Noncommutative Algebra 3: Geometric methods for representations of algebras

Bielefeld University, Winter Semester 2017/18

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Literature for algebraic geometry

G. R. Kempf, 'Algebraic varieties', Cambridge Univ. Press 1993.
R. Hartshorne, 'Algebraic geometry', Springer 1977.
D. Mumford, 'Algebraic geometry I. complex projective varieties'.
D. Mumford, 'The red book of varieties and schemes'.

1 Varieties

We fix an algebraically closed field K of arbitrary characteristic.

1.1 Topological spaces and locally closed subsets

Recall that a *topological space* is given by a set X together with a set of subsets of X, the *open sets* such that

- $\emptyset$ and X are open.

- Any union of open sets is open

- A finite intersection of open sets is open.

Any subset Y of a topological space X becomes a topological space with the *induced topology*, in which the open sets are the sets of the form $Y \cap U$ with U an open subset of X.

Definition. A subset S of a topological space X is *locally closed* if the following equivalent conditions hold:
(i) S is an open subset of a closed subset of X
(ii) S is open in its closure
(iii) S is the intersection of an open and a closed subset of X.

Proof of equivalence. Exercise.

Lemma. If $X \subseteq Y \subseteq Z$ and Y is locally closed in Z, then X is locally closed in Y iff it is locally closed in Z.

Proof. Exercise.

Definition. A topological space X is *connected* if it cannot be written as a disjoint union of two open and closed subsets.

A topological space X is *irreducible* if $X \neq \emptyset$ and $X = Y \cup Z$ with Y and Z closed subsets implies $Y = X$ or $Z = X$. Equivalently any non-empty open subset is dense.

An irreducible topological space is connected, but it is far from being Hausdorff.

1.2 Spaces with functions

If U is a set, then the set of functions $U \to K$ becomes a commutative K-algebra under the pointwise operations

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x).$$

Definition. [See Kempf] A *space with functions* consists of a topological space X and an assignment for each open set $U \subseteq X$ of a K-subalgebra $\mathcal{O}(U)$ of the algebra of functions $U \to K$, satisfying:

(a) If U is a union of open sets, $U = \bigcup U_\alpha$, then $f \in \mathcal{O}(U)$ iff $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for all α.

(b) If $f \in \mathcal{O}(U)$ then $D(f) = \{u \in U \mid f(u) \neq 0\}$ is open in U and $1/f \in \mathcal{O}(D(f))$.

Elements of $\mathcal{O}(U)$ are called *regular functions*. We sometimes write it as $\mathcal{O}_X(U)$.

A *morphism* of spaces with functions is a continuous map $\theta : X \to Y$ with the property that for any open subset U of Y, and any $f \in \mathcal{O}(U)$, the composition

$$\theta^{-1}(U) \xrightarrow{\theta} U \xrightarrow{f} K$$

is in $\mathcal{O}(\theta^{-1}(U))$. In this way one gets a category of spaces with functions.

Examples.
(1) Let X be a topological space, and choose any topology on K. Let $\mathcal{O}(U)$ be the set of continuous functions $U \to K$. Morphisms between such spaces with functions are continuous maps.
(2) X manifold, $\mathcal{O}(U)$ = infinitely differentiable functions $U \to \mathbb{R}$. Morphisms are infinitely differential maps between manifolds.
(3) X complex manifold, eg the complex plane, $\mathcal{O}(U)$ = analytic functions $U \to \mathbb{C}$.

Definition. If X is a space with functions and Y is a subset of X, one defines $\mathcal{O}(Y)$ to be the set of functions $f : Y \to K$ such that each $y \in Y$ has a neighbourhood U in X such that $f|_{Y\cap U} = g|_{Y\cap U}$ for some $g \in \mathcal{O}(U)$.

Any subset Y of a space with functions X has an *induced structure as a space with functions* by equipping Y with the subspace topology and open subsets of Y with the induced sets of functions.

We are only interested in the case where Y is locally closed in X.

Lemma. The inclusion $i : Y \to X$ is a morphism of spaces with functions, and if Z is a space with functions, then $\theta : Z \to Y$ is a morphism if and only if $i\theta : Z \to X$ is a morphism.

Proof. Exercise.

Theorem. If X and Y are spaces with functions, then the set $X \times Y$ can be given the structure of a space with functions, so that it becomes a product of X and Y in the category of spaces with functions.

Proof. See Kempf, Lemma 3.1.1. The topology is not the usual product topology. Instead a basis of open sets is given by the sets

$$\{(u, v) \in U \times V : f(u, v) \neq 0\}$$

where U is open in X, V is open in Y and $f(x, y) = \sum_{i=1}^{n} g_i(x)h_i(y)$ with $g_i \in \mathcal{O}(U)$ and $h_i \in \mathcal{O}(V)$

Lemma. The image of an open set under the projection $p : X \times Y \to X$ is open.

Proof. For $y \in Y$, the categorical product gives a morphism $i_y : X \to X \times Y$ with $i_y(x) = (x, y)$. Now if $U \subseteq X \times Y$, then $p(U) = \bigcup_{y \in Y} i_y^{-1}(U)$, which is open.

Our spaces with functions will usually not be Hausdorff. Instead the following usually holds.

Definition. A space with functions X is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$.

Note that using the product topology, the diagonal is closed if and only if X is Hausdorff.

Note that separatedness passes to subsets of a space with functions equipped with the induced structure, for if Y is a subset of X, then $\Delta_Y = (Y \times Y) \cap \Delta_X$ in $X \times X$.

1.3 Affine space

Affine n-space is $\mathbb{A}^n = K^n$ considered as a space with functions

- The topology is the *Zariski topology.* Closed sets are of the form

$$V(S) = \{(x_1, \dots, x_n) \in K^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}$$

where S is a subset of the polynomial ring $K[X_1, \dots, X_n]$. Observe that $V(S) = V(I)$, where I is the ideal generated by S.

Equivalently, the open sets are unions of sets of the form

$$D(f) = \{(x_1, \dots, x_n) \in K^n \mid f(x_1, \dots, x_n) \neq 0\}$$

with $f \in K[X_1, \dots, X_n]$.

This is a topology since $D(1) = K^n$, $D(0) = \emptyset$ and $D(f) \cap D(g) = D(fg)$, so

$$(\bigcup_\lambda D(f_\lambda)) \cap (\bigcup_\mu D(g_\mu)) = \bigcup_{\lambda,\mu} D(f_\lambda g_\mu).$$

For example, for $\mathbb{A}^1$, if $0 \neq f \in K[X]$ then $V(f)$ is a finite set. Thus the closed subsets of $\mathbb{A}^1$ are $\emptyset$, finite subsets, and $\mathbb{A}^1$. Thus the nonempty open sets in $\mathbb{A}^1$ are the cofinite subsets $\mathbb{A}^1 \setminus \{a_1, \dots, a_k\}$. This is NOT Hausdorff.

- If U is an open subset of $\mathbb{A}^n$ then $\mathcal{O}(U)$ consists of the functions $f : U \to K$ such that each point $u \in U$ has an open neighbourhood $W \subseteq U$ such that $f|_W = p/q$ with $p, q \in K[X_1, \dots, X_n]$ and $q(x_1, \dots, x_n) \neq 0$ for all $(x_1, \dots, x_n) \in W$.

Theorem.
(i) This turns $\mathbb{A}^n$ into a space with functions.
(ii) Any open subset of $\mathbb{A}^n$ is a finite union $D(f_1) \cup \cdots \cup D(f_m)$.
(iii) It is irreducible.

Proof. (i) Straightforward, since the regular functions are defined locally.

(ii) If U is an open set, say $U = \mathbb{A}^n \setminus V(S)$ then

$$V(S) = V(I) = V((f_1, \ldots, f_m)) = V(f_1) \cap \cdots \cap V(f_m)$$

since any ideal I is finitely generated, so $U = D(f_1) \cup \cdots \cup D(f_m)$.

(iii) Since K is algebraically closed it is infinite. Thus any non-zero polynomial in $K[X]$ is non-zero on some element of K. An induction on n shows that any non-zero polynomial in $K[X_1, \ldots, X_n]$ is non-zero at some element $(x_1, \ldots, x_n) \in \mathbb{A}^n$. Thus $D(f) \neq \emptyset$ iff f is non-zero. For irreduciblity it is equivalent to show that any two non-empty open subsets of $\mathbb{A}^n$ have non-empty intersection. Now one contains $D(f)$ and the other $D(g)$ with f and g nonzero polynomials. Then $fg \neq 0$ since the polynomial ring is a domain, so $D(f) \cap D(g) = D(fg) \neq \emptyset$.

Theorem. If X is a space with functions, then a mapping

$$\theta : X \to \mathbb{A}^n, \quad \theta(x) = (\theta_1(x), \ldots, \theta_n(x))$$

is a morphism of spaces with functions iff the θ_i are regular functions on X.

Proof. Since the ith projection $\pi_i : \mathbb{A}^n \to K$ is regular, if θ is a morphism then $\theta_i = \pi_i \theta$ is regular.

Suppose $\theta_1, \ldots, \theta_n$ are regular. Let U be an open subset of $\mathbb{A}^n$ and $f = p/q \in \mathcal{O}(U)$ with $q(u) \neq 0$ for $u \in U$. We need to show that $f\theta$ is regular on $\theta^{-1}(U)$. Now by assumption $p\theta = p(\theta_1(x), \ldots, p_n(x))$ and $q\theta$ are regular on U. Also $q\theta$ is non-vanishing on $\theta^{-1}(U)$. Thus $p\theta/q\theta$ is regular on $\theta^{-1}(U)$.

Corollary 1. $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$.

Corollary 2. $\mathbb{A}^n$ is separated.

Proof. The diagonal for $\mathbb{A}^n$ is

$$\Delta_{\mathbb{A}^n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{A}^{2n} : x_1 = y_1, \ldots, x_n = y_n\}$$

so it is closed.

Coordinate-free description. If V is an n-dimensional vector space, then by choosing a basis we can identify $V \cong \mathbb{A}^n$, and then V becomes a space with functions. Choosing a different basis gives an isomorphic space with functions.

1.4 Affine varieties

Definition. An *affine variety* is a space with functions which is (isomorphic to) a closed subset of $\mathbb{A}^n$.

The *coordinate ring* of an affine variety X is $\mathcal{O}(X)$. It is often denoted $K[X]$.

Example (Determinantal varieties). If V and W are f. d. vector spaces then the space $\mathrm{Hom}(V, W)_{\leq r}$ of linear maps of rank $\leq r$ is closed in $\mathrm{Hom}(V, W)$, so an affine variety. Choosing bases, $\mathrm{Hom}(V, W) \cong M_{m\times n}(K)$, and the matrices of rank $\leq r$ are exactly those for which all minors of size $r + 1$ vanish.

Definition. Given any ideal I in a commutative ring A, we define the radical of I to be

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$

It is an ideal. The ideal I is *radical* if $I = \sqrt{I}$, that is, $a^n \in I$ implies $a \in I$. Equivalently, if the factor ring A/I is *reduced*, that is, it has no nonzero nilpotent elements.

Theorem. If I is an ideal in $K[X_1, \dots, X_n]$ and $X = V(I)$ is the corresponding closed subset of $\mathbb{A}^n$, then the natural map

$$K[X_1, \dots, X_n] \to \mathcal{O}(X)$$

is surjective, and has kernel $\sqrt{I}$. In particular, if X is an affine variety, then $\mathcal{O}(X)$ is a finitely generated K-algebra which is reduced.

Proof. For surjectivity, adapt Hartshorne, Proposition II.2.2. The statement about the kernel is Hilbert's Nullstellensatz.

Theorem. If X is an affine variety, and Z is a space with functions, then the map

$$\mathrm{Hom}_{\text{spaces with functions}}(Z, X) \to \mathrm{Hom}_{K\text{-algebras}}(\mathcal{O}(X), \mathcal{O}(Z))$$

sending $\theta : Z \to X$ to the composition map $f \mapsto f\theta$, is a bijection.

Proof. Implicit in Kempf.

Corollary. There is an anti-equivalence between the categories of affine varieties and finitely generated reduced K-algebras. The variety corresponding to a finitely generated reduced K-algebra A is denoted $\operatorname{Spec} A$.

Proof. It just remains to observe that all finitely generated reduced K-algebras arise.

Proposition. An affine variety X is irreducible iff $\mathcal{O}(X)$ is a domain.

Proof. Say $X = V(I)$ with I an ideal in $K[X_1, \ldots, X_n]$. We may assume that I is radical.

If there are zero divisors, there are $f, g \notin I$ with $fg \in I$. Then $X = (V(I) \cap V(f)) \cup (V(I) \cap V(g))$.

Conversely if X is not irreducible, then it has non-empty open subsets with empty intersection, say $(V(I) \cap D(f)) \cap (V(I) \cap D(g)) = \emptyset$. Then $V(I) \cap D(fg) = \emptyset$, so fg vanishes on X, so $fg \in \sqrt{I} = I$, but $f, g \notin I$.

1.5 Abstract varieties

Definition. A *variety* is a space with functions X with a finite open covering $X = U_1 \cup \cdots \cup U_n$ by affine varieties. Usually one also includes in the definition that X must be separated.

A *subvariety* Y of a variety X is a locally closed subset equipped with the induced structure as a space with functions. A *quasi-affine variety* is an open subvariety of an affine variety, or equivalently a subvariety of affine space.

Theorem. (i) If $f \in K[X_1, \ldots, X_n]$, then the open subvariety $D(f)$ of $\mathbb{A}^n$ is isomorphic to the affine variety

$$\{(x_1, \ldots, x_n, t) \in \mathbb{A}^{n+1} : f(x_1, \ldots, x_n) \cdot t = 1\}.$$

(ii) Any subvariety is a variety.

Proof. (i) The maps are the projection $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_n)$ and the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1/f(x_1, \ldots, x_n))$. Now $1/f \in \mathcal{O}(D(f))$, so both are morphisms.

(ii) Suppose $Y \subseteq X$. We need to show that Y is a finite union of affine open subsets. Sunce X is a finite union of affine opens, we may reduce to the case when X is affine. We may also assume that Y is open in X and X is closed in $\mathbb{A}^n$. But then $Y = X \cap U$ with $U = D(f_1) \cup \cdots \cup D(f_m)$ open in $\mathbb{A}^n$, and then $Y = V_1 \cup \cdots \cup V_m$ with $V_i = X \cap D(f_i)$ a closed subset of the affine variety $D(f_i)$, hence affine.

Remarks. (i) (Added after the lecture) I should have mentioned before that any polynomial ring $K[X_1, \ldots, X_n]$ over a field is a unique factorization domain (UFD). It follows that any irreducible polynomial $f \in K[X_1, \ldots, X_n]$ is *prime*, that is, the ideal it generates (f) is a prime ideal, or equivalently the factor ring $K[X_1, \ldots, X_n]/(f)$ is a domain. In particular it is reduced, so $\sqrt{(f)} = (f)$. It follows that the coordinate ring of $V(f)$, the affine variety in $\mathbb{A}^n$ defined by f, is $K[X_1, \ldots, X_n]/(f)$ and that it is irreducible.

Given any polynomial $f \in K[X_1, \ldots, X_n]$, the set $D(f)$ is isomorphic to the closed subset $V(g) = \{(x_1, \ldots, x_n, t) \in \mathbb{A}^{n+1} : f(x_1, \ldots, x_n) \cdot t = 1\}$, where $g(X_1, \ldots, X_n, T) = f(X_1, \ldots, X_n) \cdot T - 1$. Now the polynomial g is easily seen to be irreducible, so

$$\begin{aligned}\mathcal{O}(D(f)) &\cong \mathcal{O}(V(g)) \cong K[X_1, \ldots, X_n, T]/(f(X_1, \ldots, X_n) \cdot T - 1) \\ &\cong K[X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)].\end{aligned}$$

(The last ring is alternative notation for the Ore localization of the ring $K[X_1, \ldots, X_n]$ at the multiplicative subset consisting of all powers of f.)

(ii) The example of $D(f)$ shows that some quasi-affine varieties are again affine. But this is not always true. For example $U = \mathbb{A}^2 \setminus \{0\} = D(X_1) \cup D(X_2)$ is quasi-affine but not affine.

To see this, we show first that $\mathcal{O}(U) = K[X_1, X_2]$. A function $f \in \mathcal{O}(U)$ is determined by its restrictions f_i to $D(X_i)$ $(i = 1, 2)$. Now $f_i \in \mathcal{O}(D(X_i)) = K[X_1, X_2, X_i^{-1}]$. Moreover the restrictions of f_1 and f_2 to $D(X_1) \cap D(X_2) = D(X_1 X_2)$ are equal, so f_1 and f_2 are equal as elements of $K[X_1, X_2, 1/X_1 X_2]$. But this is only possible if they are both in $K[X_1, X_2]$, and equal. Thus $f \in K[X_1, X_2]$.

Now the inclusion morphism $\theta : U \to \mathbb{A}^2$ induces a homomorphism $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(U)$ which is actually an isomorphism. Now the corollary in the last section says that the category of affine varieties is anti-equivalent to the category of finitely generated reduced K-algebras. If U were affine, then since the map on coordinate rings is an isomorphism, θ would have to be an isomorphism. But is isn't.

(iii) A coordinate-free example of a variety. If V and W are vector spaces, the set of injective linear maps $\mathrm{Inj}(V, W)$ is an open in $\mathrm{Hom}(V, W)$, since the complement is $\mathrm{Hom}_{\leq r}(V, W)$ where $r = \dim V - 1$. Thus $\mathrm{Inj}(V, W)$ is a quasi-affine variety.

Theorem. A product of varieties $X \times Y$ is a variety. If X and Y are irreducible, so is $X \times Y$.

Proof. Recall that the product $X \times Y$ exists for any two spaces with functions. It is straightforward that if $U \subseteq X$ and $V \subseteq Y$ are open (resp. closed) subsets, then $U \times V$ is open (resp. closed) in $X \times Y$. Moreover with the induced structure as a space with functions it is a categorical product.

Since any variety is a finite union of affine open subsets, decomposing X and Y it suffices to prove that a product of affine varieties is affine. Now if X is closed in $\mathbb{A}^n$ and Y is closed in $\mathbb{A}^m$ then $X \times Y$ is closed in $\mathbb{A}^n \times \mathbb{A}^m \cong A^{n+m}$, so affine.

Assuming that X and Y are separated, $\Delta_{X\times Y}$ is identified with $\Delta_X \times \Delta_Y$ which is closed in $(X \times X) \times (Y \times Y)$.

Say X, Y are irreducible and $X \times Y = \bigcup_i Z_i$, a finite union of closed subsets.

If $y \in Y$ then $X = \bigcup_i \{x \in X \mid (x, y) \in Z_i\} = \bigcup_i i_y^{-1}(Z_i)$, so by irreducibility $i_y^{-1}(Z_i) = X$ for some i.

Thus $Y = \bigcup_i Y_i$ where $Y_i = \{y \in Y \mid X \times \{y\} \subseteq Z_i\}$.

Now $Y \setminus Y_i = \{y \in Y \mid (x, y) \notin Z_i \text{ for some } x \in X\} = p_Y((X \times Y) \setminus Z_i)$ is open by the lemma at the end of Section 1.2

Thus Y_i is closed, so some $Y_i = Y$. Then $Z_i = X \times Y$.

Definition. An *embedding* or *immersion* of varieties is a morphism $\theta : X \to Y$ whose image is locally closed, and such that $X \to \mathrm{Im}(\theta)$ is an isomorphism.

For example, for any variety there is a diagonal morphism $X \to X \times X$ and X is separated if and only if the diagonal morphism is a closed embedding. The point is that the natural map $\Delta_X \to X$ is always a morphism, since it factors as the inclusion morphism into $X \times X$ followed by either projection to X.

Theorem. Any variety can be written in a unique way as a union of *irreducible components*, maximal irreducible closed subsets.

Proof. See Kempf section 2.3.

For example the node $\{(x, y) \in K^2 : xy = 0\}$ is the union of the two coordinate axes. These are each isomorphic to $\mathbb{A}^1$, so irreducible.

Another example: $\{(x, y) \in K^2 : xy^2 = x^4\}$. The set is $V(x(y^2 - x^3)) = V(x) \cup V(y^2 - x^3)$. Since x and $y^2 - x^3$ are irreducible polynomials, the varieties they define are irreducible (using that the polynomial ring $K[X, Y]$ is a UFD, as in remark (i) after the first theorem in this section).

Another example:

$$\{(x, y, z) \in K^3 : xy = xz = 0\} = \{(0, y, z) : y, z \in K\} \cup \{(x, 0, 0) : x \in K\},$$

a union of a plane and a line. This is the decomposition into irreducible components.

1.6 Projective space

Projective n-space $\mathbb{P}^n$ is the set of 1-dimensional subspaces of K^{n+1}, or equivalently the set of $(n + 1)$-tuples $[x_0 : x_1 : \cdots : x_n]$ with the $x_i \in K$, not all zero, subject to the equivalence relation

$$[x_0 : x_1 : \cdots : x_n] \sim [x'_0 : x'_1 : \cdots : x'_n]$$

iff there is some $0 \neq \lambda \in K$ with $x'_i = \lambda x_i$ for all i. It can can be considered as a space with functions

- $\mathbb{P}^n$ is equipped with its Zariski topology, in which the closed subsets are $V'(S) = \{[x_0 : \cdots : x_n] \mid F(x_1, \dots, x_n) = 0 \text{ for all } F \in S\}$ where S is a set of homogeneous polynomials. Recall that a polynomial $F \in K[X_0, \dots, X_n]$ is homogeneous of degree d provided all monomials in it have total degree d, or equivalently

$$F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n)$$

for all λ, x_i.

Equivalently the open sets are unions of sets of the form

$$D'(F) = \{[x_0 : \cdots : x_n] \mid F(x_0, \dots, x_n) \neq 0\}$$

with F a homogeneous polynomial.

- If U is an open subset of $\mathbb{P}^n$, then $\mathcal{O}(U)$ consists of the functions $f : U \to K$ such that any point $u \in U$ has an open neighbourhood W in U such that $f|_W = P/Q$ with $P, Q \in K[X_0, \dots, X_n]$ homogeneous of the same degree and $Q(x_0, \dots, x_n) \neq 0$ for all $[x_0 : \cdots : x_n] \in W$.

Theorem. (i) $\mathbb{P}^n$ is a space with functions.

(ii) For $0 \leq i \leq n$ the set $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$ is an open subset of $\mathbb{P}^n$ which is isomorphic to $\mathbb{A}^n$.

(iii) $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ and $\mathbb{P}^n$ is separated. Thus $\mathbb{P}^n$ is a variety.

(iv) The map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a morphism of varieties. A subset U of $\mathbb{P}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{A}^{n+1} \setminus \{0\}$. If so, then a function $f : U \to K$ is in $\mathcal{O}(U)$ if and only if $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

Proof. (i) Clear.

(ii) There are inverse maps between U_i and $\mathbb{A}^n$ sending $[x_0 : \cdots : x_n]$ to $(x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i)$ and $(y_1, \dots, y_n)$ to $[y_1 : \cdots : y_i : 1 : y_{i+1} : \cdots : y_n]$. One needs to check that the regular functions correspond.

(iii) The union is clear.

For separatedness, given distinct points u, w, we need to find open neighbourhoods U and W and a function $f(x, y)$ on $U \times W$ of the form $\sum_i g_i(x) h_i(y)$ with g_i and h_i regular, such that $f(u, w) \neq 0$ but $f(x, x) = 0$ for all $x \in U \cap W$.

There must be indices i, j with $u_i w_j \neq u_j w_i$, and without loss of generality $u_i w_j \neq 0$. Take $U = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$, $W = \{[y_0 : \cdots : y_n] : y_j \neq 0\}$ and

$$f(x, y) = \frac{x_j y_i - x_i y_j}{x_i y_j}.$$

(iv) It is clear that π is a morphism. We show that as subset U of $\mathbb{P}^n$ is open if and only if its inverse image $\pi^{-1}(U)$ is open in $X = \mathbb{A}^{n+1} \setminus \{0\}$. We leave the rest as an exercise. First observe that $\pi^{-1}(D'(F)) = X \cap D(F)$, so if U is open, so is $\pi^{-1}(U)$. Conversely suppose that $\pi^{-1}(U)$ is open, so

$$\pi^{-1}(U) = X \cap \bigcup_{f \in S} D(f)$$

for some subset $S \subseteq K[X_0, \dots, X_n]$. Suppose $x = [x_0 : \cdots : x_n] \notin U$. Let $f \in S$. Then $(\lambda x_0, \dots, \lambda x_n) \notin \pi^{-1}(U)$ for all $0 \neq \lambda \in K$. Thus $f(\lambda x_0, \dots, \lambda x_n) = 0$ for all $\lambda \neq 0$. Writing f as a sum of homogeneous polynomials, say $f = \sum_d f_d$ with f_d homogeneous of degree d, we have $\sum_d f_d(x_0, \dots, x_n)\lambda^d = 0$ for all $\lambda \neq 0$. This forces $f_d(x_0, \dots, x_n) = 0$ for all d. It follows that

$$U = \bigcup_{f \in S} \bigcup_d D'(f_d),$$

so U is open.

Coordinate-free description. The set $\mathbb{P}(V)$ of 1-dimensional subspaces of V a vector space of dimension $n + 1$ has a natural structure as a variety isomorphic to $\mathbb{P}^n$.

Lemma. $\mathbb{P}^n$ is a disjoint union $U_0 \cup V_0$ where
$U_0 = \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\}$ is an open subvariety isomorphic to $\mathbb{A}^n$.
$V_0 = \{[x_0 : \cdots : x_n] \mid x_0 = 0\}$ is a closed subvariety isomorphic to $\mathbb{P}^{n-1}$.

