)$ and $\lambda \in Z(KG)$, then

$$\mathcal{O}^\lambda = e_G[(K\langle x,y\rangle\#G)/(yx - xy - \lambda)]e_G.$$

If $\lambda = 0$ this is the Kleinian singularity $K[x,y]^G$. If $\lambda = 1$ it is $A_1(K)^G$. In general we consider it as a noncommutative deformation of the Kleinian singularity.

Lemma. $(K\langle x,y\rangle\#G)/(yx - xy - \lambda)$ has K-basis the elements $x^n y^m g$ with $n,m \geq 0$ and $g \in G$. (At this point I've stopped writing the # symbol).

Proof. We consider it as the algebra generated by x, y and $g \in G$ subject to the relations $y \cdot x = xy + \lambda$, $g \cdot x = (g_{11}x + g_{21}y)g$, $g \cdot y = (g_{12}x + g_{22}y)g$, $g \cdot g' = gg'$.

The ambiguities are $(gy)x = g(yx)$, $(gg')x = g(g'x)$, $(gg')y = g(g'y)$, $(gg')g'' = g(g'g'')$.

Now $(gy)x \rightsquigarrow (g_{12}x + g_{22}y)gx = g_{12}x(gx) + g_{22}y(gx) = g_{12}x(g_{11}x + g_{21}y)g + g_{22}y(g_{11}x+g_{21}y)g = g_{12}g_{11}x^2g+g_{12}g_{21}xyg+g_{11}g_{22}yxg+g_{21}g_{22}y^2g \rightsquigarrow g_{12}g_{11}x^2g+ g_{12}g_{21}xyg + g_{11}g_{22}(xy + \lambda)g + g_{21}g_{22}y^2g$. On the other hand $g(yx) \rightsquigarrow g(xy + \lambda) = gxy+g\lambda \rightsquigarrow (g_{11}x+g_{21}y)gy+g\lambda = g_{11}x(gy)+g_{21}y(gy)+g\lambda \rightsquigarrow g_{11}x(g_{12}x+ g_{22}y)g + g_{21}y(g_{12}x + g_{22}y)g + g\lambda = g_{11}g_{12}x^2g + g_{11}g_{22}xyg + g_{21}g_{12}yxg + g_{21}g_{22}y^2g + g\lambda \rightsquigarrow g_{11}g_{12}x^2g + g_{11}g_{22}xyg + g_{21}g_{12}(xy + \lambda)g + g_{21}g_{22}y^2g + g\lambda$. These are equal since $\det(g) = 1$ and $\lambda \in Z(KG)$.

The diamond condition for the other ambiguities is easy.

2.15 Graded and filtered rings

We consider algebras over K, allowing $K = \mathbb{Z}$, so that we deal with rings.

Definitions. A K-module M is *graded* or *$\mathbb{Z}$-graded* if it is equipped with a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$. We say that it is $\mathbb{N}$-*graded*, or *non-negatively graded*, if $M = \bigoplus_{n\in\mathbb{N}} M_n$. Alternatively we may consider it as $\mathbb{Z}$-graded with $M_n = 0$ for $n < 0$. A module homomorphism $\theta : M \to N$ is *graded* if $\theta(M_n) \subseteq N_n$ for all n.

An element $m \in M$ is *homogeneous* if $m \in M_n$ for some n. A submodule $N \subseteq M$ is a *graded* or *homogeneous* submodule if $N = \bigoplus_{n\in Z}(N \cap M_n)$. It is equivalent that it is generated by homogeneous elements.

A K-algebra is *graded* if it is graded as a K-module, $R = \bigoplus_{n\in\mathbb{Z}} R_n$, with

$$R_n R_m \subseteq R_{n+m}$$

for all $n, m \in \mathbb{Z}$. It follows that $1 \in R_0$ (for if $1 = \sum s_n$ and $r \in R_i$ then $r = r1 = 1r$ gives $r = rs_0 = s_0 r$, so s_0 is a unity for R).

Examples. (a) $K[x_1, \dots, x_r]$ is graded by the degree of a polynomial. More generally you can choose any $d_1, \dots, d_r \in \mathbb{Z}$ and grade it with $\deg(x_i) = d_i$.
(b) KQ (including the special case $K\langle X\rangle$) can be graded with all e_i of degree 0, by choosing a degree $d_a \in \mathbb{Z}$ for each arrow a.
(c) An ideal I in a graded algebra R is homogeneous if and only if it is generated by homogeneous elements, and if so, R/I is a graded algebra.

If R is a graded algebra, then a *graded R-module* M is an R-module with a graded K-module structure satisfying

$$R_n M_m \subseteq M_{n+m}.$$

A graded module homomorphism $\theta : M \to N$ is one with $\theta(M_n) \subseteq N_n$ for all n.

Example. Consider $K[x, y]$ with the usual grading. A graded $K[x, y]$-module is the same thing as a representation of the quiver with vertex set $\mathbb{Z}$, arrows $x_n : n \to n + 1$ and $y_n : n \to n + 1$ and relations $x_{n+1} y_n = y_{n+1} x_n$ for all n.

Remark. We can formulate this using rings with enough idempotents. If R is a graded algebra, let S be the subset of $R^{(\mathbb{Z}\times\mathbb{Z})}$ consisting of all matrices $s = (s_{ij})$ with $s_{ij} \in R_{j-i}$. This is a ring with enough idempotents $e_i = e^{ii}$, and

$e_i S e_j = R_{j-i}$. Then a graded R-module is exactly the same as an S-module, and graded homomorphisms correspond to S-module homomorphisms.

This appears in the theory of $\mathbb{Z}$-*algebras* (in a different sense to how we use this term). See for example M. van den Bergh, Noncommutative quadrics, IMRN 2011.

By a *filtered algebra* we mean an algebra S equipped with K-subspaces

$$S_{\leq 0} \subseteq S_{\leq 1} \subseteq S_{\leq 2} \subseteq \dots$$

such that $S = \bigcup_{n\in\mathbb{N}} S_{\leq n}$, $S_{\leq n} S_{\leq m} \subseteq S_{\leq n+m}$, and $1 \in S_{\leq 0}$.

Example. (a) Any $\mathbb{N}$-graded ring R is filtered by $R_{\leq n} = \bigoplus_{i\leq n} R_i$.

(b) A quotient of a filtered ring S/I is filtered by $(S/I)_{\leq n} = (I + S_{\leq n})/I$.

Definition. If S is a filtered ring, then the *associated graded ring* is

$$\operatorname{gr} S = \bigoplus_{n\in\mathbb{N}} \operatorname{gr}_n S, \quad \operatorname{gr}_n S = S_{\leq n}/S_{\leq n-1}$$

with $S_{\leq -1} = 0$. The multiplication is given by

$$\operatorname{gr}_n S \times \operatorname{gr}_m S \to \operatorname{gr}_{n+m} S, \quad (S_{\leq n-1} + x, S_{\leq m-1} + y) \mapsto S_{\leq n+m-1} + xy.$$

This is well-defined by the condition that S is a filtered algebra. The *symbol map* of degree n is the natural map

$$\sigma_n : S_{\leq n} \to \operatorname{gr}_n S, \quad \sigma_n(x) = S_{\leq n-1} + x.$$

Lemma. Grade the free algebra $K\langle X\rangle$ by choosing degrees d_x of the generators $x \in X$. Let $S = K\langle X\rangle/I$ be a ring with the induced filtration. Then $\operatorname{gr} S$ is generated as a K-algebra by the homogeneous elements $\sigma_{d_x}(x)$ $(x \in X)$.

Proof. $\operatorname{gr}_n S$ is spanned by the elements $\sigma_n(x_1x_2\dots x_k)$ where $x_1x_2\dots x_k$ is a word with degree $d_{x_1} + \dots + d_{x_k} \leq n$. If this inequality is strict, then the element is zero. Otherwise the element can be written as $\sigma_{d_{x_1}}(x_1)\dots\sigma_{d_{x_k}}(x_k)$.

Example. The *Bernstein filtration* of the Weyl algebra $S = A_n(K)$ is the filtration induced from the usual grading of $K\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle$, so $S_{\leq m}$ is the K-subspace spanned by $x^\alpha y^\beta$ with $|\alpha|+|\beta| \leq m$. Then $\operatorname{gr} S$ is generated by the elements $\sigma_1(x_i)$ and $\sigma_1(y_i)$ and they commute, since, for example,

$$\sigma_1(y_i)\sigma_1(x_i) - \sigma_1(x_i)\sigma_1(y_i) = \sigma_2(y_ix_i - x_iy_i) = \sigma_2(1) = 0$$

since $1 \in S_{\leq 1}$. This gives a surjective homomorphism

$$K[x_1, \dots, x_n, y_1, \dots, y_n] \to \operatorname{gr} S.$$

This map sends a monomial $x^\alpha y^\beta$ to the element $\sigma_{|\alpha|+|\beta|}(x^\alpha y^\beta)$. As α and β vary, these elements run through bases of $K[x_1, \dots, x_n, y_1, \dots, y_n]$ and $\operatorname{gr} S$ respectively, so the map is an isomorphism.

Theorem. Suppose S is a filtered ring.
(1) If $\operatorname{gr} S$ is a domain then so is S.
(2) If $\operatorname{gr} S$ is left or right noetherian, then so is S.

Proof. (1) Say $a, b \in S$ are nonzero and $ab = 0.$. For some $n, m \geq 0$ we have $a \in S_{\leq n} \setminus S_{\leq n-1}$ and $b \in S_{\leq m} \setminus S_{\leq m-1}$. Then $(S_{\leq n-1} + a)(S_{\leq m-1} + b) = S_{\leq n+m-1} + 0$. But this is a product of two nonzero elements of $\operatorname{gr} S$.

(2) Any right ideal I in S gets a filtration $I_{\leq n} = I \cap S_{\leq n}$, and then

$$\operatorname{gr}_n I = (I \cap S_{\leq n})/(I \cap S_{\leq n-1}) \cong [(I \cap S_{\leq n}) + S_{\leq n-1}]/S_{\leq n-1} \subseteq \operatorname{gr}_n S.$$

It is easy to see that this makes $\operatorname{gr} I$ a right ideal in $\operatorname{gr} S$. Suppose that I is not finitely generated. Choose $f_1 \in I_{\leq n_1}$ with n_1 minimal, and if we have $f_1, \dots, f_k \in I$, choose $f_{k+1} \in I_{\leq n_{k+1}} \setminus \sum_{i=1}^k f_i S$ with n_{k+1} minimal. Now the chain

$$\sigma_{n_1}(f_1) \operatorname{gr} S \subseteq \sigma_{n_1}(f_1) \operatorname{gr} S + \sigma_{n_2}(f_2) \operatorname{gr} S \subseteq \dots$$

becomes stationary, so some $\sigma_{n_{k+1}}(f_{k+1}) = \sum_{i=1}^k \sigma_{n_i}(f_i) s_i$. We may suppose the s_i are homogeneous, of the form $\sigma_{n_{k+1}-n_i}(s'_i)$. Then $f_{k+1} - \sum_{i=1}^k f_i s'_i \in S_{\leq n_{k+1}-1}$, contradicting the choice of f_{k+1}.

Examples. (1) Grade $K\overline{Q}$ by path length. Then $\Pi(Q)$ is graded, and $\Pi^\lambda(Q)$ gets a filtration. There is a surjective map $\Pi(Q) \to \operatorname{gr} \Pi^\lambda(Q)$. In general it is not an isomorphism.

(2) Grade $K[x, y]\#G$ or $K\langle x, y\rangle\#G$ with $\deg x = \deg y = 1$ and the elements of G in degree 0. Then $(K\langle x, y\rangle\#G)/(yx - xy - \lambda)$ gets a filtration. Then there is a surjective map

$$K[x, y]\#G \to \operatorname{gr}(K\langle x, y\rangle\#G)/(yx - xy - \lambda).$$

and it is an isomorphism since both have K-bases given by elements of form $x^n y^m g$. Now $\mathcal{O}^\lambda = e_G[(K\langle x, y\rangle\#G)/(yx - xy - \lambda)]e_G$ gets a filtration, and there is an isomorphism

$$K[x, y]^G = e_G(K[x, y]\#G)e_G \to \operatorname{gr} \mathcal{O}^\lambda.$$

In particular $\mathcal{O}^\lambda$ is a noetherian domain.

2.16 Localization

Let R be a ring. A subset $S \subseteq R$ is *multiplicative* if $1 \in S$ and $ss' \in S$ for all $s, s' \in S$.

Definition. If S is a multiplicative subset in R, then $\theta_S : R \to R_S$ is the univeral homomorphism to a ring with the property that $\theta_S(s)$ is invertible for all $s \in S$. That is, if $\theta : R \to A$ is a ring with $\theta(s)$ invertible for all $s \in S$ then there is a unique $\phi : R_S \to A$ with $\theta = \phi\theta_S$. For existence, take $R_S = (R * K\langle s^{-1} : s \in S\rangle)/(ss^{-1} - 1, s^{-1}s - 1)$, e.g. with $K = \mathbb{Z}$. Uniqueness follows from the universal property.

Definition. A multiplicative subset S in R satisfies the *left Ore condition* if for all $s \in S$ and $a \in R$ there exist $s' \in S$ and $a' \in R$ with $s'a = a's$, and it is *left reversible* if $as = 0$ with $a \in R$ and $s \in S$ implies that there is $s' \in S$ with $s'a = 0$. Both conditions are trivial if R is commutative or more generally if $S \subseteq Z(R)$.

Construction. If S is a left reversible left Ore set in R and M is a left R-module, then on the set of pairs $(s, m) \in S \times M$ we consider the relation

$$(s, m) \sim (s', m) \Leftrightarrow \text{there are } u, u' \in R \text{ with } um = u'm' \text{ and } us = u's' \in S.$$

Lemma 1. This is an equivalence relation.

Proof. See exercise sheet.

We write $s^{-1}m$ for the equivalence class of (s, m) and define $S^{-1}M$ to be the set of equivalence classes.

Lemma 2. Any two elements of $S^{-1}M$ can be written with a common denominator.