Repeating, we can write $\mathbb{P}^n$ as a disjoint union of copies of $\mathbb{A}^n$, $\mathbb{A}^{n-1}$, $\dots$, $\mathbb{A}^0 = \{pt\}$.

Example. $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ where $\lambda \in \mathbb{A}^1$ coresponds to $[1 : \lambda]$ and $\infty = [0 : 1]$. For $K = \mathbb{C}$ one identifies $\mathbb{P}^1$ with the Riemann sphere by stereographic projection.

The closed subsets are $\emptyset$, finite subsets, and $\mathbb{P}^1$. Thus the nonempty open sets are the cofinite subsets $\mathbb{P}^1 \setminus \{a_1, \dots, a_k\}$.

We show that $\mathcal{O}(\mathbb{P}^1) = K$. A regular function $f \in \mathcal{O}(\mathbb{P}^1)$ induces a regular functions on $U_0 \cong \mathbb{A}^1$ and on $U_1 \cong \mathbb{A}^1$. The coordinate ring of $\mathbb{A}^1$ is the polynomial ring $K[X]$. Thus there are polynomials $p, q \in K[X]$ with $f([x_0 :$

$x_1]) = p(x_1/x_0)$ for $x_0 \neq 0$ and $f([x_0 : x_1]) = q(x_0/x_1)$ for $x_1 \neq 0$. Thus $p(t) = q(1/t)$ for $t \neq 0$. Thus both are constant polynomials.

1.7 Projective varieties

Definition. A *projective variety* is (a variety isomorphic to) a closed subset in projective space. A *quasiprojective variety* is (a variety isomorphic to) a locally closed subset in projective space.

Example. A curve in $\mathbb{A}^2$, for example

$$\{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x\},$$

can be homogenized to give a curve in $\mathbb{P}^2$

$$\{[w : x : y] \in \mathbb{P}^2 : y^2 w = x^3 + xw^2\}$$

Recall that $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. On the affine space part $w \neq 0$, we recover the original curve. On the line at infinity $w = 0$ the equation is $x^3 = 0$, which has solution $x = 0$, giving rise to one point at infinity $[w : x : y] = [0 : 0 : 1]$.

For the curve $y^3 = x^3 + x$, the points at infinity are $[0 : 1 : \epsilon]$ where $\epsilon^3 = 1$.

Theorem (Segre). The is an embedding $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$ give by

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_m].$$

Proof. See Kempf, Theorem 3.2.1.

Corollary. A product of (quasi-)projective varieties is (quasi-)projective.

1.8 Schemes

More general than varieties are schemes. I only discuss affine schemes. I do it using functors rather than sheaves. See

M. Demazure and P. Gabriel, Groupes Algébriques, 1970. Partial English translation, Introduction to Algebraic Geometry and Algebraic Groups, 1980.

W. C. Waterhouse, Affine group schemes, 1979.

D. Eisenbud and J. Harris, The geometry of schemes, 2000. (Chapter VI)

Let K be a commutative ring (not necessarily a field). When discussing algebraic schemes we assume that K is noetherian. We write K-comm for the category of commutative K-algebras, or equivalently commutative rings R equipped with a homomorphism $K \to R$.

Definition. The category of *affine (K-)schemes* is the category of representable (covariant) functors

$$F : K\text{-comm} \to \text{Sets}$$

with morphisms given by natural transformations. (These are not additive categories.) Recall that a functor F is said to be *representable* if there is an object A in the category (a commutative K-algebra) such that

$$F(-) \cong \mathrm{Hom}_{K\text{-comm}}(A, -).$$

By Yoneda's lemma, the functor $A \mapsto \mathrm{Hom}_{K\text{-comm}}(A, -)$ defines an anti-equivalence from K-comm to the category of affine schemes.

Examples. (i) $\mathbf{A}^n$ (or $\mathbf{A}^n_K$ if we need to stress the base ring K) is the affine scheme with $\mathbf{A}^n(R) = R^n$. It is represented by the polynomial ring $K[X_1, \ldots, X_n]$, since

$$\mathrm{Hom}_{K\text{-comm}}(K[X_1, \ldots, X_n], R) = R^n.$$

(ii) Any subset S of $K[X_1, \ldots, X_n]$ defines a functor $\mathbf{V}(S)$ by

$$\mathbf{V}(S)(R) = \{(x_1, \ldots, x_n) \in R^n : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.$$

It is represented by the ring $K[X_1, \ldots, X_n]/(S)$ where (S) is the ideal generated by S.

Definition. An affine scheme is *algebraic* if the algebra A is finitely generated as a K-algebra (assuming that K is a noetherian ring).

It is *reduced* if A is reduced.

Lemma. Given an affine (algebraic) scheme F, there is a reduced affine (algebraic) scheme F_{red} and a morphism $F_{\text{red}} \to F$ such that for all R the map

$$F_{\text{red}}(R) \to F(R)$$

is injective, and a bijection for R reduced. This defines a functor $F \mapsto F_{\text{red}}$ which is right adjoint to the inclusion of reduced affine (algebraic) schemes into affine (algebraic) schemes.

Proof. If $F(-) = \mathrm{Hom}(A, -)$ we set $F_{\text{red}}(-) = \mathrm{Hom}(A_{\text{red}}, -)$. The natural map $A \to A_{\text{red}}$ gives a morphism $F_{\text{red}} \to F$.

For example $\mathbf{V}(S)$ is algebraic. It is reduced if and only if $K[X_1, \ldots, X_n]/(S)$ is reduced. The scheme $\mathbf{V}(S)_{\mathrm{red}}$ is represented by $K[X_1, \ldots, X_n]/\sqrt{(S)}$

Definition. If $K \to L$ is a homomorphism of commutative rings, and F is an affine K-scheme, we write F^L for the functor defined by

$$F^L(R) = F(R)$$

where R is a commutative L-algebra, considered as a K-algebra by composition $K \to L \to R$.

This is an L-scheme since if $F = \mathrm{Hom}_{K\text{-comm}}(A, -)$ then

$$F^L(R) = \mathrm{Hom}_{K\text{-comm}}(A, R) \cong \mathrm{Hom}_{L\text{-comm}}(L \otimes_K A, R)$$

This defines a functor from the category of affine (algebraic) K-schemes to affine (algebraic) L-schemes.

For example there is a unique homomorphism $\mathbb{Z} \to K$ and $K \otimes_{\mathbb{Z}} \mathbb{Z}[X_1, \ldots, X_n] \cong K[X_1, \ldots, X_n]$, so $\mathbf{A}^n_K \cong (\mathbf{A}^n)_{\mathbb{Z}})^K$. Also, if $S \subseteq \mathbb{Z}[X_1, \ldots, X_n]$ then $\mathbf{V}(S)_K \cong (\mathbf{V}(S)_{\mathbb{Z}})^K$.

Proposition. If K is an algebraically closed field, then the category of affine varieties is equivalent the category of reduced affine algebraic schemes.

Under this correspondence, an affine variety X is sent to the reduced affine algebraic scheme $\mathrm{Hom}_{K\text{-comm}}(\mathcal{O}(X), -)$.

Conversely, if F is an affine algebraic scheme, then the variety corresponding to F (or F_{red} if F is not reduced) has as underlying set $F(K)$.

Proof. Straightforward, using that the category of affine varieties is anti-equivalent to the category of finitely generated reduced K-algebras,

2 Varieties arising in representation theory

2.1 Algebraic groups

Definition. An *algebraic group* is a group which is also a variety, such that multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms of varieties.

A morphism of algebraic groups is a map which is a group homomorphism and a morphism of varieties.

When considering an action of an algebraic group on a variety X we shall suppose that the map $G \times X \to X$ is a morphism of varieties.

The *general linear group* $\mathrm{GL}_n(K)$ is the open subset $D(\det)$ of $M_n(K)$, so an affine variety. It is an algebraic group thanks to the formula $g^{-1} = \operatorname{adj} g / \det g$. It acts by left multiplication or by conjugation on $M_n(K)$.

Definition. A *linear algebraic group* is an algebraic group which is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$. For example
- the special linear group $\mathrm{SL}_n(K)$,
- the orthogonal group $O_n(K)$,
- the multiplicative group $G_m = (K^*, \times) = GL_1(K)$,
- the additive group $G_a = (K, +)$, since it is isomorphic to $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in K\}$
- any finite group.
- any finite product of copies of these, using that $\mathrm{GL}_n(K) \times \mathrm{GL}_m(K)$ embeds in $\mathrm{GL}_{n+m}(K)$.

Lemma. A connected algebraic group is an irreducible variety.

Proof. Write the group as a union of irreducible components $G = G_1 \cup \cdots \cup G_n$. Since G_1 is not a subset of the union of the other components, some element $g \in G_1$ does not lie in any other component. Now any two elements of an algebraic group look the same, since multiplication by any $h \in G$ defines an isomorphism $G \to G$. It follows that every element of G lies in only one irreducible component. Thus G is the disjoint union of its irreducible components. But then the components are open and closed, and since G is connected, there is only one component.

Remark. Clearly any linear algebraic group is an affine variety, and conversely one can show that any affine algebraic group is linear, see for example Humphreys, Linear algebraic groups, section 8.6. There are algebraic groups which are not affine varieties. One can show that a connected algebraic group which is a projective variety must be commutative. It is called an 'abelian variety'. For example elliptic curves (non-singular cubics in $\mathbb{P}^2$)

have a group law.

Affine group schemes. If K is any commutative ring, then an *affine group scheme* over K is a representable functor $F : K\text{-comm} \to \text{Groups}$. If A is the commutative K-algebra A representing F, then A becomes a Hopf algebra, see Waterhouse section 1.4.

For example $\mathbf{GL}_n$ is the affine group scheme with $\mathbf{GL}_n(R) = \mathrm{GL}_n(R)$ for all R. It is represented by the algebra $K[X_{ij}, 1/\det]$, so reduced.

Observe that $(\mathbf{GL}_n)_K = ((\mathbf{GL}_n)_{\mathbb{Z}})^K$.

2.2 The variety of algebras

Let V be a vector space of dimension n, with basis $e_1, \dots, e_n$.

We write $\mathrm{Bil}(n)$ for the set of bilinear maps $V \times V \to V$. A map $\mu \in \mathrm{Bil}(n)$ is given by its structure constants $(c_{ij}^k) \in K^{n^3}$ with

$$\mu(e_i, e_j) = \sum_k c_{ij}^k e_k.$$

Equivalently $\mathrm{Bil}(n) \cong \mathrm{Hom}(V \otimes V, V)$, Thus it is affine space $\mathbb{A}^{n^3}$.

We write $\mathrm{Ass}(n)$ for the subset consisting of associative multiplications. This is a closed subset of $\mathrm{Bil}(n)$, hence an affine variety, since it is defined by the equations

$$\mu(\mu(e_i, e_j), e_k) = \mu(e_i, \mu(e_j, e_k)),$$

that is

$$\sum_p c_{ij}^p c_{pk}^s = \sum_q c_{iq}^s c_{jk}^q$$

for all s.

We write $\mathrm{Alg}(n)$ for the subset of associative unital multiplications, so algebra structures on V.

Theorem. $\mathrm{Alg}(n)$ is an affine open subset of $\mathrm{Ass}(n)$, hence an affine variety. The algebraic group $\mathrm{GL}(V)$ acts by basis change, and the orbits correspond to isomorphism classes of algebras.

Proof. (i) We use that a vector space A with an associative multiplication has a 1 if and only if there is some $a \in A$ for which the maps $\ell_a, r_a : A \to A$ of left and right multiplication by a are invertible.

Namely, if $u = \ell_a^{-1}(a)$, then $au = a$. Thus $aub = ab$ for all b, so since ℓ_a is invertible, $ub = b$. Thus u is a left 1. Similarly there is a right 1, and they must be equal.

(ii) For the algebra V with multiplication μ, write ℓ_a^μ and r_a^μ for left and right multiplication by $a \in V$. Then $\mathrm{Alg}(n) = \bigcup_{a \in V} D(f_a)$ where $f_a(\mu) = \det(\ell_a^\mu)\det(r_a^\mu)$. Thus $\mathrm{Alg}(n)$ is open in $\mathrm{Ass}(n)$.

(iii) The map

$$\mathrm{Alg}(n) \to V, \quad \mu \mapsto \text{the 1 for } \mu$$

is a morphism of varieties, since on $D(f_a)$ it is given by $(\ell_a^\mu)^{-1}(a)$, whose components are rational functions, with $\det(\ell_a^\mu)$ in the denominator.

(iv) $\mathrm{Alg}(n)$ is affine. In fact

$$\mathrm{Alg}(n) \cong \{(\mu, u) \in \mathrm{Ass}(n) \times V \mid u \text{ is a 1 for } \mu\}.$$

The right hand side is a closed subset, hence it is affine. Certainly there is a bijection, and the maps both ways are morphisms.

(v) Last statement is clear.

Example. The structure of $\mathrm{Alg}(n)$ is known for small n. For example $\mathrm{Alg}(4)$ has 5 irreducible components, of dimensions 15, 13, 12, 12, 9. See P. Gabriel, Finite representation type is open, 1974.

2.3 Module varieties

Let A be a finitely generated associative K-algebra, and $d \in \mathbb{N}$.

Lemma 1. (i) $\mathrm{Mod}(A, d) = \mathrm{Hom}_{K\text{-algebra}}(A, M_d(K))$, the set of A-module structures on K^d, has a natural structure as an affine variety.

(ii) Given any $a \in A$, the map $\mathrm{Mod}(A, d) \to M_d(K)$, sending $\theta : A \to M_d(K)$ to $\theta(a)$, is a morphism of varieties.

(iii) There is an action of $\mathrm{GL}_d(K)$ on $\mathrm{Mod}(A, d)$ by conjugation, so given by $(g \cdot \theta)(a) = g\theta(a)g^{-1}$. The orbits correspond to isomorphism classes of d-dimensional modules.

Proof. (i) We choose a presentation $A \cong K\langle x_1, \dots, x_k\rangle/I$. A homomorphism $\theta : A \to M_d(K)$ is determined by the matrices $A_i = \theta(x_i)$, so

$$\mathrm{Mod}(A, d) = \{(A_1, \dots, A_k) \in M_d(K)^k : p(A_1, \dots, A_k) = 0 \text{ for all } p \in I\}.$$

Here any $p \in K\langle x_1, \dots, x_k\rangle$ is thought of as a noncommutative polynomial in $x_1, \dots, x_k$, and then $p(A_1, \dots, A_k)$ is a $k \times k$ matrix of ordinary polynomials in the entries of the A_i. This is a closed subset of the affine space $M_d(K)^k$, so an affine variety.

It remains to check that the structure doesn't depend on the presentation of A. For this we use (ii), which is clear. Now if $\mathrm{Mod}(A, d)'$ is the same set but with the variety structure given by a different presentation, then (ii) shows that the identity maps $\mathrm{Mod}(A, d) \to \mathrm{Mod}(A, d)'$ and $\mathrm{Mod}(A, d)' \to \mathrm{Mod}(A, d)$ are morphisms of varieties, giving (i).

(iii) Clear.

Quiver version. Suppose A is a finitely generated K-algebra and $e_1, \ldots, e_n$ is a complete set of orthogonal idempotents in A (not necessarily primitive). Thus $e_i e_j = \delta_{ij} e_i$ and $\sum e_i = 1$.

It is equivalent that A can be presented as KQ/I where Q is a (finite) quiver with vertex set $\{1, \ldots, n\}$, with the e_i corresponding to the trivial paths.

If M is any A-module, then $M = \bigoplus_{i=1}^n e_i M$. The *dimension vector* of M is the vector $\alpha \in \mathbb{N}^n$ with $\alpha_i = \dim e_i M$.

Letting $d = \sum_{i=1}^n \alpha_i$, we can define

$$\mathrm{Mod}(A, \alpha) = \{\theta : A \to M_d(K) : \theta(e_i) = \pi_i\} \subseteq \mathrm{Mod}(A, d)$$

where π_i is the projection of $K^d = K^{\alpha_1} \oplus \cdots \oplus K^{\alpha_n}$ onto its ith summand.

If $\alpha \in \mathbb{N}^n$ then

$$\mathrm{Rep}(Q, \alpha) = \prod_{a \in Q_1} \mathrm{Hom}(K^{\alpha_{t(a)}}, K^{\alpha_{h(a)}}).$$

This is a vector space, so affine space of some dimension.

Lemma 2. Let $A = KQ/I$. Then

(i) As a variety, $\mathrm{Mod}(A, \alpha) \cong \{x \in \mathrm{Rep}(Q, \alpha) : x \text{ satisfies relations in } I\}$.

(ii) There is an action of $\mathrm{GL}(\alpha) = \prod_{i=1}^n \mathrm{GL}_{\alpha_i}(K)$ on $\mathrm{Mod}(A, \alpha)$ by conjugation. The orbits correspond to isomorphism classes of modules of dimension vector α.

Scheme structure. Given a finitely generated K-algebra and $d \in \mathbb{N}$ we define a functor

$$\mathbf{Mod}(A, d) : K\text{-comm} \to \text{Sets}, \quad R \mapsto \mathrm{Hom}_{K\text{-alg}}(A, M_d(R)).$$

Lemma 3. $\mathbf{Mod}(A, d)$ is an affine algebraic scheme. The variety $\mathrm{Mod}(A, d)$ corresponds to the scheme $\mathbf{Mod}(A, d)_{\mathrm{red}}$.

Similarly there is a module schemes $\mathbf{Mod}(A, \alpha)$.

Proof. Recall that any algebra A has a dth root algebra $\sqrt[d]{A}$. Moreover for any algebra B, we write define

$$B_{\text{comm}} = B/(bb' - b'b : b, b' \in B)$$

the largest quotient of B which is commutative. Then

$$\mathbf{Mod}(A, d)(R) \cong \mathrm{Hom}_{K\text{-alg}}(\sqrt[d]{A}, R) \cong \mathrm{Hom}_{K\text{-comm}}((\sqrt[d]{A})_{\text{comm}}, R)$$

so it is represented by the commutative K-algebra $(\sqrt[d]{A})_{\text{comm}}$.

Examples.

(1) The scheme $\mathbf{Mod}(A, 1)$ is represented by A_{comm}, and $\mathrm{Mod}(A, 1)$ is the variety with regular functions $(A_{\text{comm}})_{\text{red}}$.

For example if $A = K[x]/(x^2)$ then $\mathbf{Mod}(A, 1)$ is a point since $A_{\text{red}} = K$.

(2) The *nilpotent variety* is

$$N_d = \{A \in M_d(K) : A^d = 0\} = \mathrm{Mod}(K[x]/(x^d), d)$$

(3) The *commuting variety* is

$$C_d = \{(A, B) \in M_d(K)^2 : AB = BA\} = \mathrm{Mod}(K[x, y], d).$$

(4) $\mathrm{Mod}(M_d(K), d)$ is the set of K-algebra maps $M_d(K) \to M_d(K)$. These are all automorphisms, since $M_d(K)$ is a simple algebra. Thus it is $\mathrm{Aut}(M_d(K))$. Now every automorphism of $M_d(K)$ is inner (for central simple algebras this is the Skolem-Noether Theorem). Thus the map

$$\mathrm{GL}_d(K) \to \mathrm{Aut}(M_d(K)), g \mapsto (A \mapsto gAg^{-1})$$

is onto. The kernel consists of the multiples of the identity matrix, a copy of the group G_m. Thus $\mathrm{Aut}(M_d(K))$ is in bijection with $\mathrm{PGL}_d(K) = \mathrm{GL}_d(K)/G_m$.

2.4 Geometric quotients

Suppose that an algebraic group G acts on a variety X. Let X/G be the set of orbits and let $\pi : X \to X/G$ be the quotient map. We can turn X/G into a space with functions via

- A subset U of X/G is open iff $\pi^{-1}(U)$ is open in X. (Thus also U is closed iff $\pi^{-1}(U)$ is closed in X.)

- A function $f : U \to K$ is in $\mathcal{O}(U)$ iff $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

This ensures that π is a morphism. If with this structure X/G is a variety, we call it a *geometric quotient*.

Example. The group G_m acts on $X = \mathbb{A}^{n+1} \setminus \{0\}$ by rescaling. The quotient X/G is isomorphic to $\mathbb{P}^n$, so is a variety. This was part (iv) of the theorem about projective space.

On the other hand the orbits of G_m acting on $\mathbb{A}^{n+1}$ are not all closed, so $\mathbb{A}^{n+1}/G_m$ is not a geometric quotient by the following.

Lemma. (i) If there is a geometric quotient X/G then the orbits of G must be closed in X.

(ii) A geometric quotient X/G is a *categorical quotient*, meaning that π is a morphism which is constant on G-orbits, and any morphism $\phi : X \to Z$ which is constant on G-orbits factors uniquely as a composition $X \xrightarrow{\pi} X/G \xrightarrow{\psi} Z$.

(iii) If Y is a variety and G acts on $Y \times G$ by $g(y, g') = (y, gg')$, then $(Y \times G)/G \cong Y$.

(iv) If Y is a variety and G acts on $Y \times G$ by $g(y, g') = (y, gg')$ for some action of G on Y, then $(Y \times G)/G \cong Y$.

Proof. (i) Any orbit of G in X is the inverse image of a point in X/G, and any point in a variety is closed. (We should have had this earlier. This is clear for affine space, and hence it passes to abstract varieties.)

(ii) There is a unique map $X/G \to Z$. It is straightforward that it is a morphism.

(iii) The projection map $p : Y \times G \to Y$ is open, so U is open in Y if and only if $p^{-1}(U)$ is open. Also a function f on an open set U of Y is regular if and only if fp is regular on $U \times G$. Namely, if it is regular on $U \times G$ then so is its composition with the map $U \to U \times G$, $x \mapsto (x, 1)$.

(iv) Use the automorphism $Y \times G \to Y \times G$, $(y, g) \mapsto (g^{-1}y, g)$ to pass to the case of trivial action on Y.

Remark. If the orbits aren't closed, one needs a different approach. This is 'geometric invariant theory'. More later.

Even if the orbits of G are closed, there may not be a geometric quotient. See for example H. Derksen, Quotients of algebraic group actions, in: Automorphisms of affine spaces, 1995. Maybe you need to work with algebraic spaces rather than varieties. See for example J. Kollár, Quotient spaces modulo algebraic groups, Ann. of Math. 1997.

One case that is understood, however, is quotients G/H where G is a linear algebraic group and H is a closed subgroup, acting on G by left multiplication, so G/H is the set of cosets.

It is known that:
- G/H is a quasi-projective variety, so a geometric quotient. See A. Borel, Linear Algebraic Groups, Corollary 5.5.6.
- If H is a normal subgroup, G/H is an affine variety, so a linear algebraic group. Borel, Proposition 5.5.10.
- G/H is a projective variety (in which case H is called a *parabolic subgroup*) if and only if H contains a Borel subgroup (a maximal closed connected soluble subgroup of G). Borel, Theorem 6.2.7.

In the Example (4) in the last section, I mentioned that the module variety $\mathrm{Mod}(M_d(K), d)$ is in bijection with $\mathrm{PGL}_d(K)$. This is an isomorphism of varieties, but I don't think we yet have the methods to prove this.

2.5 Grassmannians

Definition. If V is a vector space of dimension n, the Grassmannian $\mathrm{Gr}(V, d)$ is the set of subspaces of V of dimension d.

We write $\mathrm{Inj}(K^d, V)$ for the set of injective linear maps $K^d \to V$. It is open in $\mathrm{Hom}(K^d, V)$, so a quasi-affine variety.

The group $\mathrm{GL}_d(K)$ act by $g \cdot \theta = \theta g^{-1}$.

Two injective maps are in the same orbit if and only if they have the same image, so we have a natural bijection $\mathrm{Inj}(K^d, V)/\mathrm{GL}_d(K) \to \mathrm{Gr}(V, d)$.

This turns $\mathrm{Gr}(V, d)$ into a space with functions.

Fixing a basis $e_1, \dots, e_n$ of V, we identify $\mathrm{Inj}(K^d, V)$ with the set of $n \times d$ matrices of rank d.

Let I be a subset of $\{1, \dots, n\}$ with $|I| = d$. If $A \in \mathrm{Inj}(K^d, V)$, we write A_I for the square matrix obtained by selecting the rows of A in I. Then $\det(A_I)$ is a minor of A. We write A'_I for the $(n-d) \times d$ matrix obtained by deleting the rows in I.

We consider the map $\phi : \mathrm{Inj}(K^d, V) \to \mathbb{P}^N$ where $N = \binom{n}{d} - 1$, sending A to $[\det(A_I)]_I$.

The action of $g \in \mathrm{GL}_d(K)$ on $\mathrm{Inj}(K^d, V)$ sends A to Ag^{-1}, Now $\det((Ag^{-1})_I) = \det(A_I)\det(g)^{-1}$. Thus the map ϕ is constant on the orbits of $\mathrm{GL}_d(K)$.

Theorem (Plücker embedding). The induced map $\theta : \mathrm{Gr}(V, d) \to \mathbb{P}^N$ is a closed embedding, so $\mathrm{Gr}(V, d)$ is a projective variety.

We use the following facts.

Lemma 1. Given a mapping $\theta : X \to Y$ between varieties and an open covering $Y = \bigcup U_\lambda$, the map θ is a closed embedding if and only if its restrictions $\theta_\lambda : \theta^{-1}(U_\lambda) \to U_\lambda$ are closed embeddings.

Proof. First, $Y \setminus \mathrm{Im}\,\theta = \bigcup_\lambda U_\lambda \setminus \mathrm{Im}\,\theta_\lambda$ is open in Y, so $\mathrm{Im}\,\theta$ is closed. Second, there is an inverse map $g : \mathrm{Im}\,\theta \to X$. Now $\mathrm{Im}\,\theta$ has an open covering by sets of the form $\lambda \cap \mathrm{Im}\,\theta$, and the restriction of g to each of these sets is a morphism, hence so is g.

Lemma 2. If $g : X \to Y$ is a morphism, with Y separated, then the map $X \to X \times Y$, $x \mapsto (x, g(x))$ is a closed embedding.

Proof. Its image is the inverse image of the diagonal Δ_Y under the map $X \times Y \to Y \times Y$, $(x, y) \mapsto (g(x), y)$. The projection from $X \times Y \to X$ gives an inverse map from the image to X.

Proof of the theorem.

Let $X = \mathrm{Inj}(K^d, V)$, let $Y = \mathrm{Gr}(V, d)$ and let $\theta : Y \to \mathbb{P}^N$ be the Plücker map.

We write elements of $\mathbb{P}^N$ in the form $[x_I]$ with $x_I \in K$, not all zero, for I a subset of $\{1, \dots, n\}$ of size d.

Recall that $\mathbb{P}^N$ has an affine open covering by the sets $U_J = \{[x_I] : x_J \neq 0\}$.

Let X_J be the inverse image of U_J in $\mathrm{Inj}(K^d, V)$, and let $Y_J = X_J / \mathrm{GL}_d(K)$ be its inverse image in Y.

By Lemma 1 it suffices to show that $Y_J \to U_J$ is a closed embedding.

Now X_J consists of the matrices A such that A_J is invertible. Thus there is isomorphism of varieties

$$X_J \cong \mathrm{GL}_d(K) \times M_{(n-d)\times d}(K), A \mapsto (A_J, A'_J).$$

By the lemma from the last section, $Y_J = X_J / \mathrm{GL}_d(K) \cong M_{(n-d)\times d}(K)$ so it is an affine variety. Varying Y this gives an affine open covering of Y.

Given a matrix $B \in M_{(n-d)\times d}(K)$, we write $\hat{B}$ for the matrix A with $A_J = I_d$ and $A'_J = B$. We can identify U_J with $\mathbb{A}^N$ with components indexed by subsets $I \neq J$, and the map $Y_J \to U_J$ with the map

$$M_{(n-d)\times d}(K) \to \mathbb{A}^N, \quad B \mapsto (\det \hat{B}_I)_I.$$

Now observe that if we take I to be equal to J, except that we omit the jth element, and instead insert the ith element of $\{1,\dots,n\}\setminus J$, then $\det(\hat{B}_I) = \pm b_{ij}$. Thus, up to sign, this map is of the form $Y_J \to Y_J \times W$ for some W. Thus by Lemma 2 it is a closed embedding.

Alternative version. We can instead consider surjective linear maps, and realise $\mathrm{Gr}(V,d)$ as a quotient of $\mathrm{Surj}(V,K^c)$ by $\mathrm{GL}_c(K)$ where $c+d=\dim V$.

(It is not obvious that these two constructions give the same variety structure. This can no doubt be checked locally. An alternative would be to consider the variety of exact sequences $0 \to K^d \to V \to K^c \to 0$ modulo the action of $\mathrm{GL}_d(K)\times \mathrm{GL}_c(K)$. This quotient would have a natural map, which should be an isomorphism. to each of the other two quotients.)

Lemma 3 If $\theta : V \to V'$ is a linear map, then

$$\{(U,U') \in \mathrm{Gr}(V,d)\times \mathrm{Gr}(V',d') : \theta(U)\subseteq U'\}$$

is closed in the product.

Proof. For this we realise $\mathrm{Gr}(V,d)$ as the quotient of $\mathrm{Inj}(K^d,V')$ by $\mathrm{GL}(d)$. We can realise $\mathrm{Gr}(V',d')$ as a quotient of $\mathrm{Surj}(V',K^c)$ by $\mathrm{GL}_c(K)$, where $c=\dim V'-d'$. Then we have a closed subset

$$\{(f,g)\in \mathrm{Inj}(K^d,V)\times \mathrm{Surj}(V',K^c) : g\theta f = 0\}.$$

Using this, the flag variety

$$\mathrm{Flag}(V,d_1,\dots,d_k) = \{0\subseteq U_1\subseteq\cdots\subseteq U_k\subseteq V : \dim U_i = d_i\}$$

for $0\le d_1\le\cdots\le d_k\le \dim V$, is realized as a closed subset of $\prod_i \mathrm{Gr}(V,d_i)$, hence a projective variety.

Definition. Let A be an algebra and $e_1,\dots,e_n$ a complete set of orthogonal idempotents. Let M be a finite dimensional A-module and let α be a dimension vector. We write M_i for e_iM. We define

$$\mathrm{Gr}_A(M,\alpha) = \{(U_i)\in \prod_{i=1}^n \mathrm{Gr}(M_i,\alpha_i) : (U_i) \text{ defines a submodule of } M\}$$

This is a *Quiver Grassmannian.* This is a closed subvariety of the product of Grassmannians $\prod_i \mathrm{Gr}(M_i,\alpha_i)$, hence a projective variety. Namely, for all i,j and all $a\in e_jAe_i$, we need $\hat{a}(U_i)\subseteq U_j$, where $\hat{a}: M_i\to M_j$ is the homothety $\hat{a}(m)=am$. This is a closed condition.

3 Dimension theory and applications

3.1 Function fields

Recall that a variety is irreducible iff it is non-empty and any two non-empty open subsets have non-empty intersection.

Definition. Let X be an irreducible variety. A *rational function* on X is a regular function on a non-empty open subset of X. We identify $f_1 \in \mathcal{O}_X(U_1)$ with $f_2 \in \mathcal{O}_X(U_2)$ if they agree on an open subset of $U_1 \cap U_2$. Then they actually agree on all of $U_1 \cap U_2$, for

$$\{x \in U_1 \cap U_2 \mid f_1(x) = f_2(x)\}$$

is closed and dense in $U_1 \cap U_2$. It follows that a rational function is defined on a unique maximal open subset of X.

The *function field* $K(X)$ of X is the set of all rational functions on X. It is a field.

If U is a nonempty open subset of X then restriction induces an isomorphism $K(X) \to K(U)$.

If U is open in X, one can identify $\mathcal{O}(U)$ with the subset of $K(X)$ of rational functions defined on U.

Lemma. If X is irreducible and affine, then $K(X)$ is the quotient field of its coordinate ring $\mathcal{O}(X)$.

Proof. An element f/g of the quotient field gives a rational function defined on $D(g) \subseteq X$. Conversely, any rational function is regular on some open set of U. This open set contains an affine open of the form $D(g)$ with $g \in K[X]$, and the regular functions on this are of the form f/g^n.

Definition. Two irreducible varieties are said to be *birational* if they have non-empty open subsets which are isomorphic.

For example $\mathbb{A}^2$, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational, but not isomorphic.

Proposition. Irreducible varieties are birational if and only if they have isomorphic function fields.

Proof. One implication is trivial. For the other, assume that $K(X) \cong K(Y)$. We may assume that X is affine.

Take generators of $\mathcal{O}(X)$, consider as elements of $K(Y)$, and choose an affine open subset Y' of Y on which all the elements are defined. Then $\mathcal{O}(X)$ embeds in $\mathcal{O}(Y')$.

Similarly $\mathcal{O}(Y')$ embeds in $\mathcal{O}(X')$ for an affine open X' in X.

These give maps

$$X' \xrightarrow{f} Y' \xrightarrow{g} X$$

such that gf is the inclusion, so an open embedding. But then the map $X' \to g^{-1}(X')$ is an isomorphism.

3.2 Dimension

See D. Mumford, The red book of varieties and schemes.

Definition. The *dimension* of a variety is the supremum of the n such that there is a chain of distinct (non-empty) irreducible closed subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ in X. ($\dim \emptyset = -\infty$.)

Thus, if X is an affine variety, $\dim X$ is the Krull dimension of $\mathcal{O}(X)$, the maximal length of a chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$.

Lemma 1. If X is an irreducible affine variety, then $\dim X$ is the transcendence degree of the field extension $K(X)/K$.

The proof is commutative algebra. As a consequence we get the following.

Lemma 2.
(0) $\dim \mathbb{A}^n = n$.
(1) Any variety has finite dimension.
(2) If $X \subseteq Y$ is a locally closed subset, then $\dim X \leq \dim Y$, strict if Y is irreducible and X is a proper closed subset.
(3) If X is irreducible then $\dim X$ =transcendence degree of $K(X)/K$. Thus if U is nonempty open in X, $\dim U = \dim X$.
(4) If $X = Y_1 \cup \cdots \cup Y_n$, with the Y_i locally closed in X, then $\dim X = \max\{\dim Y_i\}$.

Proof. (0) By transcendence degree.

(2) If X_i is a chain of irreducible closed subsets in X, then $\overline{X_i}$ is a chain of irreducible closed subsets of Y, and if $\overline{X_i} = \overline{X_{i+1}}$ then X_i is open in $\overline{X_i}$, so

$$X_{i+1} = X_i \cup (X_{i+1} \cap (\overline{X_i} \setminus X_i))$$

a union of two closed subsets, so $X_{i+1} = X_i$.

(4) for the special case when Y_i open in X.

Take a chain $X_0 \subset X_1 \subset \cdots \subset X_n$ in X.

Then X_0 meets some Y_i.

Consider the chain $Y_i \cap X_0 \subset Y_i \cap X_1 \subset \cdots \subset Y_i \cap X_n$ in Y_i.

$Y_i \cap X_j$ is nonempty and open in X_j, hence irreducible.

The terms are distinct, for if $Y_i \cap X_j = Y_i \cap X_{j+1}$ then $X_{j+1} = X_j \cup (X_{j+1} \setminus Y_i)$ is a proper decomposition.

Thus $\dim Y_i \geq n$.

(1) Combine (0), (2) and the special case of (4).

(3) X is a union of affine opens. These all have function field $K(X)$, so dimension given by the transcendence degree.

(4) in general. Suppose F is an irreducible closed subset of X.

Then F is the union of the sets $\overline{F \cap Y_i}$.

By irreducibility, some $\overline{F \cap Y_i} = F$.

Thus $F \cap Y_i$ is open in F.

Thus $\dim F = \dim F \cap Y_i \leq \dim Y_i$.

Definition. A morphism $\theta : X \to Y$ of varieties, with X and Y irreducible, is *dominant* if its image is dense in Y.

Lemma 3. If $\theta : X \to Y$ is a morphism of varieties and X is irreducible, then $Z = \overline{\operatorname{Im} \theta}$ is irreducible, the restricted map $\theta' : X \to Z$ is dominant and it induces an injection $K(Z) \to K(X)$. Thus $\dim Z \leq \dim X$.

Proof. Straightforward.

Main Lemma. If $\pi : X \to Y$ is a dominant morphism of irreducible varieties then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least $\dim X - \dim Y$. Moreover, there is a nonempty open subset $U \subseteq Y$ with $\dim \pi^{-1}(u) = \dim X - \dim Y$ for all $u \in U$.

This can be reduced to the case when X, Y are affine, and then it is commutative algebra.

Two special cases. (1) $\dim X \times Y = \dim X + \dim Y$. Reduce to the case of irreducible varieties, and then consider the projection $X \times Y \to Y$.

(2) (Hypersurfaces in $\mathbb{A}^n$). The irreducible closed subsets of $\mathbb{A}^n$ of dimension $n-1$ are the zero sets $V(f)$ of irreducible polynomials $f \in K[X_1, \ldots, X_n]$.

Namely, if f is irreducible then $V(f)$ is irreducible, a proper closed subset of $\mathbb{A}^n$, so dimension $< n$, but a fibre of $f : \mathbb{A}^n \to K$, so dimension $\geq n-1$.

Conversely if $X \subseteq \mathbb{A}^n$ is an irreducible closed subset of dimension $n-1$ then $X = V(I)$ so $X \subseteq V(g)$ for some non-zero $g \in I$. But then $X \subseteq V(f)$ for some irreducible factor f of g. Then equal by dimensions.

Example. The commuting variety C_d is irreducible of dimension $d^2 + d$. (Theorem of Motzkin and Taussky, 1955.)

Following R. M. Guralnick, A note on commuting pairs of matrices, 1992.

A $d \times d$ matrix A is *regular* or *non-derogatory* if in it's Jordan normal form each Jordan block has a different eigenvalue. Equivalently if its minimal polynomial is equal to its characteristic polynomial. Equivalently if it defines a cyclic $K[X]$-module. Equivalently if all eigenspaces are at most one-dimensional. Equivalently the only matrices which commute with A are polynomials in A. Equivalently that $I, A, A^2, \dots, A^{d-1}$ are linearly independent. Thus the set of regular matrices is an open subset U of $M_d(K)$.

Suppose B is any matrix and R is regular. Consider the map

$$f : \mathbb{A}^1 \to M_d(K), \quad f(\lambda) = R + \lambda B.$$

The image meets U. Thus $f^{-1}(M_d(K) \setminus U)$ is a proper closed subset of $\mathbb{A}^1$, so finite. Thus $R + \lambda B$ is regular for all but finitely many λ. Thus $B + \nu R$ is regular for all but finitely many $\nu \in K$.

Every matrix A commutes with a regular matrix R, for if A has diagonal blocks $J_{n_i}(\lambda_i)$ (with the λ_i) not necessarily distinct, then it commutes with the matrix with diagonal blocks $J_{n_i}(\mu_i)$, for any μ_i, and this is regular if the μ_i are distinct.

The set $C'_d = C_d \cap (M_d \times U)$ is dense in C_d, for if $(A, B) \in C_d \setminus \overline{C'_d}$, then there is an open set W of C_d containing (A, B) but not meeting C'_d. Let $g : \mathbb{A}^1 \to C_d$, $g(\nu) = (A, B + \nu R)$. Then $g^{-1}(C'_d)$ and $g^{-1}(W)$ are non-empty open subsets of $\mathbb{A}^1$ which don't meet. Impossible.

Let P be the set of polynomials of degree $\leq d - 1$. Now the map $h : P \times U \to C_d$, $(f(t), B) \mapsto (f(B), B))$ has image C'_d. Thus $C_d = \overline{\operatorname{Im} h}$, and since $P \times U$ is irreducible, so is C_d. Also this map is injective, so $\dim C_d = \dim U + \dim P = d^2 + d$.

3.3 Constructible sets

A subset of a variety is *constructible* if it is a finite union of locally closed subsets.

Lemma. (1) The class of constructible subsets is closed under finite unions and intersections, complements, and inverse images.

(2) If V is a constructible subset of X and $\overline{V}$ is irreducible, then there is a nonempty open subset U of $\overline{V}$ with $U \subset V$.

Proof. (1) Exercise.

(2) Write V as a finite union of locally closed subsets V_i. Then $\overline{V} = \bigcup_i \overline{V_i}$. Thus some $\overline{V_i} = \overline{V}$. Then V_i is open in $\overline{V}$.

Example. The punctured x-axis is locally closed in $\mathbb{A}^2$. It's complement is not locally closed, but it is constructible: it is the union of the plane minus the x-axis, and the origin.

Chevalley's Constructibility Theorem. The image of a morphism of varieties $\theta : X \to Y$ is constructible. More generally, the image of any constructible set is constructible.

Sketch. Wma X irreducible. Wma $Y = \overline{\mathrm{Im}(\theta)}$. The main lemma says that $\mathrm{Im}(\theta)$ contains a dense open subset U of Y. Thus it suffices to prove that the image under θ of $X \setminus \theta^{-1}(U)$ is constructible. Now work by induction on dimension.

Example. The set $\{x \in \mathrm{Mod}(A, \alpha) : K_x$ is indecomposable $\}$ is constructible in $\mathrm{Mod}(A, \alpha)$. Here K_x denotes the A-module of dimension vector α corresponding to x.

If $\alpha = \beta + \gamma$, then there is a direct sum map

$$f : \mathrm{Mod}(A, \beta) \times \mathrm{Mod}(A, \gamma) \to \mathrm{Mod}(A, \alpha)$$

sending (x, y) to the module structure $A \to M_d(K)$ which has x and y as diagonal blocks. It is a morphism of varieties. Thus the map

$$\mathrm{GL}(\alpha) \times \mathrm{Mod}(A, \beta) \times \mathrm{Mod}(A, \gamma) \to \mathrm{Mod}(A, \alpha), \quad (g, x, y) \mapsto g.f(x, y)$$

has as image all modules which can be written as a direct sum of modules of dimensions β and γ. This is constructible. Thus so is the union of these sets over all non-trivial decompositions $\alpha = \beta + \gamma$. Hence so is its complement, the set of indecomposables.

3.4 Upper semicontinuity and completeness

Definition. A function $f : X \to \mathbb{Z}$ is *upper semicontinuous* if $\{x \in X \mid f(x) < n\}$ is open for all $n \in \mathbb{Z}$. Thus with $\{x \in X \mid f(x) \geq n\}$ closed.

Examples. (1) The map $\mathrm{Hom}(V,W) \to \mathbb{Z}$, $\theta \mapsto \dim \mathrm{Ker}\,\theta$ is upper semicontinuous.

The set where it is $\geq t$ is the set of maps of rank $\leq r = \dim V - t$, so identifying with matrices, the set where all minors of size $r+1$ are zero.

(2) On the variety $\{(\theta,\phi) \in \mathrm{Hom}(U,V) \times \mathrm{Hom}(V,W) : \phi\theta = 0\}$, the map $(\theta,\phi) \mapsto \dim(\mathrm{Ker}\,\phi/\,\mathrm{Im}\,\theta)$ is upper semicontinuous.

Since it is equal to $\dim \mathrm{Ker}\,\theta + \dim \mathrm{Ker}\,\phi - \dim U$.

The *local dimension* at $x \in X$, denoted $\dim_x X$ is the infemum of the dimensions of neighbourhoods of x. Equivalently it is the maximal dimension of an irreducible component containing x.

Chevalley's Upper Semicontinuity Theorem. If $\theta : X \to Y$ is a morphism then the function $X \to \mathbb{Z}$, $x \mapsto \dim_x \theta^{-1}(\theta(x))$ is upper semicontinuous.

Sketch. Wma X is irreducible. Wma $Y = \overline{\mathrm{Im}(\theta)}$. By the Main Lemma, the minimal value of the function is $\dim X - \dim Y$, and it takes this value on an open subset $\theta^{-1}(U)$ of X. Thus need to know for the morphism $X \setminus \theta^{-1}(U) \to Y \setminus U$. Now use induction.

Definition. A *cone* in a vector space is a subset which contains 0 and is closed under multiplication by $\lambda \in K$.

If C is a closed cone in V then every irreducible component of C contains 0, so $\dim_0 C = \dim C$. Namely, let D be an irreducible component of C, there is a scaling map $f : \mathbb{A}^1 \times D \to C$, so $D \subseteq \overline{\mathrm{Im}\, f} \subseteq C$. Now $\overline{\mathrm{Im}\, f}$ is irreducible, so equal to D. It contains 0.

Corollary 1. Suppose X is a variety and V a vector space. Suppose that Y is a closed subset of $X \times V$ and that for all $x \in X$ the set $V_x = \{v \in V : (x,v) \in Y\}$ is a cone in V. Then the function $X \to \mathbb{Z}$, $x \mapsto \dim V_x$ is upper semicontinuous.

Proof. If $f : Y \to \mathbb{Z}$ is upper semicontinuous and $\phi : X \to Y$ is a morphism, then the composition $f\phi : X \to \mathbb{Z}$ is upper semicontinuous.

Consider the projection $\theta : Y \to X$. This gives an upper semicontinuous function $Y \to \mathbb{Z}$, $(x,v) \mapsto \dim_{(x,v)} \theta^{-1}(\theta(x))$.

Compose with the zero section $\phi : X \to Y$, $x \mapsto (x,0)$.

The map
$$x \mapsto \dim_{(x,0)} \theta^{-1}(\theta(x)) = \dim_0 V_x$$
is upper semicontinuous.

Now since V_x is a cone, $\dim_0 V_x = \dim V_x$.

Example. The function $\mathrm{Mod}(A, \alpha) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{End}_A(K_x)$ is upper semicontinuous.

Let $d = \sum \alpha_i$. Then K_x is K^d with the action of A given by a homomorphism $A \to M_d(K)$.

Now $\mathrm{End}_A(K_x)$ is a subspace of $\mathrm{End}_K(K_x) = M_d(K)$, so a cone, and

$$Y = \{(x = (A_1, \dots, A_k), \phi) \mid A_i\phi = \phi A_i \forall i\}$$

is a closed subset of $\mathrm{Mod}(A, \alpha) \times \mathrm{End}_K(K^d)$.

A variation: for a fixed finite-dimensional module M, the maps $\mathrm{Mod}(A, \alpha) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Hom}_A(M, K_x)$ and $\dim \mathrm{Hom}_A(M, K_x)$ are upper semicontinuous.

Another variation: the map $\mathrm{Mod}(A, \alpha) \times \mathrm{Mod}(A, \beta) \to \mathbb{Z}$ given by $(x, y) \mapsto \dim \mathrm{Hom}_A(K_x, K_y)$ is upper semicontinuous.

Definition A variety X is *complete* or *proper over* K if for any variety Y, the projection $X \times Y \to Y$ is a closed map. (Sends closed sets to closed sets.)

Easy properties. (1) A closed subvariety of a complete variety is complete.

(2) A product of complete varieties is complete

(3) If X is complete and $\theta : X \to Y$ is a morphism, then the image is closed and complete. (The image is the projection of the graph, hence closed. Need separatedness.)

(4) A complete affine or quasi-projective variety is projective.

There is an embedding $X \to \mathbb{P}^n$.

Corollary 2. Projective varieties are complete.

Proof. It suffices to prove for $\mathbb{P}^n$. Let $V = K^{n+1}$ and $V_* = V \setminus \{0\}$. There is a morphism $p : V_* \to \mathbb{P}^n$ sending a nonzero vector $(x_0, \dots, x_n)$ to $[x_0 : \cdots : x_n]$.

Let C be closed in $\mathbb{P}^n \times Y$. We need to show that its image under the projection to Y is closed.

If $y \in Y$ then $V_y = \{0\} \cup \{v \in V_* \mid (p(v), y) \in C\}$ is a cone in V. Is $Z = \{(v, y) \mid v \in V_y\}$ closed in $V \times Y$? Now p gives a morphism $(p, 1) : V_* \times Y \to \mathbb{P}^n \times Y$. Then $(p, 1)^{-1}(C)$ is closed in $V_* \times Y = (V \times Y) \setminus (\{0\} \times Y)$, so $Z = (p, 1)^{-1}(C) \cup (\{0\} \times Y)$ is closed in $V \times Y$.

Thus the function $y \mapsto \dim V_y$ is upper semicontinuous. Thus $\{y \in Y \mid \dim V_y = 0\}$ is open. This is the complement of the image of C.

Remark (to add to section on geometric quotients). Let $f : X \to Y$

be a morphism of varieties.

One says f is *universally open* if for any Z the map $f' : X \times Z \to Y \times Z$ is open, so sends open sets to open sets. One says that f : is *submersive* if it is surjective and

$$U \subseteq Y \text{ is open } \Leftrightarrow f^{-1}(U) \text{ is open in } X.$$

One says that f is *universally submersive* if f' is submersive for all Z. (Strictly speaking one should allow all fibre products in the category of schemes.)

By definition any geometric quotient $\pi : X \to X/G$ is submersive.

Fact. A geometric quotient π is universally submersive $\Leftrightarrow$ π is universally open.

Namely, suppose π is universally submersive and U is open in $X \times Z$, then so is $\bigcup_{g \in G} gU$, and this is $(\pi')^{-1}(\pi'(U))$.Thus $\pi'(U)$ is open.

Conversely suppose π is universally open, so π' is open. Now π is a morphism, hence so is π', so if $U \subseteq (X/G) \times Z$ is open, so is $(\pi')^{-1}(U)$. Conversely as π' is open, if $(\pi')^{-1}(U)$ is open, so is $\pi'((\pi')^{-1}(U)) = U$, since π' is onto.

The book Mumford, Fogarty and Kirwan, Geometric Invariant Theory, 3rd edition, 1994, claims in remark (4) on page 6 that any geometric quotient is universally open. But this is probably not true. In the first edition universally submersive was included as part of the definition of a geometric quotient. When this was changed in the second edition, presumably the remark was not corrected.

Remarks (to add to section on Grassmannians).

(1) We showed that the Grassmannian is a geometric quotient by showing that it is locally a projection. Since projections are universally open, it follows that Grassmannians are universally submersive geometric quotients.

(2) Let $\dim V = c+d$. To show that the constructions $\mathrm{Gr}(V, d) = \mathrm{Inj}(K^d, V)/\mathrm{GL}_d(K)$ and $\mathrm{Surj}(V, K^c)/\mathrm{GL}_c(K)$ are isomorphic, by duality it suffices to show that the map

$$\mathrm{Surj}(V, K^c) \to \mathrm{Gr}(V, d), \quad \phi \mapsto \mathrm{Ker}\,\phi$$

is a morphism of varieties.

As I suggested before, this can be checked locally.

Identify $\mathrm{Surj}(V, K^c)$ with the set of full rank matrices $C \in M_{c \times n}(K)$.

Given a subset I of $\{1, \dots, n\}$ of size d, let C_I be the $c \times c$ matrix obtained by deleting the columns in I and C'_I the $c \times d$ matrix obtained by keeping only the columns in I.

Let W_I be the open subset of $\mathrm{Surj}(V, K^c)$ consisting of the matrices C with C_I invertible. As I varies, this gives an open cover of $\mathrm{Surj}(V, K^c)$. Thus it suffices to show that the restriction to W_I is a morphism.

Now we have a map

$$W_I \xrightarrow{f} \mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$$

where $f(C)$ is the $n \times d$ matrix A with $A_I = I_d$ and $A'_I = -(C_I)^{-1}(C'_I)$. Observe that we have an exact sequence

$$0 \to K^d \xrightarrow{A} K^n \xrightarrow{C} K^c \to 0.$$

The composition is zero since it is $C_I A'_I + C'_I A_I = 0$. Thus f gives the map we want, and clearly f is a morphism of varieties.