Proof. Given $s^{-1}m$ and $(s')^{-1}m'$, the Ore condition gives $t \in S$, $a \in R$ with $ts' = as \in S$. Then $s^{-1}m = (as)^{-1}am$ and $(s')^{-1}m' = (ts')^{-1}tm'$.

Lemma 3. $S^{-1}M$ becomes an R-module via

$$s^{-1}m + s^{-1}m' = s^{-1}(m + m'),$$
$$a(s^{-1}m) = (s')^{-1}(a'm) \quad \text{where } s'a = a's \text{ with } s' \in S \text{ and } a' \in R$$

and $s^{-1}m = 0 \Leftrightarrow$ there is $u \in R$ with $um = 0$ and $us \in S$. In particular $1^{-1}m = 0 \Leftrightarrow$ there is $u \in S$ with $um = 0$. Moreover elements of S act invertibly on $S^{-1}M$.

Proof. Most straightforward. Now $s^{-1}m = 1^{-1}0 \Leftrightarrow$ there are $u, u' \in R$ with $um = u'0$ and $us = u'1 \in S$, gives the condition. Finally, if $t \in S$ then $s^{-1}m = t[(st)^{-1}m]$. Conversely, if $ts^{-1}m = 0$, then $(s')^{-1}a'm = 0$ where $s't = a's$. Thus there is $u \in R$ with $ua'm = 0$ and $us' \in S$. Then $ua's = us't \in S$ and $ua'm = 0$, so $s^{-1}m = 0$.

Thus $S^{-1}M$ becomes an R_S-module in a unique way.

Ore's Theorem. $R_S \cong S^{-1}R$, considered as a ring with multiplication

$$(t^{-1}a)(s^{-1}b) = (s't)^{-1}a'b$$

where $s'a = a's$ with $s' \in S$ and $a' \in R$.

Proof. Since the elements of S become invertible in R_S, there is a natural map $S^{-1}R \to R_S$. Then using the action of R_S on $S^{-1}R$ we see that the multiplication for $S^{-1}R$ is well-defined. Now the map $R \to S^{-1}R$, $r \mapsto 1^{-1}r$ has the universal property, so it is identified with $R \to R_S$.

Remark. Similarly there is the notion of a right reversible right Ore set, for which R_S can be constructed as fractions of the form rs^{-1}.

Example. If σ is a K-algebra automorphism of R, then $\{1, x, x^2, \dots\}$ is a left and right reversible Ore set in $R[x;\sigma]$ The elements of $R[x;\sigma]_S$ are of the form

$$(r_0 + r_1x + \cdots + r_nx^n)x^{-m} = r_0x^{-m} + \cdots + r_nx^{n-m},$$

so Laurent polynomials.

Theorem (Special case of Goldie's Theorem). Let R be a domain which is left noetherian (or more generally has no left ideal isomorphic to $R^{(\mathbb{N})}$). Then $S = R \setminus \{0\}$ is a left reversible left Ore set, and $\theta_S : R \to R_S$ is an injective map to a division ring.

Proof. The left reversibility condition is trivial. If S fails the left Ore condition, then there are $a, b \neq 0$ with $Ra \cap Rb = 0$. Then $a, ab, ab^2, \dots$ are linearly independent, for if $\sum_i r_iab^i = 0$, then by cancelling as many factors of b on the right as possible, we get

$$r_0a + r_1ab + \cdots + r_nab^n = 0$$

with $r_0 \neq 0$. But then $0 \neq r_0a \in Ra \cap Rb$. Thus $\bigoplus_i Rab^i \subseteq R$. Now R_S is a division ring for if $s^{-1}r \neq 0$ then $r \neq 0$ and $(s^{-1}r)^{-1} = r^{-1}s$.

Examples. (1) $\mathbb{Z}$ embeds in $\mathbb{Q}$, $K[x_1, \dots, x_n]$ embeds in $K(x_1, \dots, x_n)$, etc.

(2) $R = A_n(K)$ is a noetherian domain, so it embeds in a division ring $R_S = D_n(K)$.

(3) For $R = K\langle x, y\rangle$ the set $R\backslash\{0\}$ fails the left Ore condition since $Rx \cap Ry = 0$. There do exist embeddings of R in division rings, but they are more complicated.

3 Module categories

3.1 Categories

A *category* C consists of
(i) a collection $ob(C)$ of *objects*
(ii) For any $X, Y \in ob(C)$, a set $\mathrm{Hom}(X, Y)$ (or $C(X, Y)$, or sometimes $\mathrm{Hom}_C(X, Y)$) of *morphisms* $\theta : X \to Y$, and
(iii) For any X, Y, Z, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(\theta, \phi) \mapsto \theta\phi$.
satisfying
(a) Associativity: $(\theta\phi)\psi = \theta(\phi\psi)$ for $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z \xrightarrow{\theta} W$, and
(b) For each object X there is an identity morphism $id_X \in \mathrm{Hom}(X, X)$, with $id_Y \theta = \theta = \theta id_X$ for all $\theta : X \to Y$.

Examples.
(1) The categories of Sets, Groups, Abelian groups, Rings, Commutative rings, K-algebras, etc.
(2) The category R-Mod of R-modules for a ring R. The category R-mod of finitely generated R-modules.
(3) The category of sets with $\mathrm{Hom}(X, Y) =$ the injective functions $X \to Y$. The category of linear relations, whose objects are K-modules and with $\mathrm{Hom}(X, Y) =$ linear relations from X to Y, that is, K-submodules $R \subseteq X \oplus Y$ and composition of $S \subseteq Y \times Z$ and $R \subseteq X \times Y$ being $SR = \{(x, z) : (y, z) \in S \text{ and } (x, y) \in R \text{ for some } y\}$.
(4) Given a group G or a ring R, the category with one object x, $\mathrm{Hom}(x, x) = G$ or R and composition given by multiplication.
(5) Given a ring R, the category with objects $\mathbb{N}$, $\mathrm{Hom}(m, n) = M_{n \times m}(R)$ and composition given by matrix multiplication.
(6) Path category of a quiver Q. Objects Q_0 and $\mathrm{Hom}(i, j) =$ paths from i to j. The K-linear path category of Q. Objects Q_0 and $\mathrm{Hom}(i, j) = K$-module with basis the paths from i to j.

Definition. An *isomorphism* is a morphism $\theta : X \to Y$ with an inverse $\theta^{-1} : Y \to X$, $\theta\theta^{-1} = id_Y$, $\theta^{-1}\theta = id_X$.

Remark. Recall that there is no set of all sets. Thus $ob(C)$ may be a proper class. We say that C is *small* if $ob(C)$ is a set, and *skeletally small* if there is a set S of objects such that every object is isomorphic to one in S.

Example. The category of finite sets is not small, but it is skeletally small with $S = \{\emptyset, \{1\}, \{1,2\}, \dots\}$. R-Mod is not small or skeletally small, but R-mod is skeletally small with $S = \{R^n/U : n \in \mathbb{N}, U \subseteq R^n\}$.

Definition. A *subcategory* D of C is given by a category with $ob(D) \subseteq ob(C)$ and $D(X,Y) \subseteq C(X,Y)$ for all $X, Y \in ob(D)$ such that composition in D is the same as that in C and $id_X^C \in D(X,X)$. It is a *full subcategory* if $D(X,Y) = C(X,Y)$.

Definition. If C is a category, the *opposite category* C^{op} is given by $ob(C^{op}) = ob(C)$, $C^{op}(X,Y) = C(Y,X)$, with composition of morphisms derived from that in C.

If C and D are categories, then $C \times D$ denotes the category with $ob(C \times D) = ob(C) \times ob(D)$ and $\mathrm{Hom}((X,U),(Y,V)) = C(X,Y) \times D(U,V)$.

3.2 Monomorphisms and epimorphisms

Definition. A *monomorphism* in a category is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Z \to X$, if $\theta\alpha = \theta\beta$ then $\alpha = \beta$.

An *epimorphism* is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Y \to Z$, if $\alpha\theta = \beta\theta$ then $\alpha = \beta$.

In many concrete categories a monomorphism = injective map, epimorphism = surjective map.

Lemma. In R-Mod, monomorphism = injective map and epimorphism = surjective map.

Proof. We show epi = surjection. The other is similar.

Say $\theta : X \to Y$ is surjective and $\alpha\theta = \beta\theta$ then for all $y \in Y$ there is $x \in X$ with $\theta(x) = y$. Then $\alpha(y) = \alpha(\theta(x)) = \beta(\theta(x)) = \beta(y)$. Thus $\alpha = \beta$.

Say $\theta : X \to Y$ is an epimorphism. The natural map $Y \to Y/\operatorname{Im}\theta$ and the zero map have the same composition with θ, so they are equal. Thus $\operatorname{Im}\theta = Y$.

Example. In the category of rings, a localization map $\theta_S : R \to R_S$ is an epimorphism, but usually not a surjective map, for example $\mathbb{Z} \to \mathbb{Q}$.

Namely, if $\alpha, \beta : R_S \to T$ and $\alpha\theta_S = \beta\theta_S$, then $\alpha\theta_S$ is a map $R \to T$ which inverts the elements of S, so it can be factorized uniquely through θ_S. Thus $\alpha = \beta$.

Theorem. The following are equivalent for a ring homomorphism $\theta : R \to S$.
(i) θ is an epimorphism in the category of rings
(ii) $s \otimes 1 = 1 \otimes s$ in $S \otimes_R S$ for all $s \in S$.
(iii) The multiplication map $S \otimes_R S \to S$ is an isomorphism of S-S-bimodules.
(iv) Multiplication gives an isomorphism $S \otimes_R M \to M$ for any S-module M.
(v) For any S-modules M, N we have $\mathrm{Hom}_S(M, N) = \mathrm{Hom}_R(M, N)$.

Proof. (i)⇒(ii) The ring homomorphisms $S \to S \otimes_R S$, $s \mapsto s \otimes 1$ or $1 \otimes s$ have the same composition with θ, so they are equal.

(ii)⇒(iii) $s \mapsto s \otimes 1 = 1 \otimes s$ is an inverse. For example this map sends st to $st \otimes 1 = s(t \otimes 1) = s(1 \otimes t) = s \otimes t$.

(iii)⇒(iv) $S \otimes_R M \cong S \otimes_R S \otimes_S M \cong S \otimes_S M \cong M$.

(iv)⇒(v) $\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)) \cong \mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_S(M, N)$.

(v)⇒(i). Say $f, g : S \to T$ have the same composition with θ. Then the identity map is an R-module map between the restrictions of ${}_fT$ and ${}_gT$. Thus it is an S-module map. Thus $f = g$.

3.3 Functors

If C and D are categories, a *(covariant) functor* $F : C \to D$ is an assignment of
(i) For each object $X \in ob(C)$, an object $F(X) \in ob(D)$, and
(ii) For each $X, Y \in ob(C)$ a map $F : C(X, Y) \to D(F(X), F(Y))$,
such that $F(\theta\phi) = F(\theta)F(\phi)$ and $F(id_X) = id_{F(X)}$.

A *contravariant functor* $F : C \to D$ is the same thing as a covariant functor $C^{op} \to D$. Thus it is an assignment of
(i) For each object $X \in ob(C)$, an object $F(X) \in ob(D)$, and
(ii) For each morphism $\theta : X \to Y$ in C a morphism $F(\theta) : F(Y) \to F(X)$ in D,
such that $F(\theta\phi) = F(\phi)F(\theta)$ and $F(id_X) = id_{F(X)}$.

Definitions. If for all $X, Y \in ob(C)$ the map $F : C(X, Y) \to D(F(X), F(Y))$ is injective, then F is *faithful*. It it is surjective then F is *full*. If every object in D is isomorphic to $F(X)$ for some object X in C we say F is *dense*.

The inclusion of a subcategory is a faithful functor. It is full if and only if the subcategory is full.

Representable functors. Let C be a category and let $\mathrm{Hom}(X, Y)$ denote the Hom sets for C. Suppose we fix $X \in ob(C)$. For $Y \in ob(C)$ we define $F(Y) = \mathrm{Hom}(X, Y)$ and for $\theta \in \mathrm{Hom}(Y, Y')$ we define $F(\theta) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y')$ to be the map sending ϕ to $\theta\phi$. Then F defines a functor $C \to Sets$. We denote it $\mathrm{Hom}(X, -)$.

Dually fixing Y, we get a contravariant functor $\mathrm{Hom}(-, Y)$ from C to Sets.

In fact Hom defines a functor $C^{op} \times C \to Sets$.

Examples.
(1) The forgetful functor, forgetting some structure, for example Groups to Sets, or K-Alg to K-Mod. It is faithful.
(2) Given a ring homomorphism $\theta : R \to S$ there is a restriction functor $S - \mathrm{Mod} \to R - \mathrm{Mod}$. It is faithful. It is full if and only if θ is a ring-epimorphism.
(3) The functor M_n from rings to rings sending R to $M_n(R)$. It is faithful.
(4) If M is an R-S-bimodule, then any morphism of S-modules $X \to X'$ gives a map $M \otimes_S X \to M \otimes_S X'$. Thus $M \otimes_S -$ becomes a functor from S-Mod to R-Mod.
(5) With a bimodule one also gets functors $\mathrm{Hom}_R(M, -)$ from R-Mod to S-Mod and $\mathrm{Hom}_R(-, M)$ from R-Mod^{op} to S^{op}-Mod. Special case: if K is a field, then duality $V \rightsquigarrow V^* = \mathrm{Hom}_K(V, K)$ gives a contravariant functor K-Mod to K-Mod.

3.4 Natural transformations

Definition. If F, G are functors $C \to D$, then a *natural transformation* $\Phi : F \to G$ consists of morphisms $\Phi_X \in D(F(X), G(X))$ for all $X \in ob(C)$ such that $G(\theta)\Phi_X = \Phi_Y F(\theta)$ for all $\theta \in C(X, Y)$.

The natural transformations form the morphisms for a category whose objects are the functors $C \to D$.

Clearly Φ has an inverse if and only if all Φ_X are isomorphisms. In this case we call Φ a *natural isomorphism*.

Examples. (1) If K is a field and V is a K-vector space, there is a natural map $V \to V^{**}$, $v \mapsto (\theta \mapsto \theta(v))$. This is a natural transformation $1_C \to (-)^{**}$ of functors from K-Mod to K-Mod. If we used the category of finite dimensional K-vector spaces, it would be an isomorphism.

(2) A map of R-S-bimodules $M \to N$ gives a natural transformation of functors $M \otimes_S - \to N \otimes_S -$ from S-Mod to R-Mod. It gives a natural transformation $\mathrm{Hom}_R(N, -) \to \mathrm{Hom}_R(M, -)$ of functors from R-Mod to S-Mod. It gives a natural transformation $\mathrm{Hom}_R(, M) \to \mathrm{Hom}_R(, N)$ of functors from R-Mod^{op} to S^{op}-Mod.

(3) If M is an R-S-bimodule, X an R-module and Y an S-module, one gets a map

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y).$$

Varying Y, you can consider this as a natural transformation of functors S-Mod to K-Mod. Varying X, consider this as a natural transformation of functors $R - \mathrm{Mod}^{op}$ to K-Mod. Varying both, consider as natural transformation of functors $R - \mathrm{Mod}^{op} \times S - \mathrm{Mod}$ to K-Mod.

Yoneda's Lemma. For a functor $F : C \to \mathit{Sets}$ and $X \in ob(C)$ there is a 1-1 correspondence between natural transformations $\mathrm{Hom}(X, -) \to F$ and elements of $F(X)$.

Proof. Given $\Phi : \mathrm{Hom}(X, -) \to F$ we get $\Phi_X : \mathrm{Hom}(X, X) \to F(X)$, and $\Phi_X(id_X) \in F(X)$. Conversely, given $f \in F(X)$ and $Y \in ob(C)$ we get a map $\Phi_Y : \mathrm{Hom}(X, Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$. This defines a natural transformation Φ. These constructions are inverses.

Definition. Given functors $F : C \to D$ and $G : D \to C$, we say that (F, G) is an *adjoint pair*, or that F is *left adjoint* to G or G is *right adjoint* to F if there is a natural isomorphism $\Phi : \mathrm{Hom}(F(-), -) \cong \mathrm{Hom}(-, G(-))$ of functors $C^{op} \times D \to \mathit{Sets}$.

Thus one needs bijections

$$\Phi_{X,Y} : \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, G(Y))$$

for all $X \in ob(C)$ and $Y \in ob(D)$, such that

$$\begin{array}{ccc} \mathrm{Hom}(F(X'), Y) & \xrightarrow{\Phi_{X',Y}} & \mathrm{Hom}(X', G(Y)) \\ {\scriptstyle \cdot F(\theta)}\downarrow & & {\scriptstyle \cdot\theta}\downarrow \\ \mathrm{Hom}(F(X), Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}(X, G(Y)) \end{array}$$

commutes for all $\theta : X \to X'$, and

$$\begin{array}{ccc} \mathrm{Hom}(F(X), Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}(X, G(Y)) \\ {\scriptstyle \phi\cdot}\downarrow & & {\scriptstyle G(\phi)\cdot}\downarrow \\ \mathrm{Hom}(F(X), Y') & \xrightarrow{\Phi_{X,Y'}} & \mathrm{Hom}(X, G(Y')) \end{array}$$

commutes for all $\phi : Y \to Y'$.

Examples. (1) (Hom tensor adjointness) If M is an R-S-bimodule then

$$\mathrm{Hom}_R(M \otimes_S X, Y) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(M, Y))$$

for X an S-module and Y an R-module, so $(M \otimes_S -, \mathrm{Hom}_S(M, -))$ is an adjoint pair between R-modules and S-modules.

(2) Free algebras and free modules. For K a commutative ring,

$$\mathrm{Hom}_{K\text{-alg}}(K\langle X\rangle, R) \cong \mathrm{Hom}_{Sets}(X, R),$$

for X from the category of sets and R from the category of K-algebras, so $(X \mapsto K\langle X\rangle, Forget)$ is an adjoint pair between K-algebras and sets. For R a ring

$$\mathrm{Hom}_R(R^{(X)}, M) \cong \mathrm{Hom}_{Sets}(X, M)$$

for X from the category of sets and M from the category of R-modules, so $(X \mapsto R^{(X)}, Forget)$ is an adjoint pair between R-modules and sets.

(3) $\mathrm{Hom}_{K\text{-alg}}(\sqrt[n]{R}, S) \cong \mathrm{Hom}_{K\text{-alg}}(R, M_n(S))$, so $(\sqrt[n]{-}, M_n(-))$ is an adjoint pair between K-algebras and itself.

3.5 Equivalences of categories

Definition. A functor $F : C \to D$ is an *equivalence* if there is $G : D \to C$ such that $FG \cong 1_D$ and $GF \cong 1_C$.

Theorem. F is an equivalence if and only if it is full, faithful and dense.

Proof. Suppose there is a G and natural isomorphisms $\Phi : GF \to 1_C$ and $\Psi : FG \to 1_D$. For $\theta \in C(X,Y)$ we get $\theta\Phi_X = \Phi_Y G(F(\theta))$ so if $F(\theta) = F(\theta')$ then $\theta\Phi_X = \theta'\Phi_X$, so $\theta = \theta'$ since Φ_X is an isomorphism. Thus F is faithful. Similarly G is faithful. Suppose $\phi \in D(F(X),F(Y))$. Let $\theta = \Phi_Y G(\phi)\Phi_X^{-1} \in C(X,Y)$. Then $\theta\Phi_X = \Phi_Y G(F(\theta))$ gives $G(\phi) = G(F(\theta))$, so $\phi = F(\theta)$, so F is full. Also any $Y \in ob(D)$ is isomorphic to $F(G(Y))$, so F is dense.

On the other hand, if F satisfies the stated conditions, for each $Z \in ob(D)$ choose $G(Z) \in ob(C)$ and an isomorphism $\eta_Z : Z \to F(G(Z))$. We extend it to a functor $G : D \to C$ by defining $G(\theta)$ for $\theta \in D(Z,W)$ to be the unique morphism $\alpha \in C(G(Z),G(W))$ with $F(G(\alpha)) = \eta_W\theta\eta_Z^{-1}$.

Examples. (i) If K is a field, there is an equivalence of categories from the category with objects $\mathbb{N}$ and $Hom(m,n) = M_{n\times m}(K)$ to the category of finite dimensional K-vector spaces, sending n to K^n and a matrix A to the corresponding linear map.

(ii) The assignment $V \mapsto {}^{\oplus}V$ gives an equivalence from the category of K-representations of Q to the category of KQ-modules.

(iii) If R is a graded ring, the category of graded R-modules is equivalent to the category of modules for the associated ring with enough idempotents.

3.6 Universal constructions and additive categories

Let K be a commutative ring.

Definition. A *K-category* is a category C with the additional structure that each of the sets $\mathrm{Hom}(X,Y)$ is a K-module, in such a way that composition $\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ is K-bilinear. In particular each set $\mathrm{Hom}(X,Y)$ contains a distinguished element, the zero element.

A functor $F : C \to D$ between K-categories is *K-linear*, if all of the maps $C(X,Y) \to D(F(X),F(Y))$ are K-module maps.

One uses the terminology *pre-additive category* and *additive functor* if these hold for some K (equivalently for $K = \mathbb{Z}$).

For example K-Mod is a K-category. Also, if R is a K-algebra, then R-Mod is a K-category. Note, however, that the category of K-algebras is not a K-category. Certainly not in a natural way, since the zero map between two K-algebras is not a K-algebra morphism. But not at all, since you can find examples of K-algebras that have no algebra homomorphisms between them.

If C is a K-category and X is an object in C, then the representable functor $\mathrm{Hom}(X, -)$ can be considered as a functor from C to K-Mod, and it is K-linear.

Definition. In a category C. An *initial* object is an object X with a unique morphism to any other object. A *terminal* object is an object Y with a unique morphism from any other object. If they exist, they are unique up to isomorphism.

See the exercise sheet for a discussion of initial and terminal objects.

Proposition. In a K-category, X is initial $\Leftrightarrow$ terminal $\Leftrightarrow$ $\mathrm{Hom}(X, X) = 0$. Such an object is called a *zero* object.

Proof. If X is initial or terminal, then $\mathrm{Hom}(X, X) = 0$. Conversely, if $\mathrm{Hom}(X, X) = 0$ and $\theta : X \to Y$ then $\theta = \theta id_X = \theta 0 = 0$, so X is initial.

Definition. A *product* of a family of objects Y_i $(i \in I)$ is an object Z equipped with morphisms $p_i : Z \to Y_i$ such that for all objects X, the map $\mathrm{Hom}(X, Z) \to \prod_i \mathrm{Hom}(X, Y_i)$, $\theta \mapsto (p_i\theta)$ is a bijection.

It the product exists, it is unique up to isomorphism, and denoted $\prod_i Y_i$.

A *coproduct* of a family of objects X_i is an object Z equipped with morphisms $i_i : X_i \to Z$ giving an bijection $\mathrm{Hom}(Z, Y) \to \prod_i \mathrm{Hom}(X_i, Y)$, $\theta \mapsto (\theta i_i)$ for all objects Y.

It the coproduct exists, it is unique up to isomorphism, and denoted $\coprod_i X_i$.

Examples. (a) In the category of sets, the usual product $\prod_i Y_i$ is a categorical product. The coproduct $\coprod_i X_i$ is the disjoint union of the sets X_i.
(b) In the category R-Mod, the product and direct sum of modules are the product and coproduct.
(c) In the category of K-algebras, the product and coproduct of two algebras A and B are the usual product $A \times B$ and the free product $A *_K B$.
(d) In the category of commutative K-algebras, the product and coproduct of two algebras A and B are the usual product $A \times B$ and the tensor product $A \otimes_K B$.

Proposition. For objects X, Y, Z in a K-category the following are equivalent
(i) Z is a product of X and Y for some morphisms p_X, p_Y
(ii) Z is coproduct of X and Y for some morphisms i_X, i_Y,
(iii) There are morphisms

$$X \underset{i_X}{\overset{p_X}{\leftrightarrows}} Z \underset{i_Y}{\overset{p_Y}{\rightleftarrows}} Y$$

with $p_X i_X = id_X$, $p_X i_Y = 0$, $p_Y i_Y = id_Y$, $p_Y i_X = 0$, $i_X p_X + i_Y p_Y = id_Z$.
In this case we write $Z = X \oplus Y$ and call it a *direct sum*.

Proof. (i)$\Rightarrow$(iii) Suppose Z is a product. It comes with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$. Using the zero morphism as one component and the identity morphism as the other, one gets morphisms $i_X : X \to Z$ and $i_Y : Y \to Z$. They satisfy the conditions. For example if $\phi = i_X p_X + i_Y p_Y$ then $p_X \phi = p_X i_X p_X + p_X i_Y p_Y = p_X + 0 = p_X$ and $p_X \phi = p_Y$, so $\phi = id_Z$.

(iii)$\Rightarrow$(i) For any U one has bijections

$$\mathrm{Hom}(U, Z) \underset{\theta \mapsto (p_X\theta, p_Y\theta)}{\overset{(\alpha,\beta)\mapsto i_X\alpha + i_Y\beta}{\leftrightarrows}} \mathrm{Hom}(U, X) \times \mathrm{Hom}(U, Y)$$

so p_X and p_Y turn Z into a product.

(ii)$\Leftrightarrow$(iii) Dual.

Definition. A category is *additive* if it is a K-category for some K, if it has a zero object and every pair of objects has a direct sum.

Example. R-Mod, R-mod, the category of free R-modules.

Lemma. If F is a K-linear functor between additive K-categories, then $F(0) = 0$ and $F(X \oplus Y) \cong F(X) \oplus F(Y)$.

Proof. $id_{F(0)} = F(id_0) = F(0) = 0$. Apply F to the morphisms in the direct sum.

Definition. In a K-category, a *kernel* of a morphism $\theta : X \to Y$ is a morphism $k : U \to X$ with $\theta k = 0$ and such that any morphism $\phi : Z \to X$ with $\theta\phi = 0$ factors uniquely through k. If it exists, it is a monomorphism.

A *cokernel* of a morphism $\theta : X \to Y$ is a morphism $c : Y \to Z$ with $c\theta = 0$ and such that any morphism $\phi : Y \to W$ with $\phi\theta = 0$ factors uniquely through c.

Example. For modules we take the inclusion $k : \operatorname{Ker}\theta \to X$ and the projection $c : Y \mapsto Y/\operatorname{Im}\theta$.

Definition. Given morphisms $f : X \to Z$ and $g : Y \to Z$ in a category, a *pullback* consists of an object W and morphisms $p : W \to X$ and $q : W \to Y$ giving a commutative square

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y & \xrightarrow{g} & Z \end{array}$$

and which is univeral for such commutative squares, that is for any other W', $p' : W' \to X$, $q' : W' \to Y$ with $fp' = gq'$ there is a unique $\theta : W' \to W$ with $p' = p\theta$ and $q' = q\theta$.

Dually a *pushout* of morphisms $p : W \to X$ and $q : W \to Y$, is an object Z and morphisms $f : X \to Z$ and $g : Y \to Z$ giving a commutative square such that for all Z' and $f' : X \to Z'$ and $g' : Y \to Z'$ with $g'q = f'p$ there is unique ϕ with $f' = \phi f$ and $g' = \phi g$.

More generally, there is the notion of a limit or colimit of a functor $F : G \to C$.

Properties (i) In a pullback, if f is mono, then so is q. In a pushout, if q is epi, so is f.
(ii) For an additive category, if kernels exist, so do pullbacks, if cokernels exist, so do pushouts.

Proof. (i) If $\alpha, \beta : U \to W$ and $q\alpha = q\beta$, then $gq\alpha = gq\beta$, so $fp\alpha = fp\beta$. Since f is mono, $p\alpha = p\beta$. Thus one gets the same commutative square using $p\alpha$ and $q\alpha$ or unig $p\beta$ and $q\beta$. Thus by the uniqueness property of the pullback $\alpha = \beta$.