(3) The map $\mathrm{Exact}(K^d, V, K^c) \to \mathrm{Gr}(V, d)$ is universally submersive.

Since the map $\mathrm{Surj}(V, K^c) \to \mathrm{Gr}(V, d)$ is universally submersive, the map

$$\mathrm{Inj}(K^d, V) \times \mathrm{Surj}(V, K^c) \times Z \to \mathrm{Inj}(K^d, V) \times \mathrm{Gr}(V, d) \times Z$$

is submersive.

It identifies $\mathrm{GL}_c(K)$-orbits on the LHS with points in the RHS.

Thus $\mathrm{GL}_c(K)$-stable closed subsets of the LHS correspond to closed subsets of the RHS.

Thus $\mathrm{GL}_c(K)$-stable closed subsets of $\mathrm{Exact}(K^d, V, K^c) \times Z$ (which is also of this form) corresponds to the closed subsets of

$$\{(\theta, \pi(\theta), z) : \theta \in \mathrm{Inj}(K^d), z \in Z\} \cong \mathrm{Inj}(K^d, V) \times Z.$$

Thus the map $\mathrm{Exact}(K^d, V, K^c) \to \mathrm{Inj}(K^d, V)$ is universally submersive.

Now compose it with the map $\mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$ which is universally submersive.

Example. Given A and dimension vector α and β, one want the set

$$\mathrm{Mod}\,\mathrm{Gr}(A, \alpha, \beta) = \{(x, (U_i)) \in \mathrm{Mod}(A, \alpha) \times \prod_i \mathrm{Gr}(K^{\alpha_i}, \beta_i) : (U_i) \in \mathrm{Gr}_A(K_x, \beta)\}$$

to be a closed subset of the product, so a variety. Now the map

$$\prod_i \mathrm{Exact}(K^{\beta_i}, K^{\alpha_i}, K^{\alpha_i - \beta_i}) \to \prod_i \mathrm{Gr}(K^{\alpha_i}, \beta_i)$$

is universally submersive, so it suffices to check that the lift to

$$\mathrm{Mod}(A,\alpha) \times \prod_i \mathrm{Exact}(K^{\beta_i}, K^{\alpha_i}, K^{\alpha_i-\beta_i})$$

is closed. Here it is straightforward.

Since Grassmannians are projective varieties, and projective varieties are complete, we get that

$$\{x \in \mathrm{Mod}(A,\alpha) : K_x \text{ has a submodule of dimension } \beta\}$$

which is the image of the projection

$$\mathrm{Mod}\,\mathrm{Gr}(A,\alpha,\beta) \to \mathrm{Mod}(A,\alpha)$$

is closed. Taking the union over all $\beta \neq 0, \alpha$, we get that the set

$$\mathrm{Simple}(A,\alpha) = \{x \in \mathrm{Mod}(A,\alpha) : K_x \text{ is a simple module}\}$$

is open in $\mathrm{Mod}(A,\alpha)$.

3.5 Orbits

Let G be a (linear) algebraic group. For simplicity we assume G is connected. Suppose that G acts on a variety X and $x \in X$. Then the orbit the stabilizer $\mathrm{Stab}_G(x)$ is a closed subgroup of G.

Theorem. Suppose that G acts on a variety X. Any orbit Gx is a locally closed subset of X of dimension $\dim G - \dim \mathrm{Stab}_G(x)$. Its closure $\overline{Gx}$ is the union of Gx together with orbits of strictly smaller dimension. Moreover $\overline{Gx}$ contains a closed orbit.

Proof. We show that Gx is locally closed in X.

The map $G \to X$, $g \mapsto gx$ is a morphism, so its image Gx is constructible.

Since G is irreducible, the closure $\overline{Gx}$ is irreducible. Thus Gx contains a nonempty open subset U of $\overline{Gx}$.

Left multiplication by $g \in G$ induces an isomorphism $X \to X$, so gU is an open subset of $g\overline{Gx} = \overline{Gx}$.

Thus $Gx = \bigcup_{g\in G} gU$ is an open subset of $\overline{Gx}$.

Thus Gx is locally closed.

Now the fibres of the map $G \to Gx$ are cosets of $\mathrm{Stab}_G(x)$, so all are isomorphic as varieties to $\mathrm{Stab}_G(x)$, so they have the same dimension. Then the Main Lemma gives $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.

Clearly $\overline{Gx}$ is a union of orbits. If Gy is one of them and $\dim Gy \not< \dim Gx$, then $\overline{Gy} = \overline{Gx}$, so Gy is open in $\overline{Gx}$, so $\overline{Gx} \setminus Gy$ is closed in X. If $Gy \neq Gx$ then this contains Gx, so nonsense.

Finally, for a closed orbit, take $Gy \subseteq \overline{Gx}$ of minimal dimension.

Remark. Using that G is connected, so an irreducible variety, we also get that all orbits Gx and their closures $\overline{Gx}$ are irreducible varieties.

Proposition. The map $X \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Stab}_G(x)$ is upper semicontinuous. Thus the set

$$X_{\leq s} = \{x \in X : \dim \mathrm{Stab}_G(x) \leq s\} = \{x \in X : \dim Gx \geq \dim G - s\}$$

is open and the set

$$X_s = \{x \in X : \dim \mathrm{Stab}_G(x) = s\} = \{x \in X : \dim Gx = \dim G - s\}$$

is locally closed.

Proof. Let $Z = \{(g, x) \in G \times X : gx = x\}$ and let $\pi : Z \to X$ be the projection. Now

$$\dim_{(1,x)} \pi^{-1}\pi(1, x) = \dim_1 \mathrm{Stab}_G(x) = \dim \mathrm{Stab}_G(x)$$

since $\mathrm{Stab}_G(x)$ is a group, so every point looks the same.

3.6 Orbits in $\mathrm{Mod}(A, \alpha)$ degenerations and the nilpotent variety

Notation. Recall that the orbits of $\mathrm{GL}(\alpha)$ in $\mathrm{Mod}(A, \alpha)$ correspond to isomorphism classes of modules of dimension vector α. We write $\mathcal{O}_M$ for the orbit corresponding to a module M, so $\mathcal{O}_M = \{x \in \mathrm{Mod}(A, \alpha) : K_x \cong M\}$. We have

$$\dim \mathrm{GL}(\alpha) - \dim \mathcal{O}_M = \dim \mathrm{Stab}_{\mathrm{GL}(\alpha)}(x) = \dim \mathrm{Aut}_A(M) = \dim \mathrm{End}_A(M).$$

The last holds since $\mathrm{Aut}_A(M) \subseteq \mathrm{End}_A(M)$ is a non-empty open subset of a vector space, which is an irreducible variety. Recall also that $\mathrm{GL}(\alpha) = \prod_i \mathrm{GL}_{\alpha_i}(K)$ so it has dimension $\sum \alpha_i^2$.

Definition. We say that M *degenerates to* N if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$.

This is a partial order, for if M degenerates to N and $M \ncong N$, then $\dim \mathcal{O}_N < \dim \mathcal{O}_M$.

More generally, given any G acting on a variety X, we say that $x \in X$ *degenerates to* $y \in X$ if $y \in \overline{Gx}$.

Example. Recall that the nilpotent variety is $N_d = \mathrm{Mod}(K[T]/(T^d), d) = \{A \in M_d(K) : A^d = 0\}$.

There are only finitely many orbits under the action of $\mathrm{GL}_d(K)$. They are $\mathcal{O}_{M(\lambda)}$ where $\lambda = (\lambda_1, \lambda_2, \dots,)$ is a partition of n into at $\leq d$ parts, and $M(\lambda)$ is the $K[T]/(T^d)$-module with vector space K^n with T acting as the matrix involving a Jordan block $J_i(0)$ of eigenvalue 0 and size i for each column of length i in the Young diagram of shape λ (so with rows of length λ_i).

We claim that the module $M(1^d) \cong K[T]/(T^d)$ given by a Jordan block of size d degenerates into any other module. Namely, given λ and $t \in K$, consider the module M_t which is given by the same matrix as $M(\lambda)$, so zero except for some ones on the superdiagonal, but now with the zeros on the superdiagonal changed into ts.

Clearly $M_t \cong M_1 \cong M(1^d)$ for $t \neq 0$ and $M_0 = M(\lambda)$.

Thus $N_d = \overline{\mathcal{O}_{M(1^d)}}$, so it is irreducible of dimension $d^2 - \dim \mathrm{End}(M(1^d)) = d^2 - d$.

Theorem. Given A-modules M and N (of the same dimension vector) we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(i) There are modules $M = M_0, M_1, \dots, M_n = M$ and exact sequences $0 \to L_i \to M_i \to L_i' \to 0$ with $M_{i+1} \cong L_i \oplus L_i'$.

(ii) M degenerates to N

(iii) $\dim \mathrm{Hom}(X, M) \leq \dim \mathrm{Hom}(X, N)$ for all X.

Proof. (ii) $\Rightarrow$ (iii). Use that $\dim \mathrm{Hom}_A(X, -)$ is upper semicontinuous.

(i) $\Rightarrow$ (ii). If M degenerates to N and N degenerates to L, then certainly M degenerates to L. Thus it suffices to prove that if $0 \to L \to M \to L' \to 0$ then M degenerates to $L \oplus L'$. For simplicity we do $\mathrm{Mod}(A, d)$. An element $x \in \mathrm{Mod}(A, d)$ is defined by matrices x_a where a runs through a set of generators of A. Taking a basis of L and extending it to a basis of M, there is $x \in \mathcal{O}_M$ in which each matrix x_a has the form

$$x_a = \begin{pmatrix} y_a & w_a \\ 0 & z_a \end{pmatrix}$$

with $K_y \cong L$ and $K_z \cong L'$.

For $t \in K$ define an element x^t via

$$x^t_a = \begin{pmatrix} y_a & tw_a \\ 0 & z_a \end{pmatrix}.$$

For $t \neq 0$, x^t is the conjugation of x by the diagonal matrix $\left(\begin{smallmatrix} tI & 0 \\ 0 & I \end{smallmatrix}\right) \in \mathrm{GL}_d(K)$, so $x^t \in \mathrm{Mod}(A, d)$, and moreover $x^t \in \mathcal{O}_M$. Thus $x^0 \in \overline{\mathcal{O}_M}$, and $K_{x^0} \cong L \oplus L'$.

Remark. Hopefully we will have time to do Zwara's Theorems.

- M degenerates to N iff there is an exact sequence $0 \to Z \to Z \oplus M \to N \to 0$.

- If A has finite representation type then (iii) implies (ii) inequality on Homs implies M degenerates to N.

Special case. For the nilpotent variety, so the algebra $K[T]/(T^d)$, or more generally the algebra $K[T]$, conditions (i),(ii),(iii) are all equivalent (Gerstenhaber-Hesselink). Moreover if $M = M(\lambda)$ and $N = M(\mu)$ then condition (iii) becomes that $\lambda \trianglelefteq \mu$ in the *dominance ordering* of partitions.

Firstly, $\dim \mathrm{Hom}(K[T]/(T^i), M(\lambda)) = \lambda_1 + \dots + \lambda_i$, so condition (iii) says that $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ for all i, and this is the dominance ordering.

Now the dominance order is generated by the following move: $\lambda \trianglelefteq \mu$ if μ is obtained from λ by moving a corner block from a column of length j to a column further to the right of length $i < j$, for example

$$(6, 6, 4, 2) \trianglelefteq (6, 6, 5, 1)$$

(See for example I. G. Macdonald, Symmetric functions and Hall polynomials, I, (1.16).) We want to show in this case that there is an exact sequence

$$0 \to L \to M(\lambda) \to L' \to 0$$

with $M(\mu) \cong L \oplus L'$. Now $M(\lambda) = K[T]/(T^j) \oplus K[T]/(T^{i-1}) \oplus C$ and $M(\mu) = K[T]/(T^{j-1}) \oplus K[T]/(T^i) \oplus C$, so the exact sequence

$$0 \to K[T]/(T^i) \xrightarrow{\begin{pmatrix} -1 \\ T^{j-i} \end{pmatrix}} K[T]/(T^{i-1}) \oplus K[T]/(T^j) \xrightarrow{\begin{pmatrix} T^{j-i} & 1 \end{pmatrix}} K[T]/(T^{j-1}) \to 0.$$

will do.

Lemma. If C is a finite-dimensional algebra, then the variety $N(C)$ of nilpotent elements in C is irreducible of dimension $\dim C - s$, where s is the sum of the dimensions of the simple C-modules.

Proof. Since K is algebraically closed, we can write $C = S \oplus J(C)$ where S is semisimple, so $S \cong M_{d_1}(K) \oplus \cdots \oplus M_{d_r}(K)$. Then $N(C) \cong N_{d_1} \times \dots N_{d_r} \times J(C)$, so it is irreducible of dimension

$$\dim N(C) = \sum_i d_i^2 - d_i + \dim J(C) = \dim C - \sum_i d_i.$$

Proposition. If A is a finitely generated algebra, α a dimension vector, and $r \in \mathbb{N}$ then the set

$$\mathrm{Ind}(A,\alpha)_r = \{x \in \mathrm{Mod}(A,\alpha) : K_x \text{ is indecomposable and } \dim \mathrm{End}_A(K_x) = r\}$$

is a closed subset of

$$\mathrm{Mod}(A,\alpha)_{\leq r} = \{x \in \mathrm{Mod}(A,\alpha) : \dim \mathrm{End}_A(K_x) \leq r\},$$

which is an open subset of $\mathrm{Mod}(A,\alpha)$.

Proof. By the upper semicontinuity theorem for cones, the function

$$\mathrm{Mod}(A,\alpha) \to \mathbb{Z}, \quad x \mapsto \dim N(\mathrm{End}_A(K_x))$$

is upper semicontinuous. Now by the lemma $\mathrm{Ind}(A,\alpha)_r$ is equal to

$$\{x \in \mathrm{Mod}(A,\alpha) : \dim \mathrm{End}_A(K_x) \leq r\} \cap \{x \in \mathrm{Mod}(A,\alpha) : \dim N(\mathrm{End}_A(K_x)) \geq r-1\}.$$

3.7 Closed orbits in $\mathrm{Mod}(A,\alpha)$

Lemma. Given an A-module M and a simple module S, the multiplicity of S in M is given by

$$[M : S] = \frac{1}{\dim S} \min_{a \in \mathrm{Ann}(S)} \{\text{Order of zero at } t = 0 \text{ of } \chi_{\hat{a}_M}(t)\}$$

where $\hat{a}_M$ is the homothety $M \to M$, $m \mapsto am$ and $\chi_\theta(t) = \det(t1 - \theta)$ is the characteristic polynomial of an endomorphism θ.

Proof. Given an exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, the endomorphism $\hat{a}_Y$ has uppertriangular block form, so

$$\chi_{\hat{a}_Y}(t) = \chi_{\hat{a}_X}(t)\chi_{\hat{a}_Z}(t) = \chi_{\hat{a}_{X \oplus Z}}(t).$$

Thus we may assume that M is semisimple.

Next we may assume that $M \oplus S$ is faithful. Thus A is semisimple, Now if $M \cong S^k \oplus N$ with $[N : S] = 0$, then the smallest order we could hope to get, and it can be achieved is if a acts on S as 0 and invertibly on N. This is possible, for writing A as a product of matrix algebras we can take a to correspond to 0 in the block for S and 1 in the other blocks.

Definition. Given a module M of dimension d and $a \in A$, we define $c_i^a(M) \in K$ by

$$\chi_{\hat{a}_M}(t) = t^d + c_1^a(M)t^{d-1} + \cdots + c_d^a(M)$$

Thus $c_1^a(M) = -\operatorname{tr}(\hat{a}_M)$ and $c_d^a(M) = (-1)^d \det(\hat{a}_M)$. Then c_i^a defines a regular map $\operatorname{Mod}(A, \alpha) \to K$. Moreover it is constant on the orbits of $\operatorname{GL}(\alpha)$.

By the lemma, these functions determine the multiplicities of the simples in M. In fact if K has characteristic zero, one only needs to know the trace c_1. This is character theory of groups.

Theorem. $\overline{\mathcal{O}_M}$ contains a unique orbit of semisimple modules, namely $\mathcal{O}_{\operatorname{gr} M}$ where $\operatorname{gr} M$ is the semisimple module with the same composition multiplicities as M. It follows that $\mathcal{O}_M$ is closed if and only if M is semisimple.

Proof. By the theorem, $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{\operatorname{gr} M}$. If $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$ then by continuity $c_i^a(N) = c_i^a(M)$, so M and N have the same composition multiplicities.

Remark. Also true is that $\operatorname{Ext}^1(M, M) = 0$ implies $\mathcal{O}_M$ is open, with a converse when the scheme $\mathbf{Mod}(A, \alpha)$ is reduced, for example for $A = KQ$. I hope to discuss later.

3.8 The variety $\operatorname{Alg}\operatorname{Mod}$ and global dimension

For a finite-dimensional algebra A we can identify $\operatorname{Mod}(A, d)$ with the set of K-algebra maps $A \to M_d(K)$. We set

$$\operatorname{Alg}\operatorname{Mod}(r, d) = \{(a, x) \in \operatorname{Alg}(r) \times \operatorname{Hom}_K(K^r, M_d(K)) : x \in \operatorname{Mod}(K_a, d)\}$$

where K_a denotes the algebra structure on K^r. This is a closed subset, so an affine variety. The group $\operatorname{GL}_d(K)$ acts by conjugation on the second factor.

The following is a reformulation of Lemma 3.2 in P. Gabriel, Finite representation type is open. This reformulation is mentioned in C. Geiss, On degenerations of tame and wild algebras, 1995.

Theorem (Gabriel). The projection $\pi : \operatorname{Alg}\operatorname{Mod}(r, d) \to \operatorname{Alg}(r)$ sends $\operatorname{GL}_d(K)$-stable closed subsets to closed subsets.

Lemma 1. If X is a variety, then the projection $X \times \mathrm{Inj}(K^d, V) \to X$ sends $\mathrm{GL}_d(K)$-stable closed subsets to closed subsets. Similarly for the projection $X \times \mathrm{Surj}(V, K^c) \to X$.

Proof. We factor it as

$$X \times \mathrm{Inj}(K^d, V) \to X \times \mathrm{Gr}(V, d) \to X$$

Now the map $\mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$ is universally submersive, so $\mathrm{GL}_d(K)$-stable open subsets of $X \times \mathrm{Inj}(K^d, V)$ correspond to open subsets of $X \times \mathrm{Inj}(K^d, V)$. Thus $\mathrm{GL}_d(K)$-stable closed subsets of $X \times \mathrm{Inj}(K^d, V)$ correspond to closed subsets of $X \times \mathrm{Inj}(K^d, V)$. Now use that $\mathrm{Gr}(V, d)$ is complete.

Proof of the theorem. Let

$W = \{(a, \theta) \in \mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d) : \mathrm{Ker}\,\theta \text{ is a } K_a\text{-submodule of } (K_a)^r\}.$

This is a closed subset of the product. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & \mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d) \arrow[d, "p"] \\ \mathrm{Alg}\,\mathrm{Mod}(r, d) \arrow[r, "\pi"] & \mathrm{Alg}(r) \end{tikzcd}$$

where p is the projection and g sends (a, θ) to the pair consisting of a and the induced K_a-module structure on K^d. Now g is onto since any d-dimensional K_a-module is a quotient of a free module of rank d.

One can check using the affine open covering of $\mathrm{Surj}(K^{rd}, K^d)$ that g is a morphism of varieties.

Suppose $Z \subseteq \mathrm{Alg}\,\mathrm{Mod}(r, d)$ is $\mathrm{GL}_d(K)$-stable and closed. Then $g^{-1}(Z)$ is also. Thus it is a $\mathrm{GL}_d(K)$-stable closed subset of $\mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d)$. Thus $\pi(Z) = p(g^{-1}(Z))$ is closed by the lemma.

Lemma 2. Any algebra A has a projective resolution as an A-A-bimodule

$$\to A \otimes A \otimes A \to A \otimes A \to A \to 0$$

(where tensor products are over the base field K). Here the maps are

$$b_n : A^{\otimes n+1} \to A^{\otimes n}, \quad a_0 \otimes a_1 \otimes \cdots \otimes a_n \to \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n$$

Tensoring with a left A-module X gives a projective resolution of X,

$$\to A \otimes A \otimes X \to A \otimes X \to X \to 0$$

On Wikipedia the complex of bimodules is called the *standard complex*. In MacLane, Homology, the resolution of X is called the *un-normalized bar resolution of X*.

Proof. Define a map (of right A-modules) $h_n : A^{\otimes n} \to A^{\otimes n+1}$ by $h_n(a_1 \otimes \dots \otimes a_n) = 1 \otimes a_1 \otimes \dots \otimes a_n$. One easily checks that $b_1 h_1 = 1$ and

$$b_{n+1} h_{n+1} + h_n b_n = 1 \quad (n \geq 1).$$

Also $b_1 b_2 = 0$ and then by induction $b_n b_{n+1} = 0$ for all $n \geq 1$ since

$$b_{n+1} b_{n+2} h_{n+2} = b_{n+1}(1 - h_{n+1} b_{n+1}) = b_{n+1} - b_{n+1} h_{n+1} b_{n+1} = b_{n+1} - (1 - h_n b_n) b_{n+1} = 0.$$

Now $\mathrm{Im}(h_{n+2})$ generates $\mathbb{A}^{\otimes n+2}$ as a left A-module, and the b_i are left A-module maps (in fact bimodule maps), so $b_{n+1} b_{n+2} = 0$. Finally if $x \in \mathrm{Ker}(b_n)$ then $x = (b_{n+1} h_{n+1} + h_n b_n)(x)$ implies $x \in \mathrm{Im}(b_{n+1})$, giving exactness.

Applying $- \otimes_A X$ with a left A-module X to the standard complex gives an exact sequence. This is because the terms in the standard complex are projective right A-modules, it you break it into short exact sequences of right A-modules, all of them are split.

Proposition (Schofield). For any i, the map

$$\mathrm{Alg\,Mod}(r, d) \to \mathbb{Z}, \quad (a, x) \mapsto \dim \mathrm{Ext}^i_{K_a}(K_x, K_x)$$

is upper semicontinuous.

Proof. Applying $\mathrm{Hom}_A(-, Y)$ to the projective resolution of X given by the standard complex, with Y another A-module, and using that $\mathrm{Hom}_A(A \otimes M, Y) \cong \mathrm{Hom}_K(M, Y)$, we see that $\mathrm{Ext}^i_A(X, Y)$ is computed as the cohomology of a complex

$$0 \to \mathrm{Hom}_K(X, Y) \to \mathrm{Hom}_K(A \otimes X, Y) \to \mathrm{Hom}_K(A \otimes A \otimes X, Y) \to \dots$$

Now taking $A = K_a$ and $X = Y = K_x$ for $(a, x) \in \mathrm{Alg\,Mod}(r, d)$, we see that the terms in this complex are fixed vector spaces V^i, and the maps are given by morphisms $f_i : \mathrm{Alg\,Mod}(r, d) \to \mathrm{Hom}_K(V^i, V^{i+1})$. Thus we get a morphism

$$\mathrm{Alg\,Mod}(r, d) \to \{(\theta, \phi) \in \mathrm{Hom}(V^{i-1}, V^i) \times \mathrm{Hom}(V^i, V^{i+1}) : \phi\theta = 0\}.$$

Now use that the map $(\theta, \phi) \mapsto \dim(\mathrm{Ker}\,\phi / \mathrm{Im}\,\theta)$ is upper semicontinuous.

Corollary (Schofield). The algebras of global dimension $\leq g$ form an open subset of $\mathrm{Alg}(r)$, as do the algebras of finite global dimension. There is an integer N_r, depending on r, such that any algebra of dimension r of finite global dimension has global dimension $\leq N_r$.

Proof. A has global dimension $\leq g$ if and only if $\mathrm{Ext}_A^{g+1}(M,N)=0$ for all M,N. By the long exact sequences, it is equivalent that $\mathrm{Ext}_A^{g+1}(M,N)=0$ for all simple M and N. Thus it is equivalent that $\mathrm{Ext}_A^{g+1}(M,M)=0$ for $M=\mathrm{gr}\,A$. Consider the pairs $(a,x)\in\mathrm{Alg}\,\mathrm{Mod}(r,r)$ such that $\mathrm{Ext}_{K_a}^{g+1}(K_x,K_x)\neq 0$. By upper semicontinuity this is a closed subset of $\mathrm{Alg}\,\mathrm{Mod}(r,r)$. It is also stable under $\mathrm{GL}_r(K)$, so its image in $\mathrm{Alg}(r)$ is closed. This is the set of algebras of global dimension $>g$. Thus the algebras of global dimension $\leq g$ form an open subset D_g. Now since varieties are noetherian topological spaces, the chain of open sets

$$D_0\subseteq D_1\subseteq D_2\subseteq\dots$$

stabilizes.

3.9 Number of parameters

Let G be a connected algebraic group acting on a variety X. We define

$$X_{(d)}=\{x\in X:\dim Gx=d\}=\{x\in X:\dim\mathrm{Stab}_G(x)=\dim G-d\}=X_{\dim G-d}$$

a locally closed subset of X and

$$X_{(\leq d)}=\{x\in X:\dim Gx\leq d\}=\{x\in X:\dim\mathrm{Stab}_G(x)\geq\dim G-d\}$$

a closed subset of X.

Lemma 1. If $Y\subseteq X$ is a constructible subset of X, then it can be written as a disjoint union

$$Y=Z_1\cup\dots\cup Z_n$$

with the Z_i being irreducible locally closed subsets of X. If Y is G-stable, then we may take the Z_i to be G-stable.