(ii) morphisms $f : X \to Z$ and $g : Y \to Z$ induce a morphism $\theta : X \oplus Y \to Z$ via $\theta = fp_X + gp_Y$, so $f = \theta i_X$ and $g = \theta i_Y$. Let $k : W \to X \oplus Y$ be a kernel. We define $p = p_X k$ and $q = -p_Y k$. Then $fp - gq = \theta i_X p_X k + \theta i_Y p_Y k = \theta 1_{X\oplus Y} k = \theta k = 0$, so $fp = gq$, and similarly one can show that the universal property holds.

3.7 Abelian categories and exact functors

Definition. A category is *abelian* if it is additive, has kernels and cokernels, every mono is the kernel of its cokernel, and every epi is the cokernel of its kernel.

Example. R-Mod. Also the category R-mod of finitely generated modules, for R a left noetherian ring.

A *subobject* of an object X in an abelian category is an equivalence class of monos to X, where $\alpha : U \to X$ is equivalent to $\beta : V \to X \Leftrightarrow \alpha = \beta\phi$ for some isomorphism $\phi : U \to V$.

Given a morphism $\theta : X \to Y$, the kernel can be considered as a subobject $\operatorname{Ker}\theta$ of X. We define $\operatorname{Im}\theta$ to be the kernel of the cokernel of θ, considered as a subobject of Y.

Recall that a sequence of modules

$$\dots \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \dots$$

is exact at Y if $\operatorname{Im} f = \operatorname{Ker} g$. This makes sense for an abelian category too.

Definition An exact sequence

$$0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0$$

is *split* if it satisfies the following equivalent conditions
(i) g has a *section*, a morphism $s : Y \to E$ with $gs = id_Y$.
(ii) f has a *retraction*, a morphism $r : E \to X$ with $rf = id_X$.
(iii) There are

$$X \underset{f}{\overset{r}{\leftrightarrows}} E \underset{s}{\overset{g}{\rightleftarrows}} Y$$

with $gs = id_Y$, $gf = 0$, $rs = 0$, $rf = id_X$ and $sg + fr = id_E$, so $E \cong X \oplus Y$.

Proof of equivalence. (i)$\Rightarrow$(iii). $g(id_E - sg) = 0$, so $id_E - sg$ factors through $\operatorname{Ker} g = \operatorname{Im} f$, so $id_E - sg = fr$ for some $r : E \to X$. Then $frf = f - sgf = f$, so $rf = id_X$ since f is a mono. The rest is trivial or dual.

Definition. If F is an additive functor between abelian categories, we say that F is *exact* (respectively *left exact*, respectively *right exact*) if given any exact sequence

$$0 \to X \to Y \to Z \to 0$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact (respectively left exact, respectively right exact). Similarly, if F is a contravariant functor, we want the sequence

$$0 \to F(Z) \to F(Y) \to F(X) \to 0$$

to be exact (respectively left exact, respectively right exact).

Notes. (i) Any additive functor between abelian categories sends split exact sequences to split exact sequences.

(ii) An exact functor sends any exact sequence (not just a short exact sequence) to an exact sequence. A left exact functor sends a left exact sequence $0 \to X \to Y \to Z$ to a left exact sequence $0 \to F(X) \to F(Y) \to F(Z)$. Similarly for right exact.

Lemma. (Left exactness of Hom) For an abelian category, $\mathrm{Hom}(-,-)$ gives a left exact functor in each variable. That is, if M is an object and $0 \to X \to Y \to Z \to 0$ is exact, then so are

$$0 \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,Y) \to \mathrm{Hom}(M,Z)$$

and

$$0 \to \mathrm{Hom}(Z,M) \to \mathrm{Hom}(Y,M) \to \mathrm{Hom}(X,M).$$

Proof. The first sequence is exact at $\mathrm{Hom}(M,Y)$ since $X \to Y$ is a kernel for $Y \to Z$, and it is exact at $\mathrm{Hom}(M,X)$ since $X \to Y$ is a mono.

Lemma. Pullbacks and pushouts exist in an abelian category. Moreover a pullback involving the second map in a short exact sequence gives a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & U \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

and also a pushout involving the first map

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[r] \arrow[d, "f"'] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & V \arrow[r] & 0
\end{tikzcd}$$

Proof. We already saw existence. The morphism $\alpha : U \to Y$ together with the zero morphism $U \to X$ give a morphism $\beta : U \to W$. We need to show $0 \to U \to W \to X \to 0$ is exact.

We prove it for modules by diagram chasing. For this one needs to work with module categories. The argument for a general abelian category needs some other ideas - it is omitted.

We have $W = \mathrm{Ker}(f - g) : X \oplus Y \to Z$, with p the projection onto X and q the projection onto Y. Moreover $\beta(u) = (\alpha(u), 0)$. Given $x \in X$, $f(x) \in Z$, so there is $y \in Y$ with $g(y) = f(x)$. Then $w = (x, y) \in W$, and $p(w) = x$, so p is onto. Now suppose $w = (x, y) \in W$ and $p(w) = 0$. Thus $w = (0, y)$. Clearly $g(y) = gq(w) = fp(w) = f(0) = 0$. Thus $y = \alpha(u)$ for some u. Then $w = (0, y) = \beta(u)$. Also, if $u \in U$ and $\beta(u) = 0$, then $\alpha(u) = 0$, so $u = 0$.

I did the proof differently in the lecture.

3.8 Projective modules

Definition. An object P in an abelian category is *projective* if it satisfies the following equivalent conditions.
(i) $\mathrm{Hom}(P, -)$ is exact.
(ii) Any short exact sequence $0 \to X \to Y \to P \to 0$ is split.
(iii) Given an epimorphism $\theta : Y \twoheadrightarrow Z$, any morphism $P \to Z$ factors through θ.

Proof of equivalence. (i)$\Rightarrow$(ii) $\mathrm{Hom}(P, Y) \to \mathrm{Hom}(P, P)$ is onto. A lift of id_P is a section.

(ii)$\Rightarrow$(iii) Take the pullback along the map $P \to Z$. The resulting exact sequence has P as third term, so is split. This gives a map from P to the pullback. Composing with the map to Y gives the map $P \to Y$.

(iii)$\Rightarrow$(i) Clear.

Lemma (Better in section 1.6). Given sequences $0 \to X_i \to Y_i \to Z_i \to 0$ $(i \in I)$ of R-modules, the following are equivalent.
(i) The sequences are exact for all $i \in I$.
(ii) $0 \to \prod_i X_i \to \prod_i Y_i \to \prod_i Z_i \to 0$ is exact.
(iii) $0 \to \bigoplus_i X_i \to \bigoplus_i Y_i \to \bigoplus_i Z_i \to 0$ is exact.

Proof. Straightforward.

Proposition. A direct sum of modules $\bigoplus_i M_i$ is projective $\Leftrightarrow$ all M_i are projective.

Proof. $\mathrm{Hom}(\bigoplus_i M_i, -) = \prod_i \mathrm{Hom}(M_i, -)$, so $\bigoplus_i M_i$ is projective
$\Leftrightarrow 0 \to \mathrm{Hom}(\bigoplus_i M_i, X) \to \mathrm{Hom}(\bigoplus_i M_i, Y) \to \mathrm{Hom}(\bigoplus_i M_i, Z) \to 0$ exact for all $0 \to X \to Y \to Z \to 0$
$\Leftrightarrow 0 \to \prod_i \mathrm{Hom}(M_i, X) \to \prod_i \mathrm{Hom}(M_i, Y) \to \prod_i \mathrm{Hom}(M_i, Z) \to 0$ exact
$\Leftrightarrow$ all $0 \to \mathrm{Hom}(M_i, X) \to \mathrm{Hom}(M_i, Y) \to \mathrm{Hom}(M_i, Z) \to 0$ are exact
$\Leftrightarrow$ all M_i are projective.

Theorem. Any free module is projective, and any module is a quotient of a free module. A module is projective if and only if it is a direct summand of a free module.

Proof. $Hom_R(R, X) \cong X$, so R is a projective module, hence so is any direct sum of copies of R. If $F \to P$ is onto with F free and P projective, then P is isomorphic to a summand of F.

Examples.
(i) If R is semisimple artinian, for example a field, then every submodule is a direct summand, so every short exact sequence is split, so every module is projective.
(ii) For a principal ideal domain, any finitely generated projective module is free. This follows from the usual classification of f.g. modules for a pid.
(iii) If $e \in R$ is an idempotent, then Re is a projective R-module, and if e is primitive, it is an indecomposable projective module.
(iv) Any projective module for a path algebra KQ is isomorphic to a direct sum of left ideals generated by trivial paths KQe_i. Any submodule of a projective KQ-module is projective.

An R-module is finitely generated projective if and only if it is isomorphic to a direct summand of a free module R^n for some n. We write $R - proj$ for the category of finitely generated projective left R-modules.

Lemma. The functor $\mathrm{Hom}_R(-, R)$ defines an antiequivalence between $R - proj$ and $R^{op} - proj$.

Proof. There is a natural transformation

$$X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, R), R), \quad x \mapsto (\theta \mapsto \theta(x)).$$

It is an isomorphism for $X = R$, so for finite direct sums of copies of R, so for direct summands of such modules.

Lemma. If M is an R-S-bimodule, then there is a natural transformation

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y), \quad \theta \otimes y \mapsto (x \mapsto \theta(x) \otimes y)$$

for X an R-module and Y an S-module. It is an isomorphism if X is finitely generated projective. Moreover, if id_X is in the image of the natural map $\mathrm{Hom}_R(X, R) \otimes_R X \to \mathrm{End}_R(X)$, then X is finitely generated projective.

Proof. For the first part, reduce to the case of $X = R$. Say id_X comes from $\sum_i \theta_i \otimes x_i$, then the composition of the maps

$$X \xrightarrow{(\theta_i)} R^n \xrightarrow{(x_i)} X$$

is the identity.

3.9 Injective modules

Definition. An object I in an abelian category is *injective* if it satisfies the following equivalent conditions.
(i) $\mathrm{Hom}(-, I)$ is exact.
(ii) Any short exact sequence $0 \to I \to Y \to Z \to 0$ is split.
(iii) Given an injective map $\theta : X \hookrightarrow Y$, any map $X \to I$ factors through θ.

Proof of equivalence. This is the opposite category version of the result for projectives.

Definition. An inclusion of R-modules $M \subseteq N$ is an *essential extension* of M if every non-zero submodule S of N has $S \cap M \neq 0$.

Theorem. For an R-module I, following conditions are equivalent.
(a) I is injective.
(b) (Baer's criterion) Every homomorphism $f : J \to I$ from a left ideal J of R can be extended to a map $R \to I$.
(c) I has no non-trivial essential extensions

Proof. (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c) Let $I \subseteq L$ be a non-trivial essential extension and fix $\ell \in L \setminus I$. We consider the pullback

$$\begin{array}{ccc} J & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ I & \longrightarrow & L \end{array}$$

where $R \to L$ is the map $r \mapsto r\ell$. Then $J \to R$ is injective, so J is identified with a left ideal in R. By (b), the map $J \to I$ lifts to a map $R \to I$, say sending 1 to i. Then if $r(\ell - i) \in I$, then $r\ell \in I$, so $r \in J$, so $r\ell = ri$, so $r(\ell - i) = 0$. Thus $I \cap R(\ell - i) = 0$ and $R(\ell - i) \neq 0$, contradicting that $I \subseteq L$ is an essential extension.

(c)$\Rightarrow$(a). Given $I \subseteq Y$, we need to show that I is a summand of Y. By Zorn's Lemma, the set of submodules in Y with zero intersection with I has a maximal element C. If $I + C = Y$, then C is a complement. Otherwise, $I \cong (I + C)/C \subseteq Y/C$ is a non-trivial extension. By (c) it cannot be an essential extension, so there is a non-zero submodule U/C with zero intersection with $(I + C)/C$. Then $U \cap (I + C) = C$, so $U \cap I \subseteq C \cap I = 0$. This contradicts the maximality of C.

Proposition. A direct product of modules $\prod_i M_i$ is injective $\Leftrightarrow$ all M_i are injective

Proof. Use that $\mathrm{Hom}(-, \prod_i M_i) = \prod_i \mathrm{Hom}(-, M_i)$.

Definition. If K is an integral domain, then a K-module M is *divisible* if and only if for all $m \in M$ and $0 \neq a \in K$ there is $m' \in M$ with $m = am'$.

Observe that if M is divisible, so is any quotient M/N.

Theorem. If K is an integral domain, then any injective module is divisible. If K is a principal ideal domain, the converse holds.

Proof. Divisibility says that any map $Ka \to M$ lifts to a map $K \to M$. If K is a pid these are all ideals in K.

Now suppose that K is a field or a principal ideal domain. We define $(-)^* = \mathrm{Hom}_K(-, E)$, where

$$E = \begin{cases} K & \text{(if } K \text{ is a field)} \\ F/K & \text{(if } K \text{ is a pid with fraction field } F \neq K) \end{cases}$$

Then E is divisible, so an injective K-module, so $(-)^*$ is an exact functor. It gives a functor from R-modules on one side to R-modules on the other side.

Lemma. If M is a K-module, the map $M \to M^{**}$, $m \mapsto (\theta \mapsto \theta(m))$ is injective. (It is an isomorphism if K is a field and M is a finite-dimensional K-vector space).

Proof. Given $0 \neq m \in M$ it suffices to find a K-module map $f : Km \to E$ with $f(m) \neq 0$, for then since E is injective, f lifts to a map $\theta : M \to E$. If K is a field there is an isomorphism $Km \to E$. If K is a principal ideal domain, choose a maximal ideal Ka containing $ann(m) = \{x \in K : xm = 0\}$. Then there is a map $Km \to E$ sending xm to $K + x/a$.

If K is a field, and M is of dimension d, then so is M^*, and so also M^{**} so the map $M \to M^{**}$ must be an isomorphism.

Theorem. Any R-module embeds in a product of copies of R^*, and such a product is an injective R-module. A module is injective if and only if it is isomorphic to a summand of such a product.

Proof. We have R^* injective since $\mathrm{Hom}_R(-, R^*) \cong (-)^*$ is exact. Thus any product of copies is injective. Now choose a free right R-module and a surjection $R^{(X)} \to M^*$. Then M embeds in M^{**} and this embeds in $(R^{(X)})^* \cong (R^*)^X$. The last part is clear.

Corollary. Any module over any ring embeds in an injective module.

Proof. Apply the last result with $K = \mathbb{Z}$.

3.10 Flat modules

Proposition. If M is an S-R-bimodule, then $M \otimes_R -$ defines a right exact functor from R-Mod to S-Mod which commutes with direct sums

$$M \otimes_R \left(\bigoplus_{i \in I} X_i \right) \cong \bigoplus_{i \in I} (M \otimes_R X_i) \,.$$

Moreover any right exact functor from R-Mod to S-Mod which commutes with direct sums is naturally isomorphic to a tensor product functor for some bimodule.

Proof. $M \otimes_R -$ is a functor by the discussion in the section on tensor products, it commutes with direct sums and it is right exact by item (5) in that section.

Suppose that F is a right exact functor from R-Mod to S-Mod. Then $F(R)$

is an S-module, and it becomes an S-R-bimodule via the map

$$R \xrightarrow{\cong} \mathrm{End}_R(R)^{op} \xrightarrow{F} \mathrm{End}_S(F(R))^{op}.$$

Now for any R-module X there is a R-module map

$$X \xrightarrow{\cong} \mathrm{Hom}_R(R, X) \xrightarrow{F} \mathrm{Hom}_S(F(R), F(X)).$$

By hom-tensor adjointness this gives an S-module map $F(R) \otimes_R X \to F(X)$. This is natural in X, so it Φ_X for some natural transformation $\Phi : F(R) \otimes_R - \to F$. Clearly Φ_R is an isomorphism. Then for any free module $R^{(I)}$ we have $F(R^{(I)}) = F(R)^{(I)} \cong F(R) \otimes R^{(I)}$, so $\Phi_{R^{(I)}}$ is an isomorphism. Then for any module X there is a presentation $R^{(I)} \to R^{(J)} \to X \to 0$ and the first two vertical maps in the diagram

$$\begin{array}{ccccccc}
F(R) \otimes R^{(I)} & \longrightarrow & F(R) \otimes R^{(J)} & \longrightarrow & F(R) \otimes X & \longrightarrow & 0 \\
{\scriptstyle \Phi_{R^{(I)}}} \downarrow & & {\scriptstyle \Phi_{R^{(J)}}} \downarrow & & {\scriptstyle \Phi_X} \downarrow & & \\
F(R^{(I)}) & \longrightarrow & F(R^{(J)}) & \longrightarrow & F(X) & \longrightarrow & 0
\end{array}$$

are isomorphisms. Also the rows are exact. Hence the third vertical map is an isomorphism. Thus Φ is a natural isomorphism.

Definition. A right R-module is *flat* if $M \otimes_R -$ is an exact functor (from $R - Mod$ to $K - Mod$).

Properties.
(i) A direct sum of modules is flat if and only if each summand is flat, since $M \otimes_R (\bigoplus_i X_i) \cong \bigoplus_i M \otimes_R X_i$.
(ii) Any projective module is flat, for $R \otimes_R X \cong X$, so R is flat. Now use the previous result.

Proposition. If K is a field or a pid, then an R-module M is flat if and only if M^* is injective.

Proof. We have $\mathrm{Hom}_R(X, M^*) \cong (M \otimes_R X)^*$. If M is flat, then the right hand functor is exact, so M^* is injective. Conversely, if M^* is injective then the right hand functor is exact. Suppose M is not flat. Given an exact sequence

$$0 \to X \to Y \to Z \to 0$$

we get

$$0 \to L \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0.$$

Then get

$$(M \otimes_R Y)^* \to (M \otimes_R X)^* \to L^* \to 0$$

Thus $L^* = 0$. But L embeds in L^{**}, so $L = 0$.

Proposition. A module M_R is flat if and only if the multiplication map $M \otimes_R I \to M$ is injective for every left ideal I in R.

Proof. If flat, the map is injective. For the converse we can work over $K = \mathbb{Z}$. If the map is injective, then the map $M^* \to (M \otimes_R I)^*$ is surjective. We can write this as $\mathrm{Hom}_R(R, M^*) \to \mathrm{Hom}_R(I, M^*)$. By Baer's criterion M^* is injective. Thus M is flat.

Example. A $\mathbb{Z}$-module is flat if and only if it is torsion-free. If $I = \mathbb{Z}n$ then $M \otimes I \to M$ is injective if and only if multiplication of M by n is injective. For example $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Proposition. If S is a left reversible left Ore set in R then the assignment $M \rightsquigarrow S^{-1}M$ defines an exact functor which is naturally isomorphic to the tensor product functor $M \rightsquigarrow R_S \otimes_R M$, so R_S is a flat as a right R-module.

Proof. Since a finite number of fractions can be put over a common denominator, it follows that the functor $M \rightsquigarrow S^{-1}M$ commutes with direct sums. Thus it suffices to show that it is exact. Now if $L \xrightarrow{\theta} M \xrightarrow{\phi} N$ is exact, and $s^{-1}m$ is sent to zero in $S^{-1}N$, then there is $u \in R$ with $u\phi(m) = 0$ and $us \in S$. Then $\phi(um) = 0$, so $um = \theta(\ell)$. Then $(us)^{-1}\ell \in S^{-1}L$ is sent to $s^{-1}m \in S^{-1}M$.

Definition A module M is *finitely presented* if it is a quotient of a finitely generated free module by a finitely generated submodule. Equivalently if there is an exact sequence $R^m \to R^n \to M \to 0$.

Any f.g. projective module is finitely presented. If R is left noetherian, any finitely generated left R-module is finitely presented.

Lemma. If M is an R-S-bimodule, the natural transformation

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y)$$

is an isomorphism if X is finitely presented and Y is flat.

Proof. It is clear for $X = R$. Then it follows for $X = R^n$. In general there is an exact sequence $R^m \to R^n \to X \to 0$, and in the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(X,M)\otimes_S Y & \longrightarrow & \mathrm{Hom}_R(R^n,M)\otimes_S Y & \longrightarrow & \mathrm{Hom}_R(R^m,M)\otimes_S Y \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Hom}_R(X,M\otimes_S Y) & \longrightarrow & \mathrm{Hom}_R(R^n,M\otimes_S Y) & \longrightarrow & \mathrm{Hom}_R(R^m,M\otimes_S Y)
\end{array}$$

the rows are exact and the right two vertical maps are isomorphisms, hence so is the first.

Proposition. A finitely presented flat module is projective.

Proof. The natural map $\mathrm{Hom}_R(X,R)\otimes_R X \to \mathrm{End}_R(X)$ is an isomorphism.

3.11 Envelopes and covers

Suppose $\mathcal{C}$ is a full subcategory of R-Mod, closed under finite direct sums and direct summands.

Definition. If M is an R-module, a $\mathcal{C}$-preenvelope is a homomorphism $\theta : M \to C$ with C in $\mathcal{C}$, such that any $\theta' : M \to C'$ with C' in $\mathcal{C}$ factors as $\theta' = \phi\theta$ for some $\phi : C \to C'$. It is a $\mathcal{C}$-envelope if in addition, for any $\phi \in \mathrm{End}_R(C)$, if $\phi\theta = \theta$, then ϕ is an automorphism.

If a $\mathcal{C}$-envelope exists, it is unique up to a (non-unique) isomorphism.

Dually, if M is an R-module, a $\mathcal{C}$-precover is a homomorphism $\theta : C \to M$ with C in $\mathcal{C}$, such that any $C' \to M$ factors through $C \to M$. It is a $\mathcal{C}$-cover if in addition, for any $\phi \in \mathrm{End}_R(C)$, if $\theta\phi = \theta$, then ϕ is an automorphism.

If a $\mathcal{C}$-cover exists, it is unique up to a (non-unique) isomorphism.

Theorem. Every module M has an injective envelope. Moreover $\theta : M \to I$ is an injective envelope if and only if θ is a monomorphism, I is injective and $\mathrm{Im}\,\theta \subseteq I$ is an essential extension.

Proof. Any module M embeds in an injective module E. Zorn's Lemma implies that the set of submodules of E which are essential extensions of M has a maximal element I.

Suppose that $I \subset J$ is a non-trivial essential extension. Then $M \subset J$ is an essential extension. Since E is injective the inclusion $I \to E$ can be extended to a map $g : J \to E$. Clearly $M \cap \operatorname{Ker} g = 0$, so since M is essential in J it follows that $\operatorname{Ker} g = 0$. Thus we can identify J with $g(J)$. But then M is essential in J, contradicting the maximalirty of I.

Thus I has no non-trivial essential extensions, so I is injective.

Thus the inclusion $\theta : M \to I$ satisfies the stated conditions. We show it is an injective envelope. Clearly it is a preenvelope.

Say $\phi\theta = \theta$ for some $\phi : I \to I$. Then $M \cap \operatorname{Ker} \phi = 0$, so $\operatorname{Ker} \phi = 0$. Then $\phi : I \to I$ is a monomorphism, so $I = \operatorname{Im} \phi \oplus C$ for some complement C. But then $M \cap C = 0$, so $C = 0$. Thus ϕ is an automorphism.

Remark. (i) Bass 1960 showed that modules have projective covers only for some rings, the left perfect rings.

(ii) Bican, El Bashir and Enochs 2001 showed that every module has a flat cover.

An interesting connection between these concepts is the fact that a flat module has a projective cover if and only it it is already projective. This is Exercise 4.20 in T. Y. Lam, Lectures on Modules and Rings (combined with standard facts about projective covers and Proposition 9.13 in Anderson and Fuller, Rings and Categories of Modules).

3.12 Morita Equivalence

Definitions. An abelian category A is *cocomplete* if for every set of objects M_i $(i \in I)$ there is a coproduct $\coprod_{i \in I} M_i$. If so, then an object P is *finitely generated* if $\operatorname{Hom}(P, -)$ preserves coproducts, and P is a *generator* if for every object M there is an epimorphism $P^{(I)} \to M$.

Note that a module category R-Mod is cocomplete, finitely generated is the same as the usual definition, and R is a projective generator.

Theorem. If A is an abelian category and R is a ring, then A is equivalent to R-Mod if and only if A is cocomplete, and it has a finitely generated projective generator P with $R \cong \operatorname{End}(P)^{op}$.

Proof. The module category R-Mod has these properties, with $P = R$. For sufficiency, consider the functor $F = \operatorname{Hom}(P, -)$ from A to R-Mod. Given objects X and Y choose epimorphisms $p_X : P^{(I)} \to X$ and $p_Y : P^{(J)} \to Y$. Given $\theta : X \to Y$, if $F(\theta) = 0$, then the composition $P^{(I)} \to X \to Y$ is zero, so θ is zero. Thus F is faithful.

Applying F one gets $R^{(I)} \to F(X)$ and $R^{(J)} \to F(Y)$. Any R-module map $\alpha : F(X) \to F(Y)$ lifts to an R-module map $R^{(I)} \to R^{(J)}$. This corresponds to an element of $\mathrm{Hom}(P^{(I)}, P^{(J)})$. Now the composition $\mathrm{Ker}\, p_X \to P^{(I)} \to P^{(J)} \to Y$ is sent by F to zero, so since F is faithful, it is zero itself. Thus there is an induced morphism $\theta : X \to Y$ giving a commutative square. Thus $F(\theta)$ gives a commutative square with the map $R^{(I)} \to R^{(J)}$. Thus $\alpha = F(\theta)$. Thus F is full.

Now for any R-module M there is a presentation $R^{(I)} \to R^{(J)} \to M \to 0$. The first map comes from a morphism $P^{(I)} \to P^{(J)}$. Let this have cokernel X. Then since F is exact, we get $R^{(I)} \to R^{(J)} \to F(X) \to 0$. Thus $M \cong F(X)$. Thus F is dense.

Theorem (Morita). Let R and S be two rings. The following are equivalent.
(i) The categories R-Mod and S-Mod are equivalent
(ii) There is an S-R-bimodule M such that $M \otimes_R -$ gives an equivalence R-Mod to S-Mod
(iii) $S \cong \mathrm{End}_R(P)^{op}$ for some finitely generated projective generator P in R-Mod.

Proof. (i)$\Leftrightarrow$(iii) follows from the theorem.

(ii)$\Rightarrow$(i) is trivial. For (i)$\Rightarrow$(ii) note that an equivalence is exact, and preserves direct sums, so it must be a naturally isomorphic to a tensor product functor.

Examples. (i) R is Morita equivalent to $M_n(R)$ for $n \geq 1$. Namely the module R^n is a finitely generated projective generator in R-Mod with $\mathrm{End}_R(R^n)^{op} \cong M_n(R)$.

(ii) If $e \in R$ is idempotent, and $ReR = R$, then R is Morita equivalent to eRe. Namely, the condition ensures that the multiplication map $Re \otimes_{eRe} eR \to R$ is onto. Taking a map from a free eRe-module onto eR, say $eRe^{(I)} \to eR$, we get a map $Re^{(I)} \to R$, so Re is a generator. Then $\mathrm{End}_R(Re)^{op} \cong eRe$.

Another approach to (i) is to use that $M_n(R) = M_n(R)e^{11}M_n(R)$ and $e^{11}M_n(R)e^{11} \cong R$.

(iii) A semisimple artinian ring is isomorphic to a product of matrix rings over division rings $M_{n_1}(D_1) \times \cdots \times \ldots M_{n_r}(D_r)$. This is always Morita equivalent to a product of division rings $D_1 \times \cdots \times D_r$.

Examples. I would have liked to have had more time to talk about these.