Proof. Exercise.

For the first part, by definition we can write Y as a not necessarily disjoint union $Y=Z_1\cup\dots\cup Z_n$. Replacing each Z_i by its irreducible components we may suppose the Z_i are irreducible. Then if this union is of the form $Y=Z\cup W$ where Z is irreducible of maximal dimension and W is the union of the other terms, then Y is the disjoint union of $Z\setminus\overline{W}$ and $W'=(Z\cap\overline{W})\cup W$, and if the first term is non-empty, then $(Z\cap\overline{W})$ is a proper closed subset of Z, so has strictly smaller dimension than Z, so W' can be understood by induction.

For the last part, use that G is irreducible, so if $Z \subseteq Y$ is locally closed in X and irreducible, then $GZ = \bigcup_{g \in G} gZ$ is constructible, contained in Y and its closure $\overline{GZ}$ is irreducible, so there is an open subset U of $\overline{GZ}$ with $U \subseteq GZ$. But then GU is open in $\overline{GZ}$ and $GU \subseteq GZ$.

Definition. If Y is constructible, and it is written as a disjoint union of irreducible locally closed subsets Z_i, we define the dimension and number of top-dimensional irreducible components of Y by

$$\dim Y = \max\{\dim Z_i : 1 \leq i \leq n\},$$

$$\operatorname{top} Y = |\{1 \leq i \leq n : \dim Z_i = \dim Y\}|.$$

for a decomposition of Y as in the lemma (here we can take $G = 1$). This does not depend on the decomposition of Y.

Now suppose that G acts on X and assume that Y is G-stable. We define the number of parameters and number of top-dimensional families by

$$\dim_G Y = \max\{\dim(Y \cap X_{(d)}) - d : d \geq 0\},$$

$$\operatorname{top}_G Y = \sum\{\operatorname{top}(Y \cap X_{(d)}) : d \geq 0, \dim(Y \cap X_{(d)}) - d = \dim_G Y\}.$$

Properties.

(i) If Y_1, Y_2 are G-stable then $\dim_G(Y_1 \cup Y_2) = \max\{\dim_G Y_1, \dim_G Y_2\}$.

(ii) $\dim_G Y = 0$ if and only if Y contains only finitely many orbits, and if so, $\operatorname{top}_G Y$ is the number of orbits.

(iii) If Y contains a constructible subset Z meeting every orbit, then $\dim_G Y \leq \dim Z$.

(iv) If $f : Z \to X$ is a morphism and the inverse image of each orbit has dimension $\leq d$, then $\dim_G X \geq \dim Z - d$.

(v) $\dim_G Y = \max\{\dim(Y \cap X_{(\leq d)}) - d : d \geq 0\}$.

Lemma 2. Suppose that $\pi : X \to Y$ is constant on orbits, and suppose that the image of any G-stable closed subset of X is a closed subset of Y. Then the function $Y \to \mathbb{Z}$, $y \mapsto \dim_G(\pi^{-1}(y))$ is upper semicontinuous.

Proof. We prove it forst for the function dim. By Chevalley's upper semicontinuity theorem, for any r the set

$$C_x = \{x \in X : \dim_x \pi^{-1}(\pi(x)) \geq r\}$$

is closed in X. It is also G-stable, so by hypothesis $\pi(C_x)$ is closed. Now if $y \in Y$ then $\dim \pi^{-1}(y) = \max\{\dim_x \pi^{-1}(y) : x \in \pi^{-1}(y)\}$. Thus

$$\{y \in Y : \dim \pi^{-1}(y) \geq r\} = \pi(C_r),$$

so it is closed in Y. Thus the map $y \mapsto \dim \pi^{-1}(y)$ is upper semicontinuous.

Now $X_{(\leq d)} = \{x \in X : \dim Gx \leq d\}$ is closed in X, and π_d, which is the restriction of π to this set, sends closed G-stable subsets to closed subsets, so

$$\{y \in Y : \dim \pi_d^{-1}(y) \geq r\}$$

is closed in Y. Now

$$\{y \in Y : \dim_G \pi^{-1}(y) \geq r\} = \bigcup_d \{y \in Y : \dim \pi_d^{-1}(y) \geq d + r\}$$

which is closed.

3.10 Tame and wild

Let A and B be K-algebras and $d \in \mathbb{N}$.

Observe that there is a 1-1 correspondence between K-algebra homomorphisms $\theta : A \to M_d(B)$ up to conjugacy by an element of $\mathrm{GL}_d(B)$ and A-B-bimodules M which are free of rank d over B.

If A and B are finitely generated and $s \in \mathbb{N}$, then such a homomorphism θ induces a morphism of varieties

$$f : \mathrm{Mod}(B, s) \to \mathrm{Mod}(A, ds)$$

sending a K-algebra map $B \to M_s(K)$ to the composition $A \to M_d(B) \to M_d(M_s(K)) \cong M_{ds}(K)$. In terms of the corresponding A-B-bimodule M we have $M \otimes_B K_x \cong K_{f(x)}$ for all x.

Taking B to be a commutative and reduced, and $s = 1$, we can write this as

$$f : \mathrm{Spec}\, B \to \mathrm{Mod}(A, d).$$

Conversely any morphism of varieties of this form with B f.g. commutative and reduced comes from a homomorphism $A \to M_d(B)$. Namely since $\mathrm{Mod}(A, d)$ is an affine variety, morphisms $\mathrm{Spec}\, B$ to $\mathrm{Mod}(A, d)$ correspond to K-algebra maps $\mathcal{O}(\mathrm{Mod}(A, d)) \to B$. Since B is commutative and reduced, this is the same as K-algebra maps $\sqrt[d]{A} \to B$. This is the same as K-algebra maps $A \to M_d(B)$.

Definition. An algebra A is *tame* if for any d there are A-$K[T]$-bimodules $M_1, \dots, M_N$, finitely generated and free over $K[T]$, such that all but finitely many indecomposable A-modules of dimension $\leq d$ are isomorphic to $M_i \otimes K[T]/(T-\lambda)$ for some i and λ.

Remarks. (i) Equivalently there are a finite number of morphisms $\mathbb{A}^1 \to \mathrm{Mod}(A,d)$ such that the images meet all but finitely many orbits.

(ii) In the definition of tame can delete the "but finitely many" by including additional maps $\mathbb{A}^1 \to \mathrm{Mod}(A,d)$ which are constant. In terms of bimodules it means including bimodules of the form $M = X \otimes_K K[T]$ where X is a given left A-module.

(iii) Any algebra of finite representation type is clearly tame by this definition. Sometimes the name tame representation type' is only used for algebras of infinite representation type.

Definition. Let us say that a functor F from B-module to A-modules is a *representation embedding* if

(i) F sends indecomposable modules to indecomposable modules.

(ii) If $F(X) \cong F(Y)$ then $X \cong Y$.

(iii) F is naturally isomorphic to a tensor product functor $M \otimes_B -$ for an A-B-bimodule which is finitely generated projective over B (and on which K acts centrally), or equivalently it is an exact K-linear functor which preserves products and direct sums.

An algebra A is *wild* if there is a representation embedding from $K\langle X,Y\rangle$-modules to A-modules.

Remarks. In the definition of wild, we work with categories of all A- and B-modules, following WCB, Tame algebras and generic modules, 1991. One can also restrict to the categories of finite-dimensional modules.

Lemma. (i) If I is an ideal in A then the natural functor A/I-Mod $\to$ A-Mod is a representation embedding.

(ii) For any n there is a representation embedding $K\langle X_1, \dots, X_n\rangle$-Mod $\to$ $K\langle X,Y\rangle$-Mod.

Thus if A is wild there is a representation embedding B-Mod $\to$ A-Mod for any finitely generated algebra B.

Proof. (i) is trivial. For (ii) Let $B = K\langle X_1, \dots, X_n\rangle$. Consider the A-B-bimodule M corresponding to the homomorphism $\theta : A \to M_{n+2}(B)$ sending

X and Y to the matrices C and D,

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ X_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & X_2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & X_n & 1 & 0 \end{pmatrix}$$

These matrices are in S. Brenner, Decomposition properties of some small diagrams of modules, 1974. Thus $M \cong B^{n+2}$ as a right B-module, with the action of A given by the homomorphism. Suppose Z, Z' are B-modules and $f : M \otimes_B Z \to M \otimes_B Z'$. Then f is given by an $(n+2) \times (n+2)$ matrix of linear maps $Z \to Z'$, say $F = (f_{ij})$ such that $CF = FC$ and $DF = FD$. The condition $CF = FC$ gives

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & \dots \\ f_{21} & f_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & \dots \\ f_{21} & f_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

so $f_{i+1,j} = f_{i,j-1}$ for $1 \le i, j \le n+2$, where the terms are zero if i or j are out of range. This forces F to be constant on diagonals, and zero below the main diagonal,

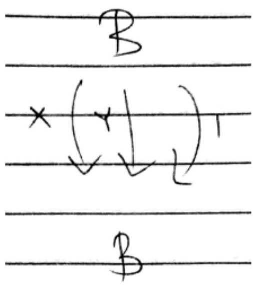
$$F = \begin{pmatrix} f_1 & f_2 & f_3 & \dots & f_{n+1} & f_{n+2} \\ 0 & f_1 & f_2 & \dots & f_n & f_{n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_1 & f_2 \\ 0 & 0 & 0 & \dots & 0 & f_1 \end{pmatrix}.$$

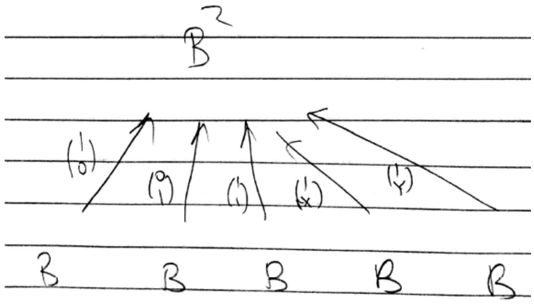
Now the condition $DF = FD$ gives $f_i = 0$ for $i > 1$ and $X_i f_1 = f_1 X_i$ for all i. Thus f_1 is a B-module map $Z \to Z'$.

If f is is an isomorphism, then so is f_1. Also, taking $Z = Z'$, if f is an idempotent endomorphism, then so is f_1. Thus is Z is indecomposable, $f_1 = 0$ or 1, so $f = 0$ or 1, so $M \otimes_B Z$ is indecomposable.

Examples. Path algebras of Dynkin and extended Dynkin quivers are tame. Other important classes of tame algebras are the tubular algebras and string algebras.

Path algebras of other quivers are wild. For example, letting $B = \langle X, Y \rangle$, for the path algebra A of the three arrow Kronecker quiver or five subspace quiver, consider the A-B-bimodule which is the direct sum of the indicated powers of B, with the natural action of B, and with the A-action given by the indicated matrices, acting as left multiplication.





The algebra $A = K[x, y, z]/(x, y, z)^2$ is wild. (This argument is taken from Ringel, The representation type of local algebras, 1975) Given a $K\langle X, Y \rangle$-module V, we send it to the A-module V^2 with

$$x = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is a tensor product functor. The image is contained in the subcategory C of A-modules M which are free over $K[z]/(z^2)$, or equivalently with $z^{-1}0_M = zM$. There is a functor from C to $K\langle X, Y \rangle$-modules, sending M to zM with X and Y given by the relations xz^{-1} and yz^{-1}. The composition

$$K\langle X, Y \rangle\text{-Mod} \xrightarrow{F} C \xrightarrow{G} K\langle X, Y \rangle\text{-Mod}$$

is isomorphic to the identity functor. Now if $G(M) = 0$ then $M = 0$. It follows that F is a representation embedding.

The algebra $K[x, y]$ is wild (Gelfand and Ponomarev), in fact even the algebra $K[x, y]/(x^2, xy^2, y^3)$ is wild (Drozd).

Drozd's Theorem. Any finite dimensional algebra is tame or wild, and not both.

The proof of the first part is difficult. The second part follows from the following.

Lemma. If A is tame then $\dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(A, d) \leq d$ for all d. If A is wild then there is $r > 0$ with $\dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(A, rd) \geq d^2$ for all d.

Proof. If M is an A-B-bimodule, free of rank r over B, then choosing a free basis of M, one obtains a homomorphism $A \to M_r(B)$. Now any element of $\mathrm{Mod}(B,d)$ is a K-algebra homomorphism $B \to M_d(K)$. Combining these, we get a K-algebra map $A \to M_{rd}(K)$. This defines a morphism of varieties

$$\mathrm{Mod}(B,d) \to \mathrm{Mod}(A,rd)$$

corresponding to the functor $M \otimes_B -$.

If A is wild we have a map

$$\mathrm{Mod}(K\langle X,Y\rangle,d) \to \mathrm{Mod}(A,rd).$$

The inverse image of any orbit is an orbit, so

$$\dim_{\mathrm{GL}_{rd}(K)} \mathrm{Mod}(A,rd) \geq \dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(K\langle X,Y\rangle,d).$$

Now $\dim \mathrm{Mod}(K\langle X,Y\rangle,d) = 2d^2$, and every orbit in $\mathrm{Mod}(K\langle X,Y\rangle,d)$ has dimension $\leq d$. Thus there is some $s \leq d$ such that the set $\mathrm{Mod}(K\langle X,Y\rangle,d)_{(s)}$ consisting of the orbits of dimension s has dimension $2d^2$. Then

$$\dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(K\langle X,Y\rangle,d) \geq 2d^2 - s \geq d^2.$$

If A is tame, we can suppose that any d-dimensional module is isomorphic to a direct sum of

$$M_{i_1} \otimes K[T]/(T-\lambda_1) \oplus \cdots \oplus M_{i_m} \otimes K[T]/(T-\lambda_m)$$

where the sum of the ranks of the M_{i_j} is d. In particular $m \leq d$. This defines a map

$$\mathbb{A}^m \to \mathrm{Mod}(A,d).$$

The union of the images of these maps, over all possible choices is a constructible subset of $\mathrm{Mod}(A,d)$ of dimension $\leq d$ which meets every orbit, giving the claim.

Theorem (Geiß). A degeneration of a wild algebra is wild.

Thus, by Drozd's Theorem, if an algebra degenerates to a tame algebra, it is tame.

Proof. By the lemma $\{x \in \mathrm{Alg}(r) : K_x \text{ is wild}\} = \bigcup_d M_d$ where

$$M_d = \{x \in \mathrm{Alg}(n) : \dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(K_x,d) > d\}.$$

Now M_d is closed by properties of Alg Mod and $\dim_{\mathrm{GL}_d(K)}$, and it is obviously $\mathrm{GL}_d(K)$-stable. Suppose $x,y \in \mathrm{Alg}(r)$ and $y \in \overline{\mathrm{GL}_d(K)x}$. If K_x wild, then

$x \in M_d$ for some d, then the orbit of x is contained in M_d, and hence so is the orbit closure. Thus $y \in M_d$, so K_y is wild.

Example. The algebra

$$A = K\langle a, b\rangle/(a^2 - bab, b^2 - aba, (ab)^2, (ba)^2)$$

degenerates to

$$B = K\langle a, b\rangle/(a^2, b^2, (ab)^2, (ba)^2)$$

and B is known to be tame, hence so is A. The degeneration is give as follows. For $t \in K$ let $x^t \in \mathrm{Alg}(7)$ have basis $1, a, b, ab, ba, aba, bab$ with multiplication as indicated, and with $a^2 = tbab$, $b^2 = taba$, $(ab)^2 = 0$, $(ba)^2 = 0$. Then for $t \neq 0$ this is isomorphic to A, and for $t = 0$ it is B.

[At the moment, I know of no classification of the indecomposable modules for this algebra A.]

Remark. In the same way, a degeneration of an algebra of infinite representation type is of infinite representation type. Gabriel used this, together with the second Brauer-Thrall conjecture to prove that the set of algebras of finite representation type is open in $\mathrm{Alg}(r)$.

4 Kac's Theorem

4.1 The fundamental set

Let Q be a finite quiver.

The *Ringel form* is the bilinear form on $\mathbb{Z}^{Q_0}$ given by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

The associated quadratic form is $q(\alpha) = \langle \alpha, \alpha \rangle$.

The associated symmetric bilinear form is $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

We write $\epsilon[i]$ for the ith coordinate vector.

Definition. The *fundamental set* F is the set of non-zero $\alpha \in \mathbb{N}^{Q_0}$ with $\mathrm{Supp}(\alpha)$ connected and $(\alpha, \epsilon[i]) \leq 0$ for all vertices i.

We define F' to be the set of non-zero $\alpha \in \mathbb{N}^{Q_0}$ such that $q(\alpha) < q(\beta^{(1)}) + \cdots + q(\beta^{(r)})$ whenever $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$ and $0 \neq \beta^{(i)} \in \mathbb{N}^{Q_0}$.

Lemma 1. If $\alpha \in F$ then either $\alpha \in F'$ or $\mathrm{Supp}(\alpha)$ is extended Dynkin and $q(\alpha) = 0$.

Proof. We may assume $Q = \mathrm{Supp}(\alpha)$, and so Q is connected. If the condition fails, then $\sum(\alpha - \beta^{(i)}, \beta^{(i)}) = (\alpha, \alpha) - \sum(\beta^{(i)}, \beta^{(i)}) \geq 0$, so there is $0 \leq \beta \leq \alpha$, with $\beta \neq 0, \alpha$ and with $(\alpha - \beta, \beta) \geq 0$. Now

$$0 \leq (\alpha - \beta, \beta) = \sum_i (\alpha, \epsilon[i]) \beta_i (\alpha_i - \beta_i)/\alpha_i + \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j (\frac{\beta_i}{\alpha_i} - \frac{\beta_j}{\alpha_j})^2$$

so $\frac{\beta_j}{\alpha_j} = \frac{\beta_j}{\alpha_j}$ whenever $(\epsilon[i], \epsilon[j]) < 0$, ie if an arrow connects i with j. Thus α is a multiple of β. Now the first sum implies that $(\alpha, \epsilon[i]) = 0$ for all i. This implies that Q is extended Dynkin.

Lemma 2. If $\alpha \in F'$, then $\mathrm{Ind}(KQ, \alpha)$ is a dense subset of $\mathrm{Mod}(KQ, \alpha)$.

Proof. If $\alpha = \beta + \gamma$ ($\beta, \gamma \neq 0$) then there is a map

$$\theta : \mathrm{GL}(\alpha) \times \mathrm{Mod}(KQ, \beta) \times \mathrm{Mod}(KQ, \gamma) \to \mathrm{Mod}(KQ, \alpha), \quad (g, x, y) \mapsto g(x \oplus y).$$

This map is constant on the orbits of a free action of $H = \mathrm{GL}(\beta) \times \mathrm{GL}(\gamma)$, so $\dim \overline{\mathrm{Im}(\theta)} \leq \dim \mathrm{LHS} - \dim H$. Now since $q(\alpha) = \dim \mathrm{GL}(\alpha) - \dim \mathrm{Mod}(KQ, \alpha)$ one deduces that

$$\dim \mathrm{Mod}(KQ, \alpha) - \dim \overline{\mathrm{Im}(\theta)} \geq q(\beta) + q(\gamma) - q(\alpha) > 0,$$

so $\overline{\mathrm{Im}(\theta)}$ is a proper subset of $\mathrm{Mod}(KQ,\alpha)$.

Notation. Let $\mathrm{End}(\alpha)=\bigoplus_{i\in Q_0} M_{\alpha_i}(K)$.

Suppose that $\underline{\lambda}=(\lambda[i])$ is a collection of partitions, one for each vertex, where $\lambda[i]$ is a partition of α_i. We say that $\theta\in\mathrm{End}(\alpha)$ is of type $\underline{\lambda}$ if the maps $\theta_i\in M_{\alpha_i}(K)$ are nilpotent of type $\lambda[i]$ (so that $\lambda[i]_r$ is the number of Jordan blocks of size $\geq r$).

The zero element of $\mathrm{End}(\alpha)$ is of type $\underline{z}$, with $z[i]$ the partition $(\alpha_i,0,\ldots)$.

We write $N_{\underline{\lambda}}$ for the set of $\theta\in\mathrm{End}(\alpha)$ of type $\underline{\lambda}$. It is a locally closed subset of $\mathrm{End}(\alpha)$.

If $\theta\in\mathrm{End}(\alpha)$ we define $\mathrm{Mod}_\theta=\{x\in\mathrm{Mod}(KQ,\alpha):\theta\in\mathrm{End}_{KQ}(K_x)\}$.

Lemma 3. (1) If $\theta\in N_{\underline{\lambda}}$ then $\dim\mathrm{Mod}_\theta=\sum_{a:i\to j}\sum_r\lambda[i]_r\lambda[j]_r$

(2) $\dim N_{\underline{\lambda}}=\dim\mathrm{GL}(\alpha)-\sum_{i\in Q_0}\sum_r\lambda[i]_r\lambda[i]_r$.

Proof. It is easy to check that if $f\in\mathrm{End}(V)$ and $g\in\mathrm{End}(W)$ are nilpotent endomorphisms of type μ and ν, then $\dim\{h:V\to W\mid gh=hf\}=\sum_r\mu_r\nu_r$. Part (1) follows immediately. For (2) note that $N_{\underline{\lambda}}$ is an orbit for the conjugation action of $\mathrm{GL}(\alpha)$ on $\mathrm{End}(\alpha)$, so if $\theta\in N_\lambda$ then

$$\dim N_{\underline{\lambda}}=\dim\mathrm{GL}(\alpha)-\dim\{g\in\mathrm{GL}(\alpha)\mid g\theta=\theta g\}$$

$$=\dim\mathrm{GL}(\alpha)-\dim\{g\in\mathrm{End}(\alpha)\mid g\theta=\theta g\}$$
$$=\dim\mathrm{GL}(\alpha)-\sum_i\sum_r\lambda[i]_r\lambda[i]_r.$$

Notation. Let $g=\dim\mathrm{GL}(\alpha)=\sum_{i\in Q_0}\alpha_i^2$. If $x\in\mathrm{Mod}(KQ,\alpha)$, then its orbit has dimension $g-\dim\mathrm{End}_{KQ}(K_x)$.

Let $I=\mathrm{Ind}(KQ,\alpha)=\bigcup_{s<g}I_{(s)}$. Recall that $I_{(s)}$ is locally closed in $\mathrm{Mod}(KQ,\alpha)$. Thus $I_{(g-1)}$ is the set of $x\in\mathrm{Rep}(\alpha)$ such that K_x is a brick (has 1-dimensional endomorphism algebra).

Lemma 4. If $\alpha\in F'$ and $s<g-1$ then $\dim_{\mathrm{GL}(\alpha)}I_{(s)}<1-q(\alpha)$.

Proof. Let N be the set of non-zero nilpotent $\theta\in\mathrm{End}(\alpha)$, so also the union $\bigcup_{\underline{\lambda}\neq\underline{z}}N_{\underline{\lambda}}$.

$MN=\{(x,\theta)\in\mathrm{Mod}(KQ,\alpha)\times N\mid\theta\in\mathrm{End}_{KQ}(K_x)\}=\bigcup_{\underline{\lambda}\neq\underline{z}}MN_{\underline{\lambda}}$.

$I_{(s)}N=\{(x,\theta)\in I_{(s)}\times N\mid\theta\in\mathrm{End}_{KQ}(K_x)\}\subseteq MN$.

We show that $\dim MN<g-q(\alpha)$. It suffices to prove that $\dim MN_{\underline{\lambda}}<g-q(\alpha)$ for all $\underline{\lambda}\neq\underline{z}$. Let $\pi:MN_{\underline{\lambda}}\to N_{\underline{\lambda}}$ be the projection. Now $\pi^{-1}(\theta)=$

Mod_θ is of constant dimension, so

$$\dim MN_{\underline{\lambda}} \leq \dim N_{\underline{\lambda}} + \dim \mathrm{Mod}_\theta = g - \sum_r q(\lambda_r) < g - q(\alpha),$$

since $\alpha = \sum_r \lambda_r$, and at least two λ_r are non-zero since $\underline{\lambda} \neq \underline{z}$. Here λ_r denotes the dimension vector whose components are the $\lambda[i]_r$.

Now suppose that $s < g-1$. If $x \in I_{(s)}$ then K_x is indecomposable and not a brick, so has a non-zero nilpotent endomorphism. Thus the projection $I_{(s)}N \xrightarrow{\pi} I_{(s)}$ is onto. Now

$$\dim \pi^{-1}(x) = \dim \mathrm{End}_{KQ}(K_x) \cap N = \dim \mathrm{Rad}\,\mathrm{End}_{KQ}(K_x) = g - s - 1.$$

Thus $\dim I_{(s)} = \dim I_{(s)}N - (g-s-1) \leq \dim MN - (g-s-1) < s+1-q(\alpha)$.

Lemma 5. For $\alpha \in F'$ the set $I_{(g-1)}$ of bricks is a non-empty open subset of $\mathrm{Mod}(KQ,\alpha)$.

Proof. It is the same as the set $\mathrm{Mod}(KQ,\alpha)_{(\geq g-1)}$, so it is open. Now I is dense and constructible in $\mathrm{Mod}(KQ,\alpha)$, so

$$\dim I = \dim \mathrm{Mod}(KQ,\alpha) = \sum_{a \in Q_1} \alpha_{h(a)}\alpha_{t(a)} = g - q(\alpha).$$

On the other hand, if $s < g-1$ we have

$$\dim I_{(s)} = \dim_G I_{(s)} + s \leq 1 - q(\alpha) + s < g - q(\alpha)$$

so $I_{(g-1)}$ must be non-empty.

Theorem. If $\alpha \in F$ then we have $\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(KQ,\alpha) = 1 - q(\alpha)$ and $\mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(KQ,\alpha) = 1$.

Proof. If $\alpha \in F'$ it follows from above, since bricks dominate. Otherwise we may assume that Q is extended Dynkin and use the classification.