(i) If Q is a quiver, K a commutative ring and $\lambda \in K^{Q_0}$, there is the deformed preprojective algebra $\Pi^\lambda(Q)$. For example if $Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ then a $\Pi^\lambda(Q)$-module is a representation X of $\overline{Q}$

$$X_1 \underset{a}{\overset{a^*}{\leftrightarrows}} X_2 \underset{b}{\overset{b^*}{\leftrightarrows}} X_3$$

satisfying the relations

$$-a^*a = \lambda_1 id_{X_1}, aa^* - b^*b = \lambda_2 id_{X_2}, bb^* = \lambda_3 id_{X_3}.$$

The usual preprojective algebra has $\lambda \neq 0$. Why is is useful to introduce the version with λ? Because if $\lambda_i \in K$ is invertible then there is a reflection functor corresponding to the vertex i

$$\Pi^\lambda(Q) - Mod \to \Pi^\mu(Q) - Mod$$

for suitable μ, which is a Morita equivalence. For example for $i = 3$ above, we have $e = \lambda_3^{-1} b^*b$ an idempotent endomorphism of X_2 with image identified with X_3. The functor sends the representation to

$$X_1 \underset{a}{\overset{a^*}{\leftrightarrows}} X_2 \underset{-\lambda_3(1-e)}{\overset{inclusion}{\leftrightarrows}} \mathrm{Im}(1-e),$$

a representation of $\Pi^\mu(Q)$ with $\mu = (\lambda_1, \lambda_2 + \lambda_3, -\lambda_3)$.

(ii) Let $K = \mathbb{C}$. McKay correspondence gives 1-1 correspondence between finite subgroups G of $\mathrm{SL}_2(K)$ (up to conjugacy) and extended Dynkin diagrams.

Cyclic group of order n	$\tilde{A}_{n-1}$ cyclic graph with n vertices
Binary dihedral group of order $4n$	$\tilde{D}_{n+2}$
Binary tetrahedral group of order 24	$\tilde{E}_6$
Binary octahedral group of order 48	$\tilde{E}_7$
Binary icosahedral group of order 120	$\tilde{E}_8$

Let Q be a quiver obtained by orienting the corresponding extended Dynkin diagram.

Let $e \in KG$ be the idempotent giving a Morita equivalence between KG and $K \times \cdots \times K$. Thus $KGeKG = KG$ and $eKGe \cong K \times \cdots \times K$.

Then one can show that $e(K[x, y]\#G)e \cong \Pi(Q)$, so $K[x, y]\#G$ is Morita equivalent to $\Pi(Q)$.

Similarly, there is a Morita equivalence between $(K\langle x, y\rangle\#G)/(yx - xy - \lambda)$ and $\Pi^{\lambda'}(Q)$.

One can show that $\mathcal{O}^{\lambda}$ is isomorphic to $e_i\Pi^{\lambda'}(Q)e_i$ for a suitable vertex i.

4 Homological algebra

Recommended book: C. A. Weibel, An introduction to homological algebra.

4.1 Complexes

Definition. Let R be a ring. A *chain complex* C (or $C.$ or C_*) consists of R-modules and homomorphisms

$$\ldots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \ldots$$

satisfying $d_n d_{n+1} = 0$ for all n. The elements of C_n are called *chains* of degree n or n-*chains*. The maps d_n are the *differential*.

If C is a chain complex, then its *homology* is defined by

$$H_n(C) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}) = Z_n(C)/B_n(C).$$

The elements of $B_n(C)$ are n-*boundaries*. The elements of $Z_n(C)$ are n-*cycles*. If x is an n-cycle we write $[x]$ for its image in $H_n(C)$.

A chain complex C is *acyclic* if $H_n(C) = 0$ for all n, that is, if it is an exact sequence. It is *non-negative* if $C_n = 0$ for $n < 0$. It is *bounded* if there are only finitely many nonzero C_n.

Definition A *cochain complex* C (or $C^\cdot$ or C^*) consists of R-modules and homomorphisms

$$\ldots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \ldots$$

satisfying $d^n d^{n-1} = 0$ for all n. The elements of C^n are called *cochains* of degree n or n-*cochains*.

The *cohomology* of a cochain complex is defined by

$$H^n(C) = \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1}) = Z^n(C)/B^n(C).$$

The elements of $B^n(C)$ are n-*coboundaries*. The elements of $Z^n(C)$ are n-*cocycles*.

Remarks. (i) There is no difference between chain and cohain complexes, apart from numbering. Pass between them by setting $C^n = C_{-n}$.

(ii) Many complexes are zero to the right, so naturally thought of as non-negative chain complexes, or zero to the left, so naturally thought of as non-negative cochain complexes.

(iii) More generally we could replace R-modules by objects in an abelian category.

Definition. The *category of cochain complexes* $C(R-Mod)$ has as objects the cochain complexes. A morphism $f : C \to D$ is given by homomorphisms $f^n : C^n \to D^n$ such that each square in the diagram commutes

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d} & C^{n-1} & \xrightarrow{d} & C^n & \xrightarrow{d} & C^{n+1} & \xrightarrow{d} & \dots \\
& & {\scriptstyle f^{n-1}}\downarrow & & {\scriptstyle f^n}\downarrow & & {\scriptstyle f^{n+1}}\downarrow & & \\
\dots & \xrightarrow{d} & D^{n-1} & \xrightarrow{d} & D^n & \xrightarrow{d} & D^{n+1} & \xrightarrow{d} & \dots
\end{array}$$

There is a *shift functor* $[i] : C(R-Mod) \to C(R-Mod)$ defined by $C[i]^n = C^{n+i}$ with the differential $d_{C[i]} = (-1)^i d_C$.

The category $C(R-Mod)$ is abelian. (It can be identified with the category of graded $R[d]/(d^2)$-modules, where R has degree 0 and d has degree 1 (or -1). Alternatively it is the category of modules for a ring with enough idempotents.)

Direct sums are computed degreewise, $(C \oplus D)^n = C^n \oplus D^n$. Also kernels and cokernels are computed degreewise. Thus a sequence $0 \to C \to D \to E \to 0$ is exact if and only if all $0 \to C^n \to D^n \to E^n \to 0$ are exact.

Lemma. A morphism of complexes $f : C \to D$ induces morphisms on cohomology $H^n(C) \to H^n(D)$, so H^n is a functor from $C(R-Mod)$ to R-Mod.

Proof. An arbitrary element of $H^n(C)$ is of the form $[x]$ with $x \in Z^n(C) = \operatorname{Ker} d^n$. We send it to $[f^n(x)] \in H^n(D)$.

Definition. A morphism of complexes $f : C \to D$ is a *quasi-isomorphism* if the map $H^n(C) \to H^n(D)$ is an isomorphism for all n.

Example. Morphism from $\mathbb{Z} \xrightarrow{a} \mathbb{Z}$ to $0 \to \mathbb{Z}/a\mathbb{Z}$ for $a \neq 0$.

Theorem. A short exact sequence of complexes $0 \to C \to D \to E \to 0$ induces a long exact sequence on cohomology

$$\dots \to H^{n-1}(E) \to H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to \dots$$

for suitable connecting maps $c^n : H^n(E) \to H^{n+1}(C)$.

Proof. For all n we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & C^n \arrow[r] \arrow[d, "d^n_C"'] & D^n \arrow[r] \arrow[d, "d^n_D"'] & E^n \arrow[r] \arrow[d, "d^n_E"'] & 0 \\ 0 \arrow[r] & C^{n+1} \arrow[r] & D^{n+1} \arrow[r] & E^{n+1} \arrow[r] & 0 \end{tikzcd}$$

and the easy part of the snake lemma gives exact sequences on kernels of the vertical maps

$$0 \to Z^n(C) \to Z^n(D) \to Z^n(E)$$

and on cokernels

$$C^{n+1}/B^{n+1}(C) \to D^{n+1}/B^{n+1}(D) \to E^{n+1}/B^{n+1}(E) \to 0$$

This holds for all n, so shows that the rows in the following diagram are exact

$$\begin{tikzcd} & C^n/B^n(C) \arrow[r] \arrow[d, "\overline{d}^n_C"'] & D^n/B^n(D) \arrow[r] \arrow[d, "\overline{d}^n_C"'] & E^n/B^n(E) \arrow[r] \arrow[d, "\overline{d}^n_E"'] & 0 \\ 0 \arrow[r] & Z^{n+1}(C) \arrow[r] & Z^{n+1}(D) \arrow[r] & Z^{n+1}(E). \end{tikzcd}$$

Here the vertical maps are induced by d^n_C, d^n_D and d^n_E, so the diagram commutes. Thus by the snake lemma one gets an exact sequence

$$\mathrm{Ker}(\overline{d}^n_C) \to \mathrm{Ker}(\overline{d}^n_D) \to \mathrm{Ker}(\overline{d}^n_E) \to \mathrm{Coker}(\overline{d}^n_C) \to \mathrm{Coker}(\overline{d}^n_D) \to \mathrm{Coker}(\overline{d}^n_E)$$

That is,

$$H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to H^{n+1}(E)$$

as required.

4.2 Ext

Definition. If M is an R-module, then a *projective resolution* of M is an exact sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

with the P_i projective modules. It is equivalent to give a non-negative chain complex P of projective modules and a quasi-isomorphism $P \to M$ (with M considered as a chain complex in degree 0),

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & {\scriptstyle\epsilon}\downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Note that every module has many different projective resolutions. Choose any surjection $\epsilon : P_0 \to M$, then any surjection $d_1 : P_1 \to \operatorname{Ker} \epsilon$, then any surjection $d_2 : P_2 \to \operatorname{Ker} d_1$, etc.

If one fixes a projective resolution of M then the *syzygies* of M are the modules $\Omega^n M = \operatorname{Im}(d : P_n \to P_{n-1})$ (and $\Omega^0 M = M$). Thus there are exact sequences

$$0 \to \Omega^{n+1} M \to P_n \to \Omega^n M \to 0.$$

Dually an *injective resolution* of a module X is an exact sequence

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

with the I^n injective modules. The cosyzygies are $\Omega^{-n} X = \operatorname{Im}(I^{n-1} \to I^n)$ (and $\Omega^0 X = X$), so

$$0 \to \Omega^{-n} X \to I^n \to \Omega^{-(n+1)} X \to 0.$$

Definition. Given modules M and X, choose a projective resolution $P_* \to M$ of M. We define $\operatorname{Ext}^n_R(M, X) = H^n(\operatorname{Hom}_R(P_*, X))$, the nth cohomology of the cohain complex of K-modules $\operatorname{Hom}_R(P_*, X)$, which is

$$\dots \to 0 \to 0 \to \operatorname{Hom}_R(P_0, X) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, X) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_2, X) \to \dots$$

where $\operatorname{Hom}_R(P_n, X)$ is in degree n.

Properties. (i) $\operatorname{Ext}^n_R(M, X)$ is a K-module, it is zero for $n < 0$, and $\operatorname{Ext}^0_R(M, X) \cong \operatorname{Hom}(M, X)$ since the exact sequence $P_1 \to P_0 \to M \to 0$ gives an exact sequence

$$0 \to \operatorname{Hom}_R(M, X) \to \operatorname{Hom}_R(P_0, X) \to \operatorname{Hom}_R(P_1, X).$$

(ii) This definition depends on the choice of the projective resolution. But we will show that $\operatorname{Ext}^n_R(M, X)$ can also be computed using an injective resolution of X, and that will show that it does not depend on the projective resolution of M.

(iii) $\operatorname{Ext}^n_R(M,X) = 0$ for $n > 0$ if X is injective. Namely, the sequence $\dots \to P_2 \to P_1 \to P_0$ is exact, hence so is the sequence

$$\operatorname{Hom}(P_0,X) \to \operatorname{Hom}(P_1,X) \to \operatorname{Hom}(P_2,X) \to \dots$$

Lemma. A map $X \to Y$ induces a map $\operatorname{Ext}^n_R(M,X) \to \operatorname{Ext}^n_R(M,Y)$, and in this way the assignment $X \rightsquigarrow \operatorname{Ext}^n_R(M,X)$ is a K-linear functor.

Proof. It induces a map of complexes $\operatorname{Hom}_R(P_*,X) \to \operatorname{Hom}_R(P_*,Y)$, and that induces a map on cohomology.

Proposition 1. A short exact sequence $0 \to X \to Y \to Z \to 0$ induces a long exact sequence

$$0 \to \operatorname{Hom}_R(M,X) \to \operatorname{Hom}_R(M,Y) \to \operatorname{Hom}_R(M,Y)$$

$$\to \operatorname{Ext}^1_R(M,X) \to \operatorname{Ext}^1_R(M,Y) \to \operatorname{Ext}^1_R(M,Z)$$

$$\to \operatorname{Ext}^2_R(M,X) \to \operatorname{Ext}^2_R(M,Y) \to \operatorname{Ext}^2_R(M,Z) \to \dots$$

Proof. One gets a sequence of complexes

$$0 \to \operatorname{Hom}_R(P_*,X) \to \operatorname{Hom}_R(P_*,Y) \to \operatorname{Hom}_R(P_*,Z) \to 0.$$

This is exact since each P_n is projective. Thus it induces a long exact sequence on cohomology.

Proposition 2. If $0 \to X \to I^0 \to I^1 \to I^2 \to \dots$ is an injective resolution of X, then one can compute $\operatorname{Ext}^n_R(M,X)$ as the nth cohomology of the complex $\operatorname{Hom}_R(M,I^*)$ if K-modules:

$$0 \to \operatorname{Hom}_R(M,I^0) \to \operatorname{Hom}_R(M,I^1) \to \operatorname{Hom}_R(M,I^2) \dots$$

Proof. Break the injective resolution into exact sequences

$$0 \to \Omega^{-i}X \to I^i \to \Omega^{-(i+1)}X \to 0$$

for $i \geq 0$ where $\Omega^0 X = X$. One gets long exact sequences

$$0 \to \operatorname{Hom}_R(M,\Omega^{-i}X) \to \operatorname{Hom}_R(M,I^i) \to \operatorname{Hom}_R(M,\Omega^{-(i+1)}X)$$

$$\to \mathrm{Ext}^1_R(M,\Omega^{-i}X) \to 0 \to \mathrm{Ext}^1_R(M,\Omega^{-(i+i)}X)$$
$$\to \mathrm{Ext}^2_R(M,\Omega^{-i}X) \to 0 \to \mathrm{Ext}^2_R(M,\Omega^{-(i+1)}X)\dots$$

so

$$\mathrm{Ext}^1_R(M,\Omega^{-i}X) \cong \mathrm{Coker}(\mathrm{Hom}_R(M,I^i) \to \mathrm{Hom}_R(M,\Omega^{-(i+1)}))$$

and

$$\mathrm{Ext}^j_R(M,\Omega^{-(i+1)}X) \cong \mathrm{Ext}^{j+1}_R(M,\Omega^{-i}X)$$

for $j \geq 1$. Thus (it is called *dimension shifting*)

$$\mathrm{Ext}^n_R(M,X) \cong \mathrm{Ext}^{n-1}_R(M,\Omega^{-1}X) \cong \dots \cong \mathrm{Ext}^1_R(M,\Omega^{-(n-1)}X)$$
$$\cong \mathrm{Coker}\left(\mathrm{Hom}_R(M,I^{n-1}) \to \mathrm{Hom}_R(M,\Omega^{-n}X)\right)$$

Now $0 \to \Omega^{-n}X \to I^n \to I^{n+1}$ is exact, hence so is

$$0 \to \mathrm{Hom}_R(M,\Omega^{-n}X) \to \mathrm{Hom}_R(M,I^n) \to \mathrm{Hom}_R(M,I^{n+1})$$

It follows that $\mathrm{Ext}^n_R(M,X)$ is the cohomology in degree n of the complex

$$\dots \to \mathrm{Hom}_R(M,I^{n-1}) \to \mathrm{Hom}_R(M,I^n) \to \mathrm{Hom}_R(M,I^{n+1}) \to \dots$$

as required.

Remarks. (i) As mentioned, it follows that $\mathrm{Ext}^n_R(M,X)$ does not depend on the projective resolution of M.

(ii) Using the description in terms of an injective resolution of X it follows that the assignment $M \rightsquigarrow \mathrm{Ext}^n(M,X)$ is a contravariant K-linear functor.

Also, if $0 \to L \to M \to N \to 0$ is an exact sequence and I^* is an injective resolution of X, then one gets an exact sequence of complexes $0 \to \mathrm{Hom}_R(N,I^*) \to \mathrm{Hom}_R(M,I^*) \to \mathrm{Hom}_R(L,I^*) \to 0$, and hence a long exact sequence

$$0 \to \mathrm{Hom}_R(N,X) \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(L,X)$$
$$\to \mathrm{Ext}^1_R(N,X) \to \mathrm{Ext}^1_R(M,X) \to \mathrm{Ext}^1_R(L,X)$$
$$\to \mathrm{Ext}^2_R(N,X) \to \mathrm{Ext}^2_R(M,X) \to \mathrm{Ext}^2_R(L,X) \to \dots$$

Example 1. If $0 \neq a \in \mathbb{Z}$ then $\mathbb{Z}/a\mathbb{Z}$ has projective resolution $0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to \mathbb{Z}/a\mathbb{Z} \to 0$. Thus $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z},X)$ is the cohomology of the complex

$$\dots \to 0 \to \mathrm{Hom}(\mathbb{Z},X) \xrightarrow{a} \mathrm{Hom}(\mathbb{Z},X) \to 0 \to \dots$$

that is,

$$\cdots \to 0 \to X \xrightarrow{a} X \to 0 \to \dots$$

so $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X) = \mathrm{Hom}(\mathbb{Z}/a\mathbb{Z}, X) \cong \{x \in X : ax = 0\}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X) \cong X/aX$.

Example 2. Let $R = K[x]/(x^2)$ with K a field. Any finitely generated module is a direct sum of copies of K (with x acting as 0) and R. The module K has projective resolution

$$\to R \xrightarrow{x} R \xrightarrow{x} R \to K \to 0.$$

Now $\mathrm{Hom}_R(R, K) = 0$, and we get $\mathrm{Ext}^n_R(K, K) \cong K$ for all $n \geq 0$.

Example 3. Consider the algebra $R = KQ/(\rho)$ given over a field K by a linearly oriented quiver Q and some zero relations ρ, e.g.

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

and $\rho = \{cba, dc\}$. Then R has basis the paths which don't go through a zero relation. Recall that R-modules correspond to representations of Q satisfying the relations.

For each vertex i there is a simple module $S(i)$ which as a representation is K at vertex i and zero elsewhere.

For each vertex i there is a projective module $P(i) = Re_i$. Considering $P(i)$ as a representation of Q, the vector space at vertex j has basis the paths from i to j which don't pass through a zero relation.

This gives representations

$$\begin{aligned}
S(1) &: K \to 0 \to 0 \to 0 \to 0, & P(1) &: K \to K \to K \to 0 \to 0,\\
S(2) &: 0 \to K \to 0 \to 0 \to 0, & P(2) &: 0 \to K \to K \to K \to 0,\\
S(3) &: 0 \to 0 \to K \to 0 \to 0, & P(3) &: 0 \to 0 \to K \to K \to 0,\\
S(4) &: 0 \to 0 \to 0 \to K \to 0, & P(4) &: 0 \to 0 \to 0 \to K \to K,\\
S(5) &: 0 \to 0 \to 0 \to 0 \to K, & P(5) &: 0 \to 0 \to 0 \to 0 \to K.
\end{aligned}$$

The simples have projective resolutions:

$$\begin{aligned}
0 \to P(5) \to{} & S(5) \to 0,\\
0 \to P(5) \to P(4) \to{} & S(4) \to 0,\\
0 \to P(5) \to P(4) \to P(3) \to{} & S(3) \to 0,\\
0 \to P(3) \to P(2) \to{} & S(2) \to 0,\\
0 \to P(5) \to P(4) \to P(2) \to P(1) \to{} & S(1) \to 0.
\end{aligned}$$

We can compute $\mathrm{Ext}^n_R(S(i), S(j))$ as the cohomology of the complex $\mathrm{Hom}_R(P_*, S(j))$ where P_* is a projective resolution of $S(i)$. Use that

$$\mathrm{Hom}_R(P(i), S(j)) = \mathrm{Hom}_R(Re_i, S(j)) = e_i S(j) = \begin{cases} K & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

For example for $\mathrm{Ext}^n(S(1), S(4))$ we have

$$0 \to \mathrm{Hom}(P_0, S(4)) \to \mathrm{Hom}(P_1, S(4)) \to \mathrm{Hom}(P_2, S(4)) \to \mathrm{Hom}(P_3, S(4)) \to \dots$$

which is

$$0 \to \mathrm{Hom}(P(1), S(4)) \to \mathrm{Hom}(P(2), S(4)) \to \mathrm{Hom}(P(4), S(4)) \to \mathrm{Hom}(P(5), S(4)) \to 0 \to \dots$$

which is

$$0 \to 0 \to 0 \to K \to 0 \to 0 \to \dots$$

so

$$\mathrm{Ext}^n_R(S(1), S(4)) = \begin{cases} K & (n = 2) \\ 0 & (n \neq 2) \end{cases}.$$

4.3 Description of Ext^1 using short exact sequences

Definition 1. Two short exact sequences ξ, ξ' with the same end terms are *equivalent* if there is a map θ (necessarily an isomorphism by the Snake Lemma) giving a commutative diagram

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\theta}\big\downarrow & & \| & & \\ \xi' : 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

It is easy to see that the split exact sequences form one equivalence class.

Definition 2. For any short exact sequence of modules

$$\xi : 0 \to L \to M \to N \to 0$$

we define an element $\hat{\xi} \in \mathrm{Ext}^1_R(N, L)$ as follows. The long exact sequence for $\mathrm{Hom}_R(N, -)$ gives a connecting map $\mathrm{Hom}_R(N, N) \to \mathrm{Ext}^1_R(N, L)$ and $\hat{\xi}$ is the image of id_N under this map.

Theorem 1. The assignment $\xi \mapsto \hat{\xi}$ gives a bijection between equivalence classes of short exact sequences $0 \to L \to M \to N \to 0$ and elements of $\mathrm{Ext}^1_R(N, L)$. The split exact sequences correspond to the element $0 \in \mathrm{Ext}^1_R(N, L)$.

Proof. Fix a projective resolution of N, and hence an exact sequence

$$0 \to \Omega^1 N \xrightarrow{\theta} P_0 \xrightarrow{\epsilon} N \to 0.$$

An exact sequence ξ gives a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathrm{Hom}(N,N) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(P_0, L) \arrow[r] \arrow[d, "\theta^*"'] & \mathrm{Hom}(P_0, M) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0, N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\Omega^1 N, L) \arrow[r] \arrow[d] & \mathrm{Hom}(\Omega^1 N, M) \arrow[r] & \mathrm{Hom}(\Omega^1 N, N) & \\
 & \mathrm{Ext}^1(N, L) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

and the connecting map $\mathrm{Hom}(N, N) \to \mathrm{Ext}^1(N, L)$ is given by diagram chasing, so by the choice of maps α, β giving a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0.
\end{tikzcd}$$

Then $\hat{\xi} = [\alpha]$ where $[\dots]$ denotes the map $\mathrm{Hom}(\Omega^1 N, L) \to \mathrm{Ext}^1(N, L)$.

Any element of $\mathrm{Ext}^1(N, L)$ arises from some ξ. Namely, write it as $[\alpha]$ for some $\alpha \in \mathrm{Hom}(\Omega^1 N, L)$. Then take ξ to be the pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Now if ξ, ξ' are equivalent exact sequences one gets a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & L \arrow[r] & M' \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

so ξ and ξ' correspond to the same map α, so $\hat{\xi} = \hat{\xi}'$. If two short exact sequences ξ, ξ' give the same element of $\mathrm{Ext}^1(N, L)$ there are diagrams with maps α, β and α', β' and with $\alpha - \alpha'$ in the image of the map θ^* : $\mathrm{Hom}(P_0, L) \to \mathrm{Hom}(\Omega^1 N, L)$. Say $\alpha - \alpha' = \phi\theta$ with $\phi : P_0 \to L$. Then there is also a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r, "\theta"] \arrow[d, "\alpha'"'] & P_0 \arrow[r] \arrow[d, "\beta - f\phi"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

This is a pushout, so by the uniqueness of pushouts, ξ and ξ' are equivalent.

Remark. Homomorphisms $L \to L'$ and $N'' \to N$ induce maps $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N, L')$ and $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N'', L)$. One can show that these maps correspond to pushouts and pullbacks of short exact sequences. For pushouts this follows directly from the definition. For pullbacks it needs more work - omitted.

Theorem 2. The following are equivalent for a module M.
(i) M is projective
(ii) $\mathrm{Ext}^n(M, X) = 0$ for all X and all $n > 0$.
(iii) $\mathrm{Ext}^1(M, X) = 0$ for all X.

The following are equivalent for a module X.
(i) X is injective
(ii) $\mathrm{Ext}^n(M, X) = 0$ for all M and all $n > 0$.
(iii) $\mathrm{Ext}^1(M, X) = 0$ for all cyclic modules M.

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are clear. (iii)$\Rightarrow$(i) using the characterization of a projective or injective module as one for which all short exact sequences ending or starting at the module split. In the injective case we use Baer's criterion: if I is a left ideal in R, the pushout of a sequence $0 \to I \to R \to$

$R/I \to 0$ along any map $I \to X$ spits. Using the splitting one gets a lift of the map to a map $R \to X$, and then by Baer's criterion X is injective.

4.4 Global dimension

Proposition/Definition 1. Let M be a module and $n \geq 0$. The following are equivalent.
(i) There is a projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$
(ii) $\mathrm{Ext}^m(M, X) = 0$ for all $m > n$ and all X.
(iii) $\mathrm{Ext}^{n+1}(M, X) = 0$ for all X.
(iv) For any projective resolution of M, we have $\Omega^n M$ projective.
The *projective dimension*, $\mathrm{proj.dim}\, M$, is the smallest n with this property (or ∞ if there is none).

Let X be a module and $n \geq 0$. The following are equivalent.
(i) There is an injective resolution $0 \to X \to I^0 \to \cdots \to I^n \to 0$
(ii) $\mathrm{Ext}^m(M, X) = 0$ for all $m > n$ and all X.
(iii) $\mathrm{Ext}^{n+1}(M, X) = 0$ for all **cyclic** M.
(iv) For any injective resolution of X, we have $\Omega^{-n} X$ injective.
The *injective dimension*, $\mathrm{inj.dim}\, X$, is the smallest n with this property (or ∞ if there is none).

Proof (i)⇒(ii)⇒(iii) are trivial. For (iii)⇒(iv) let $P_* \to M$ be a projective resolution. For any X, dimension shifting gives

$$0 = \mathrm{Ext}^{n+1}(M, X) \cong \mathrm{Ext}^n(\Omega^1 M, X) \cong \ldots \cong \mathrm{Ext}^1(\Omega^n M, X),$$

so $\Omega^n M$ is projective. Then

$$0 \to \Omega^n M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is also a projective resolution of M, giving (i).

Lemma. If $0 \to L \to M \to N \to 0$ is exact, then

$$\mathrm{proj.dim}\, M \leq \max\{\mathrm{proj.dim}\, L, \mathrm{proj.dim}\, N\},$$
$$\mathrm{inj.dim}\, M \leq \max\{\mathrm{inj.dim}\, L, \mathrm{inj.dim}\, N\}.$$

Proof. For any X the long exact sequence for $\mathrm{Hom}(-, X)$ gives an exact sequence

$$\mathrm{Ext}^{n+1}(N, X) \to \mathrm{Ext}^{n+1}(M, X) \to \mathrm{Ext}^{n+1}(L, X)$$

and the outer terms are zero for $n = \max$.

Definition. The *(left) global dimension* of R (in $\mathbb{N} \cup \{\infty\}$) is

$$\begin{aligned}
\operatorname{gl.dim} R &= \sup\{\operatorname{proj.dim} M : M \in R - Mod\} \\
&= \inf\{n \in \mathbb{N} : \operatorname{Ext}^{n+1}(M, X) = 0 \forall M, X\} \\
&= \sup\{\operatorname{inj.dim} X : X \in R - Mod\} \\
&= \inf\{n \in \mathbb{N} : \operatorname{Ext}^{n+1}(M, X) = 0 \forall M, X,\ M \text{ cyclic}\} \\
&= \sup\{\operatorname{proj.dim} M : M \text{ cyclic}\}.
\end{aligned}$$

Example. $\operatorname{gl.dim} R = 0 \Leftrightarrow$ all modules are projective $\Leftrightarrow$ all short exact sequences split $\Leftrightarrow$ every submodule has a complement $\Leftrightarrow$ R is semisimple artinian.