4.2 The generating function of representations

Besides Kac's original papers, especially V. Kac, Root systems, representations of quivers and invariant theory. Invariant theory (Montecatini, 1982), 1983, in this section we cover material from J. Hua, Counting representations of quivers over finite fields, 2000. I also used notes of A. Hubery. [I use the conjugate partition to Hua, so some formulas look different.]

In this subsection we consider the representations of Q over a finite field $K = \mathbb{F}_q$. For notational simplicity we assume that $Q_0 = \{1, 2, \dots, n\}$, so dimension vectors are elements of $\mathbb{N}^n$.

Let $r(\alpha, q)$ be the number of isomorphism classes of representations of dimension vector α. Let $i(\alpha, q)$ be the number of isomorphism classes of indecomposable representations of dimension vector α.

We consider the generating function

$$\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha \in \mathbb{Z}[[X_1, \dots, X_n]]$$

where $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$.

Example. For the quiver consisting of a vertex 1 and no arrows, there is a unique representation of each dimension, so this is

$$1 + X_1 + X_1^2 + X_1^3 = 1/(1 - X_1)$$

For the quiver $1 \to 2$ a dimension vector is a pair (a, b) and the number of representations is $1 + \min(a, b)$. So the generating function is

$$\sum_{a,b \geq 0} (1 + \min(a, b)) X_1^a X_2^b$$

This is

$$\sum_{m \geq 0} (1 + m) X_1^m X_2^m + \sum_{m \geq 0, k > 0} (1 + m) X_1^{m+k} X_2^m + \sum_{m \geq 0, k > 0} (1 + m) X_1^m X_2^{m+k}$$

This works out as

$$\frac{1}{(1 - X_1 X_2)^2} + \sum_{k > 0} \frac{X_1^k}{(1 - X_1 X_2)^2} + \sum_{k > 0} \frac{X_2^k}{(1 - X_1 X_2)^2}$$

$$= \frac{1}{(1 - X_1 X_2)^2} \left(1 + \frac{X_1}{1 - X_1} + \frac{X_2}{1 - X_2} \right)$$

$$= \frac{1}{(1 - X_1)(1 - X_2)(1 - X_1 X_2)}.$$

Proposition. We have

$$\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha = \prod_{\beta \in \mathbb{N}^n} (1 - X^\beta)^{-i(\beta, q)}.$$

Proof. This follows from Krull-Remak-Schmidt, since if M_i $(i \in I)$ are a complete set of non-isomorphic indecomposable representations, we can write both sides as

$$\prod_{i\in I}(1 + X^{\underline{\dim} M_i} + X^{2\underline{\dim} M_i} + \dots).$$

Notation. Recall that $K = \mathbb{F}_q$. Let

$$X = \operatorname{Mod}(KQ, \alpha) = \prod_{a\in Q_1} M_{\alpha(h(a)\times\alpha(t(a)}(K)$$

and

$$G = \mathrm{GL}(\alpha) = \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i}(K).$$

Thus $r(\alpha, q) = |X/G|$. Recall that Burnside's Lemma says that if a group G acts on a finite set X, then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

where X^g is the fixed points of g on X. Thus

$$|X/G| = \sum_{g\in G/\sim} \frac{|X^g|}{|C_G(g)|}$$

where the sum is over conjugacy classes and $C_G(g)$ is the centraliser of g in G.

Lemma 1. The conjugacy classes in G are in 1-1 correspondence with collections (M_i) of $K[T, T^{-1}]$-modules, with M_i of dimension α_i, up to isomorphism. For g in the corresponding conjugacy class, one has

$$X^g \cong \bigoplus_{a\in Q_1} \operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)}, M_{h(a)})$$

and

$$C_G(g) \cong \prod_{i\in Q_0} \operatorname{Aut}_{K[T,T^{-1}]}(M_i).$$

Proof. An element of $\mathrm{GL}_d(K)$ turns K^d into a $K[T, T^{-1}]$-module, and conjugate elements correspond to isomorphic modules. The rest follows.

Notation. Recall that the finite-dimensional indecomposable $K[T, T^{-1}]$-modules are the modules $K[T, T^{-1}]/(f^r)$ where $r \geq 1$ and f runs through the set Φ' of monic irreducible polynomials in $K[T]$, excluding the polynomial T. We write C_f for the full subcategory consisting of the direct sums of copies modules of the form $K[T,T^{-1}]/(f^r)$ with $r \geq 1$. Given a partition λ we define

$$M_f(\lambda) = \bigoplus_{i\geq 1} \left(K[T,T^{-1}]/(f^r)\right)^{\oplus \lambda_i - \lambda_{i+1}}$$

so the number of copies of $K[T,T^{-1}]/(f^r)$ is the number of columns of length r in the Young diagram for λ. These modules parameterize the isomorphism classes in C_f.

Lemma 2.
(i) $\dim M_f(\lambda) = d\,|\lambda|$ where $|\lambda| = \lambda_1 + \lambda_2 + \dots$ is the weight of λ and d is the degree of f.
(ii) We have

$$\dim \operatorname{Hom}(M_f(\lambda), M_g(\mu)) = \begin{cases} 0 & (f \neq g) \\ d\langle \lambda, \mu\rangle & (f = g) \end{cases}$$

where by definition $\langle \lambda, \mu \rangle = \sum_i \lambda_i \mu_i$.
(iii) $|\operatorname{Aut}(M_f(\lambda))| = q^{d\langle\lambda,\lambda\rangle} b_\lambda(q^{-d})$, where $b_\lambda(T) = \prod_{i\geq 1}\prod_{j=1}^{\lambda_i-\lambda_{i+1}}(1-T^j)$.

Proof. (iii) For all $i > 0$, the module $M_f(\lambda)$ has $\lambda_i - \lambda_{i+1}$ copies of the indecomposable module $K[T,T^{-1}]/(f^i)$ of length i. Thus

$$\operatorname{End}(M_f(\lambda))/\operatorname{Rad}\operatorname{End}(M_f(\lambda)) \cong \prod_i M_{\lambda_i-\lambda_{i+1}}(\mathbb{F}_{q^d}).$$

Thus

$$\dim \operatorname{Rad}\operatorname{End}(M_f(\lambda)) = d\left(\langle\lambda,\lambda\rangle - \sum_i (\lambda_i - \lambda_{i+1})^2\right).$$

Then

$$|\operatorname{Aut}(M_f(\lambda))| = |\operatorname{Rad}\operatorname{End}(M_f(\lambda)|.\prod_i |\operatorname{GL}_{\lambda_i-\lambda_{i+1}}(\mathbb{F}_{q^d})|$$

and $|\operatorname{GL}_n(\mathbb{F}_q)| = (q^n-1)(q^n-q)(q^n-q^2)\dots(q^n-q^{n-1})$.

Theorem (Kac-Stanley-Hua). We have

$$\sum_{\alpha\in\mathbb{N}^n} r(\alpha,q)X^\alpha = \prod_{d=1}^{\infty} P(X_1^d,\dots,X_n^d,q^d)^{\phi'_d(q)}$$

where $\phi'_d(q)$ is the number of polynomials in Φ' of degree d, so the number of monic irreducible polynomials in $K[T]$ of degree d, excluding T, and

$$P(X_1,\dots,X_n,q) = \sum_{\underline{\lambda}} \frac{\prod_{a\in Q_1} q^{\langle\lambda[t(a)],\lambda[h(a)]\rangle}}{\prod_{i\in Q_0} q^{\langle\lambda[i],\lambda[i]\rangle} b_\lambda(q^{-1})} X_1^{|\lambda[1]|}\dots X_n^{|\lambda[n]|}$$

where the sum is over collections of partitions $\underline{\lambda} = (\lambda[1], \dots, \lambda[n])$.

Proof. Burnside's Lemma and Lemma 1 give

$$r(\alpha, q) = \sum_{(M_i)} \frac{\prod_{a \in Q_1} |\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)}, M_{h(a)})|}{\prod_{i \in Q_0} |\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|}$$

where the sum is over collections (M_i) of dimension α up to isomorphism. Thus the generating function is

$$\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha = \sum_{(M_i)} \frac{\prod_{a \in Q_1} |\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)}, M_{h(a)})|}{\prod_{i \in Q_0} |\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|} X_1^{\dim M_1} \dots X_n^{\dim M_n}$$

where the sum is over all collections (M_i) of $K[T, T^{-1}]$-modules, up to isomorphism.

Since every $K[T, T^{-1}]$-module can be written uniquely as a direct sum of modules in C_f $f \in \Phi'$) and there are no non-zero maps between the different C_f we obtain

$$\sum_\alpha r(\alpha, q) X^\alpha = \prod_{f \in \Phi} P_f$$

where

$$P_f = \sum_{(M_i) \in C_f} \frac{\prod_{a \in Q_1} |\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)}, M_{h(a)})|}{\prod_{i \in Q_0} |\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|} X_1^{\dim M_1} \dots X_n^{\dim M_n}$$

where the sum is over all collections (M_i) in C_f, up to isomorphism. Now by Lemma 2, if $f \in \Phi$ is of degree d, then $P_f = P(X_1^d, \dots, X_n^d, q^d)$.

Notation. The power series $P(X_1, \dots, X_n, q) \in \mathbb{Q}(q)[[X_1, \dots, X_n]]$ has constant term 1, so there are $h(\alpha, q) \in \mathbb{Q}(q)$ with

$$\log P(X_1, \dots, X_n, q) = \sum_{\alpha \in \mathbb{N}^n} \frac{h(\alpha, q)}{\overline{\alpha}} X^\alpha$$

where $\overline{\alpha}$ is the highest common factor of the coefficients of α.

Corollary 1. Letting $e(\alpha, q) = \sum_{d | \overline{\alpha}} d \phi'_d(q) h(\alpha/d, q^d)$, we have

$$\log(\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha) = \sum_{\alpha \in \mathbb{N}^n} \frac{e(\alpha, q)}{\overline{\alpha}} X^\alpha$$

and

$$e(\alpha, q) = \sum_{d | \overline{\alpha}} \frac{\overline{\alpha}}{d} i(\alpha/d, q), \quad i(\alpha, q) = \frac{1}{\overline{\alpha}} \sum_{d | \overline{\alpha}} \mu(d) e(\alpha/d, q).$$

Proof. Observe that

$$\log P(X_1^d, \dots, X_n^d, q^d) = \sum_{\alpha \in \mathbb{N}^n} \frac{h(\alpha, q^d)}{\overline{\alpha}} X^{d\alpha}$$

so the theorem gives the first part. Then by the proposition

$$\sum_{\alpha \in \mathbb{N}^n} \frac{e(\alpha, q)}{\overline{\alpha}} X^\alpha = \log(\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha) = \sum_{\beta \in \mathbb{N}^n} i(\beta, q) \log \frac{1}{1 - X^\beta}$$

$$= \sum_{\beta \in \mathbb{N}^n} \sum_{d=1}^{\infty} \frac{i(\beta, q)}{d} X^{d\beta}.$$

Comparing coefficients of X^α gives one equality. The other follows by Möbius inversion.

Lemma 3. $\phi_n'(q) \in \mathbb{Q}[q]$.

Proof. Any monic irreducible polynomial in $\mathbb{F}_q[T]$ of degree d corresponds to d elements which lie in $\mathbb{F}_{q^d}$ but not in any intermediate field between $\mathbb{F}_q$ and $\mathbb{F}_{q^d}$. Thus if the are $\phi_d(q)$ such polynomials, then

$$q^n = \sum_{d|n} d\phi_d(q).$$

By induction on d, or Möbius inversion

$$\phi_d(q) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}$$

one deduces that $\phi_d(q) \in \mathbb{Q}[q]$. Then also $\phi_d'(q) \in \mathbb{Q}[q]$ since

$$\phi_d'(q) = \begin{cases} q - 1 & (d = 1) \\ \phi_d(q) & (d > 1) \end{cases}$$

(or $\phi_d'(q) = \frac{1}{n} \sum_{d|n} \mu(d)(q^{n/d} - 1)$).

Corollary 2. $i(\alpha, q)$ and $r(\alpha, q) \in \mathbb{Q}[q]$, and are independent of the orientation of Q.

Proof. Corollary 1 shows that $i(\alpha, q) \in \mathbb{Q}(q)$.

It takes integer values for q any prime power, so it must be a polynomial. (Note that you cannot deduce that it is in $\mathbb{Z}[q]$, for example $\frac{1}{2}q(q + 1)$.)

It is independent of the orientation since $P(X_1, \dots, X_n, q)$ only involves an arrow a through the bracket $\langle \lambda[t(a)], \lambda[h(a)] \rangle$, and this bracket is symmetric.

By Corollary 1 we then have $e(\alpha, q) \in \mathbb{Q}[q]$ and then $r(\alpha, q) \in \mathbb{Q}[q]$ since

$$\sum_{\alpha} r(\alpha, q) X^{\alpha} = \exp\left(\sum_{\alpha \in \mathbb{N}^n} \frac{e(\alpha, q)}{\overline{\alpha}} X^{\alpha} \right)$$

4.3 Field extensions

Let L/K be a field extension. We consider the relationship between representations of Q over K and over L.

More generally we consider a K-algebra A and $A^L = A \otimes L$. (Unadorned tensor products are over K.) Since L is commutative, A^L-modules can be thought of as A-L-bimodules (with K acting centrally).

Any finite-dimensional A-module M gives a finite-dimensional A^L-module $M^L = M \otimes L$.

Lemma 1. We have $\mathrm{Hom}_{A^L}(M^L, (M')^L) \cong \mathrm{Hom}_A(M, M') \otimes L$. Moreover $\mathrm{top}\, \mathrm{End}_{A^L}(M^L) \cong \mathrm{top}\left((\mathrm{top}\, \mathrm{End}_A(M))^L\right)$.

Proof. We use that M is finite dimensional. There is a natural map

$$\mathrm{Hom}_A(M, M') \otimes L \to \mathrm{Hom}_{A^L}(M^L, (M')^L)$$

which is easily seen to be injective. We need to show it is onto. Say $\theta \in \mathrm{Hom}_{A^L}(M^L, (M')^L)$. Choose a basis ξ_i of L over K. Define θ_i by $\theta(m \otimes 1) = \sum_i \theta_i(m) \otimes \xi_i$. Clearly $\theta_i \in \mathrm{Hom}_A(M, M')$ and since M is finite-dimensional, only finitely many θ_i are non-zero. Then θ is the image of the element $\sum_i \theta_i \otimes \xi_i$.

For the last part we just observe that $(\mathrm{Rad}\, \mathrm{End}_A(M)) \otimes L$ is a nilpotent ideal in $\mathrm{End}_A(M) \otimes L \cong \mathrm{End}_{A^L}(M^L)$.

Lemma 2. Assume L/K is finite of degree n. Any finite-dimensional A^L-module N gives a finite-dimensional A-module N_K by restriction. If M is an A-module then $(M^L)_K \cong M^n$. If M, M' are A-modules and $M^L \cong (M')^L$, then $M \cong M'$.

Proof. Clear. For the last part use Krull-Remak-Schmidt, since $M^n \cong (M')^n$.

Lemma 3. Assume L/K is a finite separable extension. Then $\mathrm{top}\, \mathrm{End}(M^L) \cong (\mathrm{top}\, \mathrm{End}(M))^L$. If N is an A^L-module, then N is a direct summand of $(N_K)^L$. Any indecomposable A^L-module N arises as a direct summand summand of an induced module M^L with M indecomposable. The module M is unique up to isomorphism.

Proof. The first part holds since, for a separable field extension, inducing up a semisimple K-algebra gives a semisimple L-algebra.

Since $L \otimes L$ is a semisimple algebra, the multiplication map $L \otimes L \to L$ is a split epimorphism of L-L-bimodules, so L is a direct summand of $L \otimes L$. It follows that if N is an A^L-module, then N is a direct summand of $(N_K)^L$.

If N arises as a summand of M^L and $(M')^L$ with M, M' indecomposable, then N_K is a summand of M^n and $(M')^n$. By Krull-Remak-Schmidt this implies $M \cong M'$.

Lemma 4. Assume L/K is Galois of degree n with group G. The map

$$L \otimes L \to \bigoplus_{g \in G} L, \quad a \otimes b \mapsto (ag(b))_g$$

is an isomorphism as K-algebras, and gives an isomorphism of L-L-bimodules $L \otimes L \cong \bigoplus_{g \in G} L_g$, where the L-action on the right is given by restriction via g.

Example. $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

Proof. I am indebted to Andrew Hubery for his help in many places in these notes, and especially with this lemma. The map is a map of K-algebras, and also a bimodule map for the indicated action. Thus we need it to be a bijection.

By the theorem of the primitive element we can write $L \cong K[x]/(f(x))$ with $f(x)$ irreducible over K. Let x correspond to an element $\alpha \in L$. Since G acts faithfully on L and α generates L over K, the elements $g(\alpha)$ are distinct, and in $L[x]$ we can factorize $f(x) = \prod_{g \in G}(x - g(\alpha))$.

Now we can identify $L \otimes L \cong L \otimes K[x]/(f(x)) \cong L[x]/(f(x))$, and the map sends elements of L (identified with $L \otimes 1$) to themselves, and x (identified with $1 \otimes \alpha$) to $(g(\alpha))_g$, so it sends any polynomial $p(x) \in L[x]$ to $(p(g(\alpha)))_g$. Thus if $p(x)$ is sent to zero, then $p(g(\alpha)) = 0$ for all $g \in G$. Thus $p(x)$ is divisible by $f(x)$. Thus $p(x) = 0$ in $L \otimes L$. Thus the map is injective, hence by dimensions a bijection.

Theorem. Suppose L/K is Galois with group G.

Then induction and restriction give a 1-1 correspondence between isomorphism classes of

- indecomposable A-modules M, and

- G-orbits of indecomposable A^L-modules.

Explicitly if M is an indecomposable A-module then the indecomposable summands of M^L form an orbit under G, perhaps occuring with multiplicity, and if N is an indecomposable A^L-module, then $N_K \cong M^r$ for some indecomposable A-module M and some r, and the modules in the orbit of N give the same module M.

Example. For the field extension $\mathbb{C}/\mathbb{R}$:

A	A^L	indec A-mods	G-orbits of indec A^L-mods
$A = \mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\{\mathbb{C}\}$
$A = \mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	$\{\mathbb{C}_1, \mathbb{C}_2\}$
$A = \mathbb{H}$	$M_2(\mathbb{C})$	$\mathbb{H}$	$\{\mathbb{C}^2\}$

Proof. The key formula is that if N is an A^L-module, then

$$(N_K)^L \cong N \otimes_L (L \otimes_K L) \cong N \otimes_L (\bigoplus_{g \in G} L_g) \cong \bigoplus_{g \in G} N_g.$$

where N_g is the A^L-module obtain from N with the L-action given by restriction via g.

Induction. If N is one of the summands of M^L, then N_K is a summand of $(M^L)_K \cong M^n$, so $N_K \cong M^r$, some r. Then $(M^L)^r \cong (N_K)^L \cong \bigoplus_{g \in G} N_g$.

Restriction. If $N_K = \bigoplus_i M_i$, then $\bigoplus_i M_i^L \cong (N_K)^L \cong \bigoplus_{g \in G} N_g$. Thus

$$M_i^L \cong \bigoplus_{g \in S_i} N_g$$

where the S_i are a partition of G. Then

$$M_i^n \cong (M_i^L)_K \cong N_K^{|S_i|}.$$

Thus N_K is isomorphic to a direct sum of copies of M_i, so all the summands M_i are isomorphic, say to M, and the sets S_i all have the same size s with $s|n$. Then $M^{n/s} \cong N_K$.

Definition. We say that an A-module M is *absolutely indecomposable* if M^L is an indecomposable A^L-module for any field extension L/K.

If $\operatorname{top} \operatorname{End}(M) \cong K$ then M is absolutely indecomposable. If the base field is finite, then the converse holds, for $\operatorname{top} \operatorname{End}(M)$ is necessarily a field L, and the extension L/K is necessarily Galois. Then as an algebra $L \otimes L \cong L \times \cdots \times L$ ($\dim L$ copies), showing that M^L has $\dim L$ indecomposable summands.

Corollary 1. Suppose that A is an algebra over $K = \mathbb{F}_q$. Consider the field extension L/K where $L = \mathbb{F}_{q^n}$, and let $s|n$. Then induction and restriction give a 1-1 correspondence between isomorphism classes of

- indecomposable A-modules M with $\operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$, and

- G-orbits of size s of absolutely indecomposable A^L-modules.

Explicitly M^L is the direct sum of one copy of each of the modules in the orbit, and if N is in the orbit then $N_K \cong M^{n/s}$.

Proof. If $\dim \operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$ then $\operatorname{top}\operatorname{End}(M) \otimes_K \mathbb{F}_{q^s} \cong (\mathbb{F}_{q^s})^s$, so $\operatorname{top}\operatorname{End}(M) \otimes_K L \cong L^s$, so M^L splits as a direct sum of s non-isomorphic indecomposables with $\operatorname{top}\operatorname{End}(N) \cong L$.

Conversely if N comes from an orbit of size s of absolutely indecomposables, then $N_K \cong M^r$ for some indecomposable A-module M and some r. Now $(M^L)^r \cong (N_K)^L \cong \bigoplus_{g\in G} N_g$. Suppose $\operatorname{top}\operatorname{End}(M) = D$. Since there are no finite division algebras, D is a field. Thus $\operatorname{top}\operatorname{End}(M^L) = D^L$ is commutative. Thus M^L consists of one copy of each indecomposable in the orbit of N, so $r = n/s$. Then also $D^L \cong L^s$. Thus $\dim D = s$, so $D \cong \mathbb{F}_{q^s}$.

We return to representations of quivers. We write $a(\alpha, q)$ for the number of absolutely indecomposable representations of Q of dimension α over $\mathbb{F}_q$.

Corollary 2. [Hua, Corollary 4.2]. We have

$$\sum_{d|\alpha} \frac{1}{d} i(\alpha/d, q) = \sum_{d|\alpha} \frac{1}{d} a(\alpha/d, q^d).$$

Proof. If M is an indecomposable representation of Q of dimension α, then for each vertex i, the vector space at i becomes a module for $\operatorname{End}(M)$. It follows that if $\operatorname{top}\operatorname{End}(M) = \mathbb{F}_{q^s}$, then $s|\overline{\alpha}$. Thus apply Corollary 1 with $n = \overline{\alpha}$.

Namely take n to be the hcf of components of α. An indecomposable of dimension α/r over $\mathbb{F}_q$ with $\operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$ contributes $1/r$ to the LHS for $d = r$. For any n divisible by s it corresponds to an orbit of size s of absolutely indecomposable reps over $\mathbb{F}_{q^n}$ of dimension α/rs. This contributes $1/r$ to the term $d = rs$ on the RHS.

Corollary 3. We have

$$i(\alpha, q) = \sum_{d|\alpha} \frac{1}{d} \sum_{r|d} \mu(\frac{d}{r}) a(\frac{\alpha}{d}, q^r), \quad a(\alpha, q) = \sum_{d|\alpha} \frac{1}{d} \sum_{r|d} \mu(r) i(\frac{\alpha}{d}, q^r)$$

where μ is the Möbius function.

Proof. The formula

$$\sum_{d|\alpha} \frac{1}{d} i(\alpha/d, q) = \sum_{d|\alpha} \frac{1}{d} a(\alpha/d, q^d)$$

can be written for $\alpha = n\beta$ with β coprime as follows (multiplying it by n)

$$h(n) = \sum_{d|n} \frac{n}{d} i(n\beta/d, q) = \sum_{d|n} \frac{n}{d} a(n\beta/d, q^d)$$

The first of these can be written as

$$\sum_{e|n} e i(e\beta, q).$$

Then by Möbius inversion

$$ni(n\beta, q) = \sum_{d'|n} \mu(\frac{n}{d'}) h(d')$$

$$= \sum_{d'|n} \mu(\frac{n}{d'}) \sum_{r|d'} \frac{d'}{r} a(d'\beta/r, q^r).$$

Now rewrite this as a sum over $r|d|n$ where $d/r = n/d'$, and it becomes

$$\sum_{d|n} \sum_{r|d} \mu(\frac{d}{r}) \frac{n}{d} a(n\beta/d, q^r).$$

Giving the first formula. The second formula follows by another Möbius inversion.

4.4 Kac's Theorem

Let X be a variety over an algebraically closed field K, and let k be a subfield of K. There is the notion of X being *defined over* k.

For example if X is a (quasi) affine or projective variety in $\mathbb{A}^n$ or $\mathbb{P}^n$ this means that X can be defined using polynomials with coefficients in k.

(Equivalently X is isomorphic to $(Y^K)_{\mathrm{red}}$ for some reduced algebraic k-scheme Y.)

Definition. The Zeta function of a variety X defined over $\mathbb{F}_q$ is

$$Z(X;t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})|.t^r/r \right) \in \mathbb{Q}[[t]].$$

Example. $Z(\mathbb{A}^n;t) = \exp(\sum q^{rn}t^r/r) = \exp\log 1/(1-q^n t) = 1/(1-q^n t)$.

$Z(\mathbb{P}^1;t) = \exp(\sum(q^r+1)t^r/r) = 1/(1-qt)(1-t)$.

Weil conjectures 1949. If X is a smooth projective variety of dimension n then

Rationality: $Z(X;t)$ is a rational function of t.

Functional equation: $Z(X;1/q^n t) = \pm q^{nE/2}t^E Z(X;t)$ for suitable E.

Analogue of Riemann hypothesis:

$$Z(X;t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}$$

where $P_0(t) = 1-t$, $P_{2n}(t) = 1-q^n t$ and the other $P_i(t) \in \mathbb{Z}[t]$ and have roots which are algebraic integers with absolute value $q^{i/2}$.

Theorem of Dwork 1960. Rationality holds for any X defined over $\mathbb{F}_q$ (not necessarily smooth or projective).

Later work of Grothendieck and Deligne gives the rest of the Weil conjectures, and much more.

Proposition. If X is a variety defined over $\mathbb{F}_q$, and $|X(\mathbb{F}_{q^r})| = P(q^r)$ for some $P(t) \in \mathbb{Q}(t)$ then $P(t) \in \mathbb{Z}[t]$.

Proof. As argued before, since $P(q^r) \in \mathbb{Z}$ for all r, we must have $P(t) \in \mathbb{Q}[t]$. Say $P(t) = \sum_i a_i t^i$. Then

$$Z(X;t) = \exp(\sum_r \sum_i a_i q^{ri}t^r/r) = \prod_i \frac{1}{(1-q^i t)^{a_i}}$$

Since this is a rational function, $a_i \in \mathbb{Z}$.

Theorem 1. $a(\alpha,q) \in \mathbb{Z}[q]$.

Proof. Let K be an algebraically closed field. Recall that we have an action of $G = \mathrm{GL}(\alpha)$ on the variety $M = \mathrm{Mod}(KQ,\alpha)$ and its constructible subset $I = \mathrm{Ind}(KQ,\alpha)$.

These are defined over the prime subfield of K, and for any subfield k of K we have that $G(k) = \mathrm{GL}(\alpha)(k)$, $M(k) = \mathrm{Mod}(KQ,\alpha)(k)$ and $I(k)$ is the absolutely indecomposable representations of Q over k.

We would like to apply the proposition to I/G, but this is not a variety. Kac quotes a theorem of Rosenlicht. We would like to avoid this complication.

We consider have $I = \bigcup_s I_{(s)}$ and set

$$I_{(s)}G = \{(x,g) \in I_{(s)} \times G : gx = x\}.$$

This is a locally closed subset of $M_{(s)} \times G$. Let X be the disconnected union of these as s varies. This makes sense for any K, in particular for characteritic p, and it is defined over the prime field, and

$$|X(\mathbb{F}_q)| = |G(\mathbb{F}_q)|.a(\alpha, q).$$

Now $|G(\mathbb{F}_q)| \in \mathbb{Z}[q]$ and it is monic (for example $|\operatorname{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$). By the generating function, this is in $a(\alpha, q) \in \mathbb{Q}[q]$. Thus $|X(\mathbb{F}_q)| \in \mathbb{Q}[q]$. Thus by the proposition it is in $\mathbb{Z}[q]$. But then $a(\alpha, q) \in \mathbb{Z}[q]$ by Gauss's Lemma.

Theorem of Lang-Weil 1954. There is a constant $A(n, k, d)$ depending only on n, k, d, such that if X is an irreducible closed subvariety of projective space $\mathbb{P}^n$ of degree k and dimension d, defined over $\mathbb{F}_q$, then

$$||X(\mathbb{F}_q)| - q^d| \leq (k-1)(k-2)q^{d-\frac{1}{2}} + A(n, k, d)q^{d-1}.$$

The degree of a projective variety is defined using the Hilbert series of its coordinate ring. They remark that for curves, this is equivalent to the Riemann Hypothesis for function fields.

Corollary. Suppose X is a variety which is defined over a finite field. Then

$$|X(\mathbb{F}_q)| \sim tq^d$$

where $d = \dim X$ and $t = \operatorname{top} X$, meaning that for all $\epsilon > 0$ there is some finite field $\mathbb{F}_{q_0}$ over which X is defined, such that

$$1 - \epsilon < \frac{|X(\mathbb{F}_q)|}{tq^d} < 1 + \epsilon$$

for all $\mathbb{F}_q$ containing $\mathbb{F}_{q_0}$.

Sketch. One proves this by induction on the dimension.

It is true for irreducible projective varieties. It follows for all projective varieties. Note that the irreducible components of X are defined over a (possibly larger) finite field.

Any irreducible affine variety X can be embedded in projective space, and then we know the result for it's closure $\overline{X}$ and for the complement $\overline{X} \setminus X$.

Now any irreducible variety is the union of an affine open and a variety of smaller dimension. Then get it for all varieties.

Theorem 2. For any algebraically closed field K, $\dim_{\operatorname{GL}(\alpha)} \operatorname{Ind}(Q, \alpha)$ is the degree of $a(\alpha, q)$ and $\operatorname{top}_{\operatorname{GL}(\alpha)} \operatorname{Ind}(Q, \alpha)$ is its leading coefficient.

Sketch.. For K of characteristic p this follows from Lang-Weil applied to the disconnected union X of the $I_{(s)}G$.

For K of characteristic 0, one needs to argue that X comes from a scheme over $\mathbb{Z}$, and that the behavour over 0 is the same as the behavoiur over large primes p.

Roots. Let Q be a quiver. There is an associated set of roots in $\mathbb{Z}^{Q_0}$.

The *simple roots* are the coordinate vectors $\epsilon[i] \in \mathbb{Z}^{Q_0}$ with i a loopfree vertex. Thus $q(\epsilon[i]) = 1$. Observe that $s_i(\epsilon[i]) = -\epsilon[i]$. The corresponding *reflection* $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ is defined by

$$s_i(\alpha) = \alpha - (\alpha, \epsilon[i])\epsilon[i].$$

The *Weyl group* W is the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the s_i.

A *real root* is a vector in $\mathbb{Z}^{Q_0}$ in the orbit of a simple root. An *imaginary root* is a vector in the orbit of $\pm\alpha$ with α in the fundamental region (and non-zero).

Clearly α is a root iff $-\alpha$ is root. A root is *positive* if all components are ≥ 0, negative if all are ≤ 0. In fact every root is positive or negative.

Reflection Functors. Let i be a sink in Q and let Q' be the quiver obtained by reversing all arrows incident at i. There there is a bijection between isomorphism classes

Indecomposables of Q except $S_i \leftrightarrow$ Indecomposables of Q' except S_i

It acts on dimension vectors as s_i.

Theorem 3. $i(\alpha, q)$, $a(\alpha, q)$, $r(\alpha, q)$ are invariant under reflections.

Kac's Theorem. Suppose K is an algebraically closed field. $\mathrm{Ind}(Q, \alpha)$ is non-empty if and only if α is a positive root. If so, then $\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q, \alpha) = 1 - q(\alpha)$ and $\mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q, \alpha) = 1$. Equivalently $a(\alpha, q) \in \mathbb{Z}[q]$ is nonzero if and only if α is a positive root, and if so, it is monic of degree $1 - q(\alpha)$.

Proof. The equivalence holds by Theorem 2. By Theorem 3 we can replace α by anything in its Weyl group orbit.

If α is a root, we can assume it is a simple root, or in the fundamental region. For a simple root it is clear that $a(\alpha, q) = 1$. For the fundamental region we have the theorem in section 4.1.

If α is not a root, we can reflect until it either has positive or negative components, or it has disconnected support. Either way, it is clear that $a(\alpha, q) = 0$.

5 More about group actions and quotients

5.1 Representations of algebraic groups

Again K is an algebraically closed field. Let G be a linear algebraic group.

Definition. A KG-module is *rational* provided that any finite-dimensional subspace is contained in a finite dimensional submodule U such that the corresponding representation $G \to \mathrm{GL}(U)$ is a morphism of algebraic groups.

Theorem 1. Any submodule or quotient of a rational KG-module is rational.

Proof. Suppose W is a submodule and U is a finite-dimensional submodule as in the definition. Then $U \cap W$ is a submodule of U and the representation takes block triuangular

$$R(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & D(g) \end{pmatrix}$$

with $A(g) \in \mathrm{GL}(U \cap W)$ and $D(g) \in \mathrm{GL}(U/(W \cap U))$. Now if R is a map of algebraic groups, so are A and D.

Definition. A *coalgebra* C is a vector space C equipped with a comultiplication $\mu : C \to C \otimes C$ and a counit $\epsilon : C \to K$ satisfying coassociativity and counitality axioms.

$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$, $(\epsilon \otimes 1)\mu = 1$, $(1 \otimes \epsilon)\mu = 1$.

A C-*comodule* is a vector space V equipped with a coaction $\rho : V \to V \otimes C$ such that

$(1 \otimes \mu)\rho = (\rho \otimes 1)\rho$, $(1 \otimes \epsilon)\rho = 1$.

There is a category of comodules with suitably defined morphisms. Also subcomodules, etc.

Lemma. Any comodule is a union of finite-dimensional ones.

Proof. A sum of subcomodules is again a subcomodule, so it suffices to show that each $v \in V$ is contained in a finite-dimensional subcomodule. Let (c_i) be a basis of C. Write

$$\rho(v) = \sum v_i \otimes c_i$$

with all but finitely many of the v_i zero. Write $\mu(c_i) = \sum a_{ijk} c_j \otimes c_k$. Then

$$\sum_i \rho(v_i) \otimes c_i = (\rho \otimes 1)\rho(v) = (1 \otimes \mu)\rho(v) = \sum_{i,j,k} a_{ijk} v_i \otimes c_j \otimes c_k.$$

Comparing coefficients of c_k we get $\rho(v_k) = \sum_{i,j} a_{ijk} v_i \otimes c_j$, so the subspace spanned by the v_i is a subcomodule.

Recall that $A = \mathcal{O}(G)$ becomes a Hopf algebra. In particular it is a coalgebra.

Any $g \in G$ defines a map $ev_g : \mathcal{O}(G) \to K$ by $ev_g(f) = f(g)$.

We have $ev_{g_1 g_2}(f) = f(g_1 g_2) = (ev_{g_1} \otimes ev_{g_2})\mu(f)$.

Theorem 2. Any $\mathcal{O}(G)$-comodule V becomes a KG-module via

$$g.v = (1 \otimes ev_g)\rho(v).$$

This defines an equivalence from the category of $\mathcal{O}(G)$-comodules to the category of rational KG-modules.

Sketch. We have

$$\begin{aligned} g_1.(g_2.v) = g_1.(1 \otimes ev_{g_2})\rho(v) &= (1 \otimes ev_{g_1})\rho((1 \otimes ev_{g_2})\rho(v)) \\ &= (1 \otimes ev_{g_1})(\rho \otimes ev_{g_2})\rho(v) \\ &= (1 \otimes ev_{g_1})(1 \otimes 1 \otimes ev_{g_2}(\rho \otimes 1)\rho(v) \\ &= (1 \otimes ev_{g_1} \otimes ev_{g_2}(1 \otimes \mu)\rho(v) \\ &= (1 \otimes ev_{g_1 g_2})\rho(v) = (g_1 g_2).v. \end{aligned}$$

Similarly for $1.v$.

This clearly defines a faithful functor. If $x \in U \otimes \mathcal{O}(G)$ and $(1 \otimes ev_g)(x) = 0$ for all $g \in G$, then $x = 0$. Namely, write $x = \sum u_i \otimes f_i$ with the u_i linearly independent. Then $\sum f_i(g)u_i = 0$ for all g, so $f_i(g) = 0$, so $f_i = 0$. It follows that the functor is full. Namely, if V and V' are comodules and $\theta : V \to V'$ satisfies $g.\theta(v) = \theta(g.v)$ for all g, then

$$(1 \otimes ev_g)\rho(\theta(v)) = \theta((1 \otimes ev_g)(\rho(v)) = (\theta \otimes ev_g)(\rho(v))$$

so

$$(1 \otimes ev_g)(\rho(\theta(v) - (\theta \times 1)\rho(v))) = 0,$$

so $\rho(\theta(v) - (\theta \times 1)\rho(v)) = 0$, so θ is a comodule map.

To show that any comodule V is sent to a rational KG-module, by the lemma, we may suppose that V is finite dimensional. Take a basis of $e_1, \dots e_n$ of V.

Let $\rho(e_j) = \sum_i e_i \otimes f_{ij}$ for suitable $f_i j \in \mathcal{O}(G)$. Then the matrix (f_{ij}) corresponds to a morphism of varieties $\theta : G \to M_n(K)$.

Once checks easily that this is the representation given by V, so it actually goes into $\mathrm{GL}_n(K)$, and is a morphism of varieties.

Conversely if V is a rational KG-module, we want to show that it comes from some comodue structure on V, so we need to define $\rho : V \to V \otimes \mathcal{O}(G)$.

Given $v \in V$, choose a finite-dimensional submodule U containing V such that $G \to \mathrm{GL}(U)$ is a morphism. Take a basis $e_1, \dots, e_n$ of U. Then the map $G \to M_n(K)$ is a morphism, so given by a matrix of regular maps (f_{ij}). Then we define ρ on U by $\rho(e_j) = \sum_i e_i \otimes f_{ij}$.

Example. If G acts on an affine variety X then it acts as algebra automorphism on $\mathcal{O}(X)$. This turns $\mathcal{O}(X)$ into a rational G-module, because the action $G \times X \to X$ gives a coaction $\mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$.

Theorem 3. Any rational representation of the multiplicative group G_m is a direct sum of copies of the one-dimensional representations

$$\theta_n : G_m \to \mathrm{GL}_1(K), \quad \rho_n(\lambda) = \lambda^n \quad (n \in \mathbb{Z}).$$

In this way one gets an equivalence between the category of rational representations of G_m and the category of vector spaces V equipped with a direct sum decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$.

Proof. We have $\mathcal{O}(G_m) = K[T, T^{-1}]$ with $\mu(T^n) = T^n \otimes T^n$ and $\epsilon(T^n) = 1$.

Then θ_n corresponds to the comodule $K \to K \otimes \mathcal{O}(G_m)$, $1 \mapsto 1 \otimes T^n$.

If V is a comodule, let $V_n = \{v \in V : \rho(v) = v \otimes T^n\}$. This is a subcomodule isomorphic to a direct sum of copies of θ_n.

We show that $V = \bigoplus_{n \in Z} V_n$. The sum is clearly direct. Now if $v \in V$, write $\rho(v) = \sum_n v_n \otimes T^n$. Then

$$(1 \otimes \mu)\rho(v) = \sum_n v_n \otimes T^n \otimes T^n$$

and it also equals

$$(\rho \otimes 1)\rho(v) = \sum_n \rho(v_n) \otimes T^n$$

so $\rho(v_n) = v_n \otimes T^n$. Thus $v_n \in V_n$ and $v = (1 \otimes \epsilon)\rho(v) = \sum v_n$.

Conversely given a graded vector space, we make G_m act on V_n by $\lambda.v = \lambda^n v$.

Example. If R is a K-algebra, a grading of R is the same thing as rational action of G_m as algebra automorphisms of R. Let R_n is the subspace on which $\lambda \in G_m$ acts as λ^n.

If G_m acts as automorphisms then for $x \in R_n$, $y \in R_m$ we have $\lambda.(xy) = (\lambda.x)(\lambda.y) = (\lambda^n x)(\lambda^m y) = \lambda^{n+m}(xy)$ so $xy \in R_{n+m}$. Conversely if we have a grading, so $R_n R_m \subseteq R_{n+m}$ and we make G_m act via $\lambda.x = \lambda^n x$ for $x \in R_n$, then G_m acts as algebra automorphisms on R.

Thus the actions of G_m on an affine variety X correspond to the gradings of $\mathcal{O}(X)$.

5.2 Reductive groups

Let G be a linear algebraic group. If V acts on a set X we write X^G for the fixed points. If V is a G-module then V^G is a submodule. If G acts on an algebra R then R^G is a subalgebra.

Definition. G is *reductive* if its radical (its unique maximal connected normal solvable subgroup) is isomorphic to an algebraic torus $(G_m)^r$. (See Borel, Linear algebraic groups, §11.21).

Example. Classical groups like $\mathrm{GL}_n(K)$, $SL_n(K)$, $SO_n(K)$ are reductive. Products of reductive groups are reductive.

G is *linearly reductive* if any rational G-module is semisimple. It follows (and is in fact equivalent) that the functor $V \to V^G$ from rational G-modules to vector spaces is exact.

The multiplicative group G_m and more generally tori (products of copies of G_m) are linearly reductive. In characteristic zero, reductive groups are linearly reductive. (Weyl).

G is *geometrically reductive* if for any finite-dimensional rational G-module V and non-zero $v \in V^G$ there is a G-invariant homogeneous polynomial function $f : V \to K$ with $f(v) \neq 0$.

Linearly reductive implies geometrically reductive: Namely, consider the map $\mathrm{Hom}_K(V, K) \to \mathrm{Hom}_K(V^G, K)$. This is a surjective map of rational G-modules. Thus the map $\mathrm{Hom}_G(V, K) \to \mathrm{Hom}_K(V^G, K)$ is onto. Now there is a linear map $V^G \to K$ which doesn't kill v. Hence there is a G-module homomorphism $V \to K$ which doesn't kill v. This gives a G-invariant homogeneous polynomial of degree 1.

Theorem (Haboush/Nagata/Popov). Given G, the following are equiv.
- G is reductive
- G is geometrically reductive
- R^G is finitely generated for all finitely generated commutative K-algebras R with rational G-action.

Reynolds operator. If G is linearly reductive and V is a rational KG-

module, then $V = V^G \oplus W$ where W is a direct sum of non-trivial simple modules. The *Reynolds operator* is the unique KG-module map $E : V \to V$ which is the identity on V^G and zero on W. Thus $E^2 = E$ and $E(v) = v$ iff $v \in V^G$.

For characteristic $p > 0$ there is the following replacement: See M. Nagata, Invariants of a group in an affine ring, 1964, Lemma 5.1.B and 5.2.B. See also P. E. Newstead, Introduction to moduli problems and orbit spaces, Tata notes, 1978 Lemmas 3.4.1 and 3.4.2.

Nagata Lemmas. Suppose G is geometrically reductive acting on a commutative K-algebra R, as a rational KG-module.

(1) If I is a G-stable ideal in R then $I + r \in (R/I)^G$ implies $r^d \in I + R^G$ for some positive integer d.

(2) If $I = \sum_{i=1}^s R^G r_i$ is a finitely generated ideal in R^G and $r \in RI \cap R^G$, then $r^d \in I$ for some positive integer d.

If G is linearly reductive then both hold with $d = 1$. For example in (1), since the map $R \to R/I$ is surjective, linear reductivity gives that $R^G \to (R/I)^G$ is surjective.

Proof. (1) We may suppose $r \notin I$. Choose a finite dimensional rational submodule Y of R containing r. Let $X = Kr + (Y \cap I)$. Since $(I + r) \in (R/I)^G$ it follows that X is a G-submodule of R. Now $X/(Y \cap I)$ is a one-dimensional trivial KG-module so there is a KG-module map $\lambda : X \to K$ with $\lambda(r) = 1$ and $\lambda(Y \cap I) = 0$. Apply the geometric reductivity hypothesis to $\lambda \in (DX)^G$. Let $y_1, \dots, y_m$ be a basis of $Y \cap I$. Then $r, y_1, \dots, y_m$ is a basis for X. Polynomial functions $DX \to K$ are given by elements of $f \in K[r, y_1, \dots, y_m]$, where the evaluation at $\xi \in DX$ is given by applying ξ to each indeterminate. In particular $f(\lambda)$ is the sum of the coefficients of the powers of r. Now we have a G-invariant homogeneous f of degree d whose evaluation at λ is non-zero (so wlog 1). Thus $f = r^d +$ terms of lower degree in r. Now there is a natural map $p : K[r, y_1, \dots, y_m] \to R$ and it is G-equivariant. It sends each indeterminate to the corresponding element of R, and a polynomial to the corresponding linear combination of products. Then $p(f) \in R^G$ and $p(y_i) \in I$, giving result.

(2) We work by induction on s. For $s = 1$, let $r \in Rr_1 \cap R^G$. Then $r = r'r_1$ and $({}^g r' - r')r_1 = 0$. So by (1) applied to the ideal $J = \{h \in R : hr_1 = 0\}$, we obtain $r'' \in R^G$ and d with $(r'' - (r')^t)r_1 = 0$. Hence

$$r^t = (r')^t r_1^t = r'' r_1^t \in R^G r_1.$$

Now suppose $s > 1$. We write $\overline{R} = R/Rr_1$. If $r \in RI \cap R^G$ by induction we get a positive integer d with

$$(\overline{r})^d \in \sum_{i=1}^{s} \overline{R}^G \overline{r_i}.$$

Thus we can write

$$r^t = \sum_{i=1}^{s} h_i r_i$$

with $h_i \in R$ and $\overline{h_2}, \dots, \overline{h_s} \in \overline{R}^G$. Now by (1) applied to the ideal $J = Rr_1$ there is a positive integer d' and $h_s' \in R^G$ such that $\overline{h_s}^{d'} = \overline{h_s'}$. It follows that

$$r^{dd'} - h_s' r_s^{d'} \in (\sum_{i=1}^{s-1} Rr_i) \cap R^G.$$

Again by induction there is a positive integer d'' with

$$(r^{dd'} - h_s' r_s^{d'})^{d''} \in \sum_{i=1}^{s-1} R^G r_i).$$

Thus $r^{dd'd''} \in I$ as required.

5.3 Good quotients and affine quotients

Let an algebraic group G act on a variety X.

We don't try to turn X/G into a variety. Instead we use the set of closed orbits, which we denote $X \,/\!/\, G$.

Recall that each orbit closure $\overline{Gx}$ contains a closed orbit.

Good example. If A is a finitely generated algebra and α is dimension vector, then $\mathrm{GL}(\alpha)$ acts on an affine variety $\mathrm{Mod}(A, \alpha)$. The closed orbits are those of semisimple modules. Each orbit closure contains a unique closed orbit. The quotient $\mathrm{Mod}(A, \alpha) \,/\!/\, \mathrm{GL}(\alpha)$ classifies the semisimple modules of dimension vector α.

Bad example.

$$G = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix} : \lambda \in K, \mu \in K^* \right\} \subseteq \mathrm{GL}_2(K)$$

acting by conjugation on K^2. The orbits are $K \times K^*$ and $\{(x, 0)\}$. The closure of the first orbit contains all the others.

Definition. We say that an action of G on a variety X has a *good quotient* if

(1) For any $x \in X$, the orbit closure $\overline{Gx}$ contains a unique closed orbit.

Assuming this, we get a mapping $\phi : X \to X /\!/ G$, and we can turn $X /\!/ G$ into a space with functions:

Topology: $U \subseteq X /\!/ G$ is open iff $\phi^{-1}(U)$ is open in X.

Functions: $\mathcal{O}_{X /\!/ G}(U) = \mathcal{O}_X(\phi^{-1}(U))^G$.

Thus $\phi : X \to X /\!/ G$ is a morphism of spaces with functions.

(2) The space with functions $X /\!/ G$ is a variety.

(3) If W is a closed G-stable subset of X then $\phi(W)$ is closed in $X /\!/ G$. Equivalently $\{x \in X : \overline{Gx} \cap W \neq \emptyset\}$ is closed in X.

(4) We may also demand (Newstead, Geometric invariant theory, 2009, but not all others) that ϕ is an affine morphism, that is, $\phi^{-1}(U)$ is affine for any affine open subset U of $X /\!/ G$, or equivalently for the sets U in an affine open covering of $X /\!/ G$.

Proposition. If the action of G on X has a good quotient, then

(i) Disjoint closed G-stable subsets of X have disjoint images under ϕ.

(ii) ϕ is a good quotient in the sense of Newstead, 2009. (Conversely, any Good quotient in that sense arises this way).

(iii) ϕ is a categorical quotient of X by G.

(iv) If G acts on X with closed orbits, then $Y = X/G$ is a geometric quotient of X by G.

Proof. (i) If a closed orbit Gu is in the image of closed G-stable subsets Z and Z', then there must be z, z' with $\overline{Gz}$ and $\overline{Gz'}$ both containing Gu. But then $u \in Z \cap Z'$.

(ii) Trivial.

(iii) I think it is straightforward. Let $\psi : X \to Z$ be a morphism which is constant on G-orbits. If $\phi(x) = z$, then $\overline{Gx} \subseteq \psi^{-1}(z)$. It follows that $\psi = \chi\phi$ where $\chi : X /\!/ G \to Z$ sends a closed orbit Gu to $\psi(u)$. Now χ is a morphism by the definition of $X /\!/ G$ as a space with functions.

(iv) Clear.

Lemma. If a reductive group G acts on an affine variety X, and if W_1, W_2 are disjoint closed G-stable subsets of X, then there is a function $f \in \mathcal{O}(X)^G$

with $f(W_1) = 0$ and $f(W_2) = 1$.

Proof. First we find a function in $\mathcal{O}(X)$. If the ideals defining W_i are I_i then since the W_i are disjoint, $I_1 + I_2 = \mathcal{O}(X)$. Thus we can write $1 = f_1 + f_2$ with $f_i \in I_i$. Thus f_1 is zero on W_1 and 1 on W_2.

Let V be the KG-submodule of $\mathcal{O}(X)$ generated by f_1. It has basis $h_i = {}^{g_i}f_1$ for some elements $g_1, \dots, g_n \in G$. Consider the map $\alpha : X \to DV$, $x \mapsto (f \mapsto f(x))$. Then $\alpha(W_1) = 0$ and $\alpha(W_2) = \xi$ where ξ is the element with $\xi(h_i) = 1$ for all i.

Since G is geometrically reductive, there is an invariant homogeneous polynomial function $p : DV \to K$, so $p \in K[h_1, \dots, h_n]^G$, with $p(0) = 0$, $p(\xi) = 1$. Then $f = p\alpha$ has the required properties.

Theorem. A reductive group G acting on an affine variety X has a good quotient, and $X /\!/ G$ is the affine variety with coordinate ring $\mathcal{O}(X)^G$.

Proof. By Haboush and Nagata, the algebra $\mathcal{O}(X)^G$ is finitely generated. It also has no nilpotent elements, so defines a variety Y, and the inclusion give a morphism $\psi : X \to Y$.