Proposition/Definition 2. A ring R is said to be *(left) hereditary* if it satisfies the following equivalent conditions
(i) $\operatorname{gl.dim} R \leq 1$.
(ii) Every submodule of a projective module is projective.
(iii) Every left ideal in R is projective.

Proof of equivalence. (i)$\Rightarrow$(ii) If N is a submodule of P then for any X, by the long exact sequence, $\operatorname{Ext}^1(N, X) \cong \operatorname{Ext}^2(P/N, X) = 0$.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) For any X and left ideal I we have $\operatorname{Ext}^2(R/I, X) \cong \operatorname{Ext}^1(I, X) = 0$, so X has injective dimension ≤ 1.

Examples. A principal ideal domain or a path algebra KQ of a quiver over a field is hereditary.

Theorem. Consider a skew polynomial ring $S = R[x; \sigma, \delta]$ with σ an automorphism of R and δ a σ-derivation.
(i) For any S-module M there is a an exact sequence

$$0 \to S \otimes_R ({}_{\sigma^{-1}}M) \xrightarrow{f} S \otimes_R M \xrightarrow{g} M \to 0$$

where g is multiplication and $f(s \otimes m) = sx \otimes m - s \otimes xm$.
(ii) $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$.
(iii) $\operatorname{gl.dim} S = 1 + \operatorname{gl.dim} R$ if $\delta = 0$.

Proof. (i) Since σ is an automorphism, S is a free right R-module with basis $\{1, x, x^2, \dots\}$, so the elements of $S \otimes_R N$ can be written uniquely as expressions $\sum x^i \otimes n_i$.

The map f is well-defined: define $f' : S \otimes_K M \to S \otimes_R M$ by $f'(s \otimes m) = sx \otimes m - s \otimes xm$. Then since $xr = \sigma(r)x + \delta(r)$ we get

$$\begin{aligned}
f'(sr \otimes m) - f'(s \otimes \sigma^{-1}(r)m) &= srx \otimes m - sr \otimes xm - sx \otimes \sigma^{-1}(r)m + s \otimes x\sigma^{-1}(r)m \\
&= s(rx - x\sigma^{-1}(r)) \otimes m + s \otimes (x\sigma^{-1}(r) - rx)m \\
&= -s\delta(\sigma^{-1}(r)) \otimes m + s \otimes \delta(\sigma^{-1}(r))m = 0.
\end{aligned}$$

Thus f' descends to a map f.

Exact in middle. Clearly $gf = 0$. Choose an element of $\operatorname{Ker} g$ of the form $x^i \otimes m+$ lower powers of x, with $m \neq 0$ and i minimal. Then $i = 0$, for otherwise one can cancel the leading term by subtracting $f(x^{i-1} \otimes m)$. Thus the element is $1 \otimes m$. But then since the element is in $\operatorname{Ker} g$, it is zero.

Exact on left: an element of the form $x^i \otimes m+$ lower powers of x with $m \neq 0$ is sent by f to $x^{i+1} \otimes m+$ lower powers of x, which cannot be zero.

(ii) S_R is free, so flat, so a projective resolution $P_* \to N$ of an R-module N gives an S-module projective resolution $S \otimes_R P_* \to S \otimes_R N$. Using that $\operatorname{Hom}_S(S \otimes_R -, X) \cong \operatorname{Hom}_R(-, X)$ for an S-module X, it follows that $\operatorname{Ext}^n_S(S \otimes_R N, X) \cong \operatorname{Ext}^n_R(N, X)$.

By the long exact sequence for $\operatorname{Hom}_S(-, X)$ we get

$$\operatorname{Ext}^n_S(S \otimes M, X) \xrightarrow{h} \operatorname{Ext}^n_S(S \otimes_{\sigma^{-1}} M, X) \to \operatorname{Ext}^{n+1}_S(M, X) \to \operatorname{Ext}^{n+1}_S(S \otimes M, X).$$

For $n > \operatorname{gl.dim} R$, the second and fourth terms are zero, so also the third term is zero, so $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$.

(iii) Let X be an R-module and $X \to I^*$ an injective resolution. We get cosyzygies $0 \to \Omega^{-(i-1)}X \to I^i \to \Omega^{-i}X \to 0$. Since $\delta = 0$, we can consider all of these as S-modules with x acting as 0, so for any S-module U, we get a long exact sequence

$$0 \to \operatorname{Hom}_S(U, \Omega^{-(i-1)}X) \to \operatorname{Hom}(U, I^i) \to \operatorname{Hom}_S(U, \Omega^{-i}X) \to \operatorname{Ext}^1_S(U, \Omega^{-(i-1)}X) \to \dots$$

Now suppose $U = S \otimes_R N$. If $j > 0$ we have $\operatorname{Ext}^j_S(U, I^i) \cong \operatorname{Ext}^j_R(N, I^i) = 0$, so as in dimension shifting, we get

$$\operatorname{Hom}_S(U, \Omega^{-n}X) \twoheadrightarrow \operatorname{Ext}^1_S(U, \Omega^{-(n-1)}X) \cong \dots \cong \operatorname{Ext}^n_S(U, X).$$

Applying this to the map f we get a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_S(S \otimes M, \Omega^{-n}X) & \longrightarrow & \mathrm{Ext}^n_S(S \otimes M, X) \\
{\scriptstyle f'}\big\downarrow & & {\scriptstyle h}\big\downarrow \\
\mathrm{Hom}_S(S \otimes {}_{\sigma^{-1}}M, \Omega^{-n}X) & \longrightarrow & \mathrm{Ext}^n_S(S \otimes {}_{\sigma^{-1}}M, X)
\end{array}$$

where f' is composition with f.

Since x acts as zero on M and $\Omega^{-n}X$ it follows that f' is zero. Namely $f'(\phi)(s\otimes m) = \phi f(s\otimes m) = \phi(sx\otimes m - s\otimes xm) = \phi(sx\otimes m) = sx\phi(1\otimes m) = 0$.

Since the horizontal maps are onto, h is zero. Thus for $n = \mathrm{gl.\,dim}\, R$ we get $\mathrm{Ext}^{n+1}_S(M,X) \cong \mathrm{Ext}^n_R({}_{\sigma^{-1}}M, X)$, and for suitable M, X this is non-zero. Thus $\mathrm{gl.\,dim}\, S = 1 + \mathrm{gl.\,dim}\, R$.

Corollary. If K is a field, then $\mathrm{gl.\,dim}\, K[x_1, \ldots, x_n] = n$.

4.5 Tor

Given a right R-module M and a left R-module X, choose a projective resolution $P_* \to M$ (or more generally a *flat resolution*, where we only require the P_n to be flat). We define $\mathrm{Tor}^R_n(M,X)$ to be the nth homology of the complex

$$P_* \otimes_R X : \cdots \to P_2 \otimes_R X \to P_1 \otimes_R X \to P_0 \otimes_R X \to 0$$

Since the tensor product is a right exact functor, it follows that $\mathrm{Tor}^R_0(M,X) \cong M \otimes_R X$. Moreover a short exact sequence $0 \to X \to Y \to Z \to 0$ gives a long exact sequence

$$\to \mathrm{Tor}^R_2(M,Z) \to \mathrm{Tor}^R_1(M,X) \to \mathrm{Tor}^R_1(M,Y) \to \mathrm{Tor}^R_1(M,Z) \to$$

$$\to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0.$$

Using this one can show that Tor can be computed using a projective or flat resolution of X. Thus the two modules M, X play a symmetrical role; Tor_n is a covariant functor in both arguments. This shows independence of the resolution.

Theorem. The following are equivalent for a module M.
(i) M is flat

(ii) $\mathrm{Tor}_n^R(M, X) = 0$ for all X and all $n > 0$.
(iii) $\mathrm{Tor}_1^R(M, X) = 0$ for all X.

Proposition/Definition. Let M be a module and $n \geq 0$. The following are equivalent.
(i) There is a flat resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$
(ii) $\mathrm{Tor}_m^R(M, X) = 0$ for all X and $m > n$
(iii) $\mathrm{Tor}_{n+1}^R(M, X) = 0$ for all X.
(iv) For any flat resolution of M, we have $\Omega^n M$ flat.
The *flat dimension* $\mathrm{flatdim}\, M$ is the smallest n with this property (or ∞ if there is none).

Definition. The *weak dimension* of R is

$$\mathrm{w.dim}\, R = \sup\{\mathrm{flatdim}\, M : \forall M\} = \inf\{n \in \mathbb{N} : \mathrm{Tor}_{n+1}^R(M, X) = 0 \forall M, X\}.$$

It is left/right symmetric.

4.6 Global dimension for noetherian rings

Proposition 1. (i) For M an R-module, $\mathrm{flatdim}\, M \leq \mathrm{proj.dim}\, M$, with equality if M is a finitely generated and R is left noetherian.
(ii) Thus $\mathrm{w.dim}\, R \leq \mathrm{gl.dim}\, R$, with equality if R is left noetherian.
(iii) If R is (left and right) noetherian, the left and right global dimensions or R are equal.

Proof. (i) The inequality holds since any projective resolution is also a flat resolution. If R is left noetherian and M is f.g., we have a projective resolution with all P_n finitely generated. Then $\mathrm{flatdim}\, M \leq n$ implies $\Omega^n M$ is flat. Since it is also finitely presented, it is projective. Thus $\mathrm{proj.dim}\, M \leq n$.

(ii) Use that $\mathrm{gl.dim}\, R = \sup\{\mathrm{proj.dim}\, M : M \text{ cyclic}\}$.

(iii) Clear.

Let K be a field of characteristic zero. For simplicity suppose it is algebraically closed. Recall that the first Weyl algebra is

$$R = A_1(K) = K[x][y; d/dx] = K\langle x, y\rangle/(yx - xy - 1).$$

We know $\mathrm{gl.dim}\, R \leq 2$. In fact more is true.

Theorem. The first Weyl algebra is hereditary.

Lemma 1. $S = k[x] \setminus \{0\}$ is a left and right Ore set in R and $R_S \cong K(x)[y; d/dx]$. Thus $\mathrm{gl.\,dim}\, R_S \leq 1$.

Proof. To show S is a left Ore set, given $a \in R$ and $s \in S$ we need to find a', s' with $a's = s'a$. We do this by induction on the order of a as a differential operator. Now $[a, s]$ has smaller order, so there is a'', s'' with $a''s = s''[a, s]$. Then $(s''a - a'')s = s''sa$, so we can take $a' = s''a - a''$ and $s' = s''s$. The rest is straightforward.

Lemma 2. If M is a finitely generated R-module which is torsion-free as a $k[x]$-module, then $\mathrm{proj.\,dim}\, M \leq 1$.

Proof. Since M is torsion-free over $k[x]$, the natural map $M \to S^{-1}M$ is injective. Now $S^{-1}M$ is a module for $K(x)[y; d/dx]$ so it has a projective resolution $0 \to Q_1 \to Q_0 \to S^{-1}M \to 0$. As R_S is flat as left R-module, Q_0 and Q_1 are flat R-modules, so $\mathrm{flatdim}_R\, S^{-1}M \leq 1$.

Now M embeds in $S^{-1}M$ and $\mathrm{w.\,dim}\, R = \mathrm{gl.\,dim}\, R \leq 2$, so for any L the long exact sequence gives an exact sequence

$$\to \mathrm{Tor}_3^R(L, (S^{-1}M)/M) \to \mathrm{Tor}_2^R(L, M) \to \mathrm{Tor}_2^R(L, S^{-1}M) \to$$

The outside terms are zero, so $\mathrm{flatdim}\, M \leq 1$. Now use that M is finitely generated.

Lemma 3. If $\lambda \in K$, then the R-module $S_\lambda = R/R(x - \lambda)$ is simple and $\mathrm{proj.\,dim}\, S_\lambda \leq 1$.

Proof. Any element of R can be written uniquely as a sum $\sum_n y^n p_n(x)$, so as a K-linear combination of elements $y^n(x-\lambda)^m$. Thus S_λ can be identified with $K[y]$, with y acting by multiplication and the action of x given by $xq(y) = \lambda q(y) - q'(y)$.

To show simplicity, note that the action of $(\lambda - x)$ on $K[y]$ is as differentiation by y, so the submodule generated by any non-zero element of $K[y]$ contains 1, and hence this submodule is all of $K[y]$.

Now we have projective resolution $0 \to R \xrightarrow{\cdot(x-\lambda)} R \to S_\lambda \to 0$.

Proof of the theorem. It suffices to show that $\mathrm{proj.\,dim}\, M \leq 1$ for M cyclic.

If M is not torsion-free over $K[x]$, then some non-zero element of M is killed by a non-zero polynomial $p(x)$. Since K is algebraically closed, we can factorize this polynomial, and hence find $0 \neq m \in M$ and $\lambda \in K$ with $(x -$

$\lambda)m = 0$. Then m generates a submodule of M isomorphic to S_λ. Repeating with the quotient module, we get an ascending chain of submodules of M, and since M is noetherian this terminates. Thus we get submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq M$$

such that each $M_i/M_{i-1} \cong S_{\lambda_i}$ and M/M_k is torsion-free as a $K[x]$-module.

The quotients M_i/M_{i-1} and M/M_k all have projective dimension ≤ 1, and hence $\operatorname{proj.dim} M \leq 1$.

Some other facts about noetherian rings.

(i) R is left noetherian $\Leftrightarrow$ any direct sum of injective modules is injective $\Leftrightarrow$ any injective module is a direct sum of indecomposable modules. See for example Lam, Lectures on modules and rings.

(ii) (Chase) Any product of flat modules is flat if and only if any finitely generated left ideal is finitely presented. In particular this holds if R is left noetherian, or left hereditary.

(iii) If R is left noetherian ring and $\operatorname{gl.dim} R < \infty$ then

$$\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim} S : S \text{ simple}\}.$$

For a proof see McConnell and Robson, Noncommutative noetherian rings, Corollary 7.1.14.

Some relevant books

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