First, ψ is constant on orbits, for if $\psi(gx) \neq \psi(x)$ then since Y is affine there is $f \in \mathcal{O}(Y)$ with $f(\psi(gx)) \neq f(\psi(x))$. But this contradicts that $f \in \mathcal{O}(X)^G$.

Next we show that ψ is onto. Let $y \in Y$ and let the maximal ideal in $\mathcal{O}(Y) = \mathcal{O}(X)^G$ corresponding to y be generated by $f_1, \dots, f_s$. Now Nagata's Lemma (2) implies that

$$\sum_i f_i \mathcal{O}(X) \neq \mathcal{O}(X).$$

Hence some maximal ideal of $\mathcal{O}(X)$ contains this ideal. Let x be the corresponding point of X. Then $f_i(x) = 0$ for all i. Thus $\psi(x) = y$.

Now if W_1, W_2 are disjoint closed G-stable subsets of X, then there is $f \in \mathcal{O}(X)^G$ with $f(W_1) = 0$ and $f(W_2) = 1$. Considering f as a map $Y \to K$ we see that $\psi(W_1)$ and $\psi(W_2)$ are disjoint.

It follows that every orbit closure contains a unique closed orbit, and the induced map ϕ, as a map of sets, coincides with ψ.

If W is closed G-stable then $\psi(W)$ is closed, for if $y \in \overline{\psi(W)} \setminus \psi(W)$, then $W_1 = W$ and $W_2 = \psi^{-1}(y)$ are disjoint G-stable closed sets, but there is no function $f \in \mathcal{O}(X)^G$ with $f(W_1) = 0$ and $f(W_2) = 1$.

It follows that the topology on Y coincides with that on $X /\!/ G$.

To identify Y with $X /\!/ G$ as a space with functions, we need to show that

$\mathcal{O}_{X\,/\!/\,G}(U) \cong \mathcal{O}_Y(U)$ for any open set U in Y. It suffices to do this for $U = D(f)$ with $f \in \mathcal{O}_Y(Y) = \mathcal{O}(X)^G$. Then the LHS is $\mathcal{O}(X)[f^{-1}]^G$ and the RHS is $\mathcal{O}(X)^G[f^{-1}]$, and these are isomorphic.

5.4 GIT (=Geometric invariant theory) quotients

First we need to know about $\operatorname{Proj} R$ for an commutative $\mathbb{N}$-graded ring, as defined, for example, in Hartshorne. Suppose R is a finitely generated K-algebra and reduced. I think we can understand $\operatorname{Proj} R$ as follows. There is an action of G_m on R with $g.r = g^n r$ for $r \in R_n$. Then G_m acts on the corresponding affine variety $Y = \operatorname{Spec} R$ and the set of fixed points Y^{G_m} is the zero set of the ideal $R_{>0}$, so is isomorphic to $\operatorname{Spec} R_0$. The complement is

$$Y' = \bigcup_{n>0} \bigcup_{f \in R_n} D(f)$$

This is a union of affine G_m-stable open subsets with good quotients. Moreover all orbits in Y' are closed. Thus Y' has a good quotient. It is a geometric quotient, Y'/G_m. This is $\operatorname{Proj} R$, I think. Moreover the morphism $Y' \to Y \to Y \,/\!/\, G_m = \operatorname{Spec} R_0$ induces a morphism $\operatorname{Proj} R \to \operatorname{Spec} R_0$ which is a projective morphism.

Let G be reductive. GIT is really for actions of G on projective varieties - you need to choose a linearization. Following King, Moduli of representations of finite-dimensional algebras, 1994, we consider an action of G on an affine variety X and choose a character $\chi \in \operatorname{Hom}(G, G_m)$.

We fix a closed subgroup Δ which acts trivially on X and we assume that $\chi(\Delta) = 1$.

We write χ^n for the nth power of this character. We write $\mathcal{O}(X)^{G,\chi^n}$ for the relative invariants of weight χ^n, that is, functions $f \in \mathcal{O}(X)$ with $f(g.x) = \chi(x)^n f(x)$ for all g, x.

There is a corresponding moduli space

$$X \,/\!/ (G,\chi) = \operatorname{Proj} R$$

where R is the $\mathbb{N}$-graded algebra

$$R = \bigoplus_{n=0}^{\infty} \mathcal{O}(X)^{G,\chi^n}.$$

It is finitely generated because of the following.

Lemma 1. Let G act on $X \times K$ via $g.(x,z) = (g.x, \chi(g)^{-1}z)$. Then the map

$$\bigoplus_{n\geq 0} \mathcal{O}(X)^{G,\chi^n} \to \mathcal{O}(X \times K)^G$$

sending $f \in \mathcal{O}(X)^{G,\chi^n}$ to the map $X \times K \to K$, $(x,z) \mapsto z^n f(x)$ is an isomorphism of algebras. Thus the ring $\bigoplus_{n\geq 0} \mathcal{O}(X)^{G,\chi^n}$ is the coordinate ring of $(X \times K) /\!/ G$, and the grading corresponds to the action of G_m induced from the action of G_m on $X \times K$ given by $t.(x,z) = (x, tz)$.

Proof. $\mathcal{O}(X \times K) \cong \mathcal{O}(X) \otimes \mathcal{O}(K) \cong \mathcal{O}(X) \otimes K[t]$. If f corresponds to $\sum_i f_n \otimes t^n$, then $f(x,z) = \sum f_n(x) z^n$, so

$$f(g.(x,z)) = f(g.x, \chi(g)^{-1}z) = \sum_n f_n(g.x) \left(\frac{z}{\chi(g)}\right)^n$$

so $f \in \mathcal{O}(X \times K)^G$ iff $f_n(g.x) = \chi(g)^n f_n(x)$, i.e., $f_n \in \mathcal{O}(X)^{G,\chi^n}$ for all n.

Definition. Recall that if $f \in \mathcal{O}(X)$ then $D(f) = \{x \in X : f(x) \neq 0\}$ is an affine open subset of X. If $f \in \mathcal{O}(X)^{G,\chi^n}$ then $D(f)$ is G-stable.
(i) A point $x \in X$ is χ-*semistable* if $x \in D(f)$ for some $f \in \mathcal{O}(X)^{G,\chi^n}$ with $n \geq 1$.
(ii) A point $x \in X$ is χ-*stable* if there is f as above, the G-action on $D(f)$ is closed, and $\dim \mathrm{Stab}_G(x) = \dim \Delta$.

Theorem.
(i) $X^{\chi-ss}$ and $X^{\chi-s}$ are G-stable open subsets of X.
(ii) Let $(x,z) \in X \times K$ with $z \neq 0$. Then x is χ-semistable iff in $X \times K$ we have $\overline{G(x,z)} \cap (X \times \{0\}) = \emptyset$.
(iii) The action of G on $X^{\chi-ss}$ has a good quotient, and $X^{\chi-ss} /\!/ G \cong X /\!/ (G,\chi)$.
(iv) The action of G on $X^{\chi-s}$ has a good geometric quotient.
(v) Let $(x,z) \in X \times K$ with $z \neq 0$. Then x is χ-stable iff $\dim \mathrm{Stab}_G(x,z) = \dim \Delta$ and $G(x,z)$ is closed in $X \times K$.
(vi) If $x \in X^{\chi-ss}$ the x is χ-stable iff $\dim \mathrm{Stab}_G(x) = \dim \Delta$ and Gx is closed in $X^{\chi-ss}$.

Proof. (i) Use that $D(f)$ is open and the dimension of the stabiliser of x is upper semicontinuous.

(ii) Use Lemma 1 and the lemma in the last section showing that disjoint closed G-stable subsets of the affine variety $X \times K$ are separated by an invariant function.

(iii) First we need to show that each orbit closure $\overline{Gx}$ in $X^{\chi-ss}$ contains a unique closed orbit. I didn't find a nice proof of this, so omit it. We will check it later for moduli spaces of modules.

Let $R = \bigoplus_{n\geq 0} \mathcal{O}(X)^{G,\chi^n}$ and let $Y = \operatorname{Spec} R$. As above, $Y^{G_m} = V(R_{>0})$, $Y' = Y \setminus Y^{G_m}$ and $\operatorname{Proj} R = Y'/G_m$.

The inclusion of R in $\mathcal{O}(X)$ gives a map $\sigma : X \to Y$. If $x \in X^{\chi-ss}$ then there is $f \in \mathcal{O}(X)^{G,\chi^n}$ with $f(x) \neq 0$. Then $f \in R_{>0}$ is a function on $\operatorname{Spec} R$ which is non-zero on $\sigma(x)$. Thus $\sigma(x) \notin V(R_{>0})$. Thus we get a map π

$$X^{\chi-ss} \to Y' \to Y'/G_m = \operatorname{Proj} R = X /\!/ (G, \chi).$$

To show that the quotient of $X^{\chi-ss}$ by G has a good quotient, and that it is π, it suffices to show that for U in an affine open covering of $\operatorname{Proj} R$, the open sets $\pi^{-1}(U)$ are affine, and that $U \cong \pi^{-1}(U) /\!/ G$.

An element $f \in R_n$ $(n > 0)$ defines an affine G_m-stable open set $D(f)$ of Y' and hence an affine open subset $U \subseteq \operatorname{Proj} R$. Now $\mathcal{O}(D(f)) \cong R[f^{-1}]$, so since the quotient by G_m is good,

$$\mathcal{O}(U) \cong R[f^{-1}]^{G_m} = R[f^{-1}]_0 \cong \mathcal{O}(X)[f^{-1}]^G.$$

On the other hand $\pi^{-1}(U) \cong D(f)$ so it has coordinate ring $\mathcal{O}(X)[f^{-1}]$.

(iv) We have a good quotient $\phi : X^{\chi-ss} \to Z$ where $Z = X /\!/ (G, \chi)$. Let $Z^s = \phi(X^{\chi-s})$. Now Z is a union of open affine sets Z_f. Let Z^0 be the union of the Z_f for which G acts on $D(f)$ with closed orbits. Clearly $X^{\chi-s} \subseteq \phi^{-1}(Z^0)$, and so $Z^s \subseteq Z^0$. Let $X^0 = \phi^{-1}(Z^0)$. Then $X^0 \to Z^0$ is a geometric quotient. It follows that $X^{\chi-s} = \phi^{-1}(Z^s)$ and $Z^0 \setminus Z^s = \phi(X^0 \setminus X^{\chi-s})$. Hence $Z^0 \setminus Z^s$ is closed in Z^0 by one of the properties of a geometric quotient. Thus Z^s is open in Z^0, and hence also in Z. It follows that $X^s \to Z^s$ is a geometric quotient.

(v) (cf. Newstead 2009, p105, Prop 2.1(ii).) Either condition implies that x is χ-semistable, so there is $f \in \mathcal{O}(X)^{G,\chi^n}$ with $f(x) \neq 0$. Let $\alpha = z^n f(x)$ and let $W = \{(x'z') \in X \times K : (z')^n f(x') = \alpha\}$. Consider the projection $p : W \to D(f)$. This is an affine map which is surjective with finite fibres. In fact a finite morphism. It follows that it is a closed map.

Suppose x is χ-stable. Then $\Delta \subseteq \operatorname{Stab}_G(x, z) \subseteq \operatorname{Stab}_G x$ so $\dim \operatorname{Stab}_G(x, z) = \dim \Delta$. Also Gx is closed in $D(f)$. Then $p^{-1}(Gx)$ is closed, and the union of a finite number of G-orbits. Since all have the same dimension these orbits are closed in W, and hence in $X \times K$.

Conversely suppose that $G(x, z)$ is closed and $\dim \operatorname{Stab}_G(x, z) = \dim \Delta$. Then $Gx = p(G(x, z))$ is closed in $D(f)$. Since this holds for all f with $x \in D(f)$, it follows that Gx is closed in $X^{\chi-ss}$.

(vi) (cf. Newstead 2009, proof of Thm 1.7(iv) on p113.) If x is χ-stable we need to show that Gx is closed in $X^{\chi-ss}$. Let $\phi : X^{\chi-ss} \to Z$ be the quotient map. Now $\phi^{-1}(\phi(x)) \subseteq \phi^{-1}(Z^s) = X^{\chi-s}$. Since $\phi^{-1}(\phi(x))$ is closed in $X^{\chi-ss}$, it follows that

$$\overline{Gx} \cap X^{\chi-ss} \subseteq X^{\chi-s}.$$

But G acts on $X^{\chi-s}$ with closed orbits, so Gx is closed in $X^{\chi-s}$, and therefore also in $X^{\chi-ss}$.

Conversely, if the hypotheses hold, we need to find f with $x \in D(f)$ and such that the action of G on $D(f)$ has closed orbits. We can find f with $x \in D(f)$. Then since stabilizer dimensions are upper semicontinuous, the set

$$T = \{x' \in D(f) : \dim \mathrm{Stab}_G(x') > \dim \Delta\}$$

is closed in $D(f)$. Hence Gx and T are disjoint closed G-stable subsets of the affine variety $D(f)$. Hence there is a G-invariant function f' on $D(f)$ with $f'(T) = 0$ and $f'(Gx) = 1$. Now $f' \in \mathcal{O}(X)[f^{-1}]^G$, so $f' = h/f^r$ for some $h \in \mathcal{O}(X)^{G,\chi^m}$, some m. Then $f'' = fh$ satisfies that $x \in D(f'') \subseteq D(f) \setminus T$ and all orbits of G on $D(f'')$ have the same dimension, so the action is closed.

Notation. If $a : G_m \to X$ is a morphism, we write $\lim_{t\to 0} a(t) = x$ if a extends to a morphism $a' : K \to X$ and $a'(0) = x$. If so, then the fact that X is separated implies that x is unique (for if a', a'' are extensions, then they define a morphism $(a', a'') : K \to X \times X$. Then the inverse image of the diagonal is closed and contains G_m, so it must be all of K).

Kempf's Fundamental Theorem. Let X be an affine variety with an action of G, a connected reductive group. Let x be a point of X; Let Gu be the closed orbit contained in $\overline{Gx}$. Then there is a 1-psg $\lambda \in \mathrm{Hom}(G_m, G)$ such that limit $\lim_{t\to 0} \lambda(t).x$ exists and is contained in Gu.

We have a pairing between characters and 1-psgs defined by $\langle \chi, \lambda \rangle = m$ where $\chi(\lambda(t)) = t^m$.

Theorem (Hilbert-Mumford numerical criterion). Let $(x, z) \in X \times K$ with $z \neq 0$. Then

(a) The following are equivalent:
(i) x is χ-semistable
(ii) for all 1-psgs $\lambda \in \mathrm{Hom}(G_m, G)$, if $\lim_{t\to 0} \lambda(t).(x, z)$ exists, it is not in $X \times \{0\}$.
(iii) for all 1-psgs $\lambda \in \mathrm{Hom}(G_m, G)$, if $\lim_{t\to 0} \lambda(t).x$ exists, then $\langle \chi, \lambda \rangle \geq 0$.

(b) The following are equivalent:
(i) x is χ-stable

(ii) the only 1-psgs $\lambda \in \mathrm{Hom}(G_m, G)$ for which $\lim_{t\to 0} \lambda(t).(x,z)$ exists, are in Δ.
(iii) the only 1-psgs $\lambda \in \mathrm{Hom}(G_m, G)$ for which $\lim_{t\to 0} \lambda(t).x$ exists and $\langle \chi, \lambda \rangle = 0$, are in Δ.

Proof. (a) Clear.

(b) (ii) iff (iii) and (ii) implies (i) are clear.

I didn't find a nice proof that (i) implies (ii), so omit it. We will avoid using this part of the theorem later when we talk about moduli spaces of modules.

5.5 Moduli spaces of representations

Let A be a finitely generated K-algebra and $e_1, \dots, e_n$ a complete set of orthogonal idempotents.

Definition. Let $\theta \in \mathbb{Z}^n$. An A-module M is *θ-semistable* if $\theta. \underline{\dim} M = 0$ and $\theta. \underline{\dim} M' \geq 0$ for all submodules $M' \subseteq M$.

Moreover M is *θ-stable* if the inequality is strict for $M' \neq 0, M$.

Apart from some changes of convention, we studied this last semester. The θ-semistable modules form a wide subcategory of the category of A-modules, in particular it is abelian. The θ-stables are the simple objects in this category. For θ-semistable M we write $\mathrm{gr}_\theta M$ for the direct sum of the quotients in a composition series of M in this category.

Let $\alpha \in \mathbb{N}^n$ a dimension vector. The group $G = \mathrm{GL}(\alpha)$ acts on $\mathrm{Mod}(A, \alpha)$. The element θ defines a character

$$\chi_\theta : \mathrm{GL}(\alpha) \to \mathcal{G}_m, \quad \chi_\theta(g) = \prod_i \det(g_i)^{\theta_i}.$$

Let Δ be the subgroup of G consisting of the elements g such that each g_i is the same multiple of the identity (so $\Delta \cong G_m$). Clearly Δ acts trivially on $\mathrm{Mod}(A, \alpha)$, and $\chi_\theta(\Delta) = 1$ iff $\theta.\alpha = 0$.

The components of a 1-psg λ are 1-psgs $\lambda_i : G_m \to \mathrm{GL}(\alpha_i)$, so they correspond to $\mathbb{Z}$-gradings

$$K^{\alpha_i} = \bigoplus_{s \in \mathbb{Z}} V_{i,s}$$

where $\lambda_i(t).v = t^s v$ for $t \in K^*$ and $v \in V_{i,s}$. This defines filtrations

$$K^{\alpha_i} \supseteq \dots \supseteq V_{i,\geq -1} \supseteq V_{i,\geq 0} \supseteq V_{i,\geq 1} \supseteq \dots$$

where $V_{i,\geq s} = \bigoplus_{p\geq s} V_{i,p}$. Thus $V_{i,\geq s} = K^{\alpha_i}$ for $s \ll 0$ and $V_{i,\geq s} = 0$ for $s \gg 0$. Conversely any such filtrations arise from some λ.

Lemma. Let $x \in \mathrm{Mod}(A,\alpha)$. Then $\lim_{t\to 0} \lambda(t).x$ exists if and only if the filtrations define A-submodules $V_{\geq s} = \bigoplus_i V_{i,\geq s}$ of K_x for all s. In this case $\langle \chi_\theta, \lambda\rangle = \sum_{s\in Z} \theta.\underline{\dim} V_{\geq s}$ and $\lim_{t\to 0} \lambda(t).x$ corresponds to the associated graded module $\bigoplus_s V_{\geq s}/V_{\geq s+1}$.

Proof. $a \in e_i A e_j$ gives a linear map $x_a : K^{\alpha_j} \to K^{\alpha_i}$, so corresponds to linear maps $x_{apq} : V_{jq} \to V_{ip}$. Now the action of $g = (g_i) \in \mathrm{GL}(\alpha)$ on $\mathrm{Mod}(A,\alpha)$ is given by $(g.x)_a = g_i x_a g_j^{-1}$. Thus $(\lambda(t).x)_a = \lambda_i(t) x_a \lambda_j(t)^{-1}$. Thus $(\lambda(t).x)_{apq} = t^{p-q} x_{apq}$.

Thus $\lim_{t\to 0} \lambda(t).x$ exists

iff $x_{apq} = 0$ for all a and $p < q$

iff $V_{\geq q}$ is a submodule of K_x for all q.

Then

$$\langle \chi_\theta, \lambda\rangle = \sum_i \theta_i \sum_{s\in\mathbb{Z}} n \dim V_{i,s} = \sum_{s\in\mathbb{Z}} s(\theta.\underline{\dim}(V_{\geq s}/V_{\geq s+1})) = \sum_{s\in\mathbb{Z}} \theta.\underline{\dim} V_{\geq s}.$$

Theorem. Let $x \in \mathrm{Mod}(A,\alpha)$ and let K_x be the corresponding A-module.
(i) K_x is θ-semistable iff x is χ_θ-semistable.
(ii) K_x is a direct sum of θ-stables iff the orbit of x is closed in $\mathrm{Mod}(A,\alpha)^{\chi_\theta - ss}$. Moreover every orbit closure $\overline{Gx}$ in $\mathrm{Mod}(A,\alpha)^{\chi_\theta - ss}$ contains a unique closed orbit, corresponding to the module $\mathrm{gr}_\theta K_x$.
(iii) K_x is θ-stable iff x is χ_θ-stable.

Proof. (i) If x is χ_θ-semistable and M' is a submodule of K_x with $\theta.\underline{\dim} M' < 0$ then it defines a filtration with $V_{\geq 1} = 0$, $V_{\geq 0} = M'$ and $V_{\geq -1} = K_x$. Let λ be the corresponding 1-psg. Then the limit exists, so $\langle \chi_\theta, \lambda\rangle = \theta.\underline{\dim} M' < 0$, contradicting the Hilbert-Mumford numerical criterion.

Conversely if K_x is θ-semistable and λ is a 1-psg such that the limit exists, then it corresponds to a filtration, and so $\langle \chi_\theta, \lambda\rangle = \sum_{n\in Z} \theta.\underline{\dim} V_{\geq n} \geq 0$ since K_x is θ-semistable. Thus x is χ_θ-semistable by the Hilbert-Mumford numerical criterion.

(ii) If K_x is θ-semistable, then it has a filtration with associated graded module $\mathrm{gr}_\theta K_x$, so the orbit of this module is contained in $\overline{Gx}$.

If Gx is closed, it follows that K_x is a direct sum of θ-stables.

If Gx is not closed, then $\overline{Gx}$ contains a closed orbit Gy in $\mathrm{Mod}(A,\alpha)^{\chi_\theta - ss}$. Then $y \in D(f)$ for some $f \in \mathcal{O}(\mathrm{Mod}(A,\alpha))^{G,\chi^n}$. Then also $x \in D(f)$. Thus by Kempf's Fundamental Theorem, applied to the affine variety $D(f)$, there is a 1-psg with $\lim_{t\to 0}\lambda(t).x$ in the orbit of y. Thus K_y is an associated graded module for some filtration of K_x. But since K_y is a direct sum of θ-stables, $K_y \cong \operatorname{gr}_\theta K_x$.

(The fact that every orbit closure contains a unique closed orbit, therefore, comes down to the fact that $\operatorname{gr}_\theta K_x$ is well-defined, which is essentially the Jordan-Hölder theorem.)

(iii) Straightforward, using that the θ-stable modules form an open subset of $\operatorname{Mod}(A,\alpha)$, and a direct sum of θ-stables is θ-stable iff it has automorphism group Δ.

Summary. Let's write $M(A,\alpha)_\theta$ for $\operatorname{Mod}(A,\alpha)\,/\!/(\operatorname{GL}(\alpha,\theta)$. We get a projective morphism

$$M(A,\alpha)_\theta \to M(A,\alpha)_0 = \operatorname{Mod}(A,\alpha)\,/\!/\,\operatorname{GL}(\alpha).$$

The space on the RHS classifies semisimple modules of dimension vector α. If A is finite dimensional, this is finite, so $M(A,\alpha)_\theta$ is a projective variety.

The stable points form an open subset $M(A,\alpha)^s_\theta$ of $M(A,\alpha)_\theta$ which is a geometric quotient

$$\operatorname{Mod}(A,\alpha)^{\theta-s}/\operatorname{GL}(\alpha).$$

But in general it might be empty.

If $\theta.\beta \neq 0$ for all β with $0<\beta<\alpha$, then $M(A,\alpha)^s_\theta = M(A,\alpha)_\theta$. We say "s=ss".

Examples. (i) Let Q be a quiver and α a dimension vector. One would like to study $M(KQ,\alpha)^{(s)}_\theta$. The cohomology of these moduli spaces is studied by M. Reineke, The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli, Invent. Math. 2003.

(ii) If there is a vertex i with $\alpha_i = 1$, $\theta_i = -\sum_{j\neq i}\alpha_j$ and $\theta_j = 1$ for all $j\neq i$, then a module M of dimension α has e_iM 1-dimensional, and M is θ-stable iff it is θ-semistable iff M is generated by e_iM.

(iii) For the quiver with vertices 1 and 2, n arrows from 1 to 2, dimension vector $(1,r)$ and $\theta = (-r,1)$, a representation is given by n linear maps $K\to K^r$, so by a map $K^n\to K^r$. The stability condition is that this map is onto. Thus the moduli space is $\operatorname{Gr}(n-r,K^n)$.

(iv) K of characteristic 0. If Q is an extended Dynkin quiver and $\Pi(Q)$ is its preprojective algebra, then $M(\Pi(Q),\delta)_0$ is isomorphic to the corresponding Kleinian singularity K^2/Γ, Γ a finite subgroup of $\operatorname{SL}_2(K)$. The space $M(\Pi(Q), \delta)_\theta$ for suitable θ, eg as in (ii), is the minimal resolution of singularities.

(v) If Q has vertices 1, 2, an arrow 1 to 2 and a loop at 2, $\alpha = (1, n)$ and $\theta = (n, -1)$ then $M(\Pi(Q), \alpha)_\theta$ is isomorphic to the Hilpert scheme of n points in the plane and $M(\Pi^\theta(Q), \alpha)_0$ is Calogero-Moser space.

(vi) If Q is a quiver without oriented cycles and α, β are two dimension vectors, then the Nakajima quiver variety can be defined to be $M(\Pi(Q'), \alpha')_\theta$ where Q' is Q with a new vertex ∞, and β_i arrows $\infty \to i$ for all i, $\alpha' = \alpha$ with $\alpha'_\infty = 1$ and θ as in (ii).

See for example A. Kirillov Jr., Quiver representations and quiver varieties, 2016.

Omissions.
- I wanted to talk about tangent spaces, smoothness of varieties and moment maps. This is one of the explanations of why preprojective algebras come up.
- I would have liked to talk about vector bundles, so as to discuss, for example, universal bundles in GIT.
- It would have been nice to explain the connection to McKay correspondence and Kleininan singularities in much more detail.

It seems that 4 hours per week for 3 semesters of 15 weeks each is not enough!