Noncommutative Algebra 2: Representations of finite-dimensional algebras

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1 Introduction

Let K be a field. Often it will be algebraically closed. Let A be a K-algebra. Any A-module M becomes a vector space. We are interested in the classification of finite-dimensional A-modules, especially in the case when A is finite-dimensional.

References:
D. J. Benson, Representations and cohomology
Assem, Simson and Skowronski, Elements of the representation theory of associative algebras I
Auslander, Reiten and Smalo, Representation theory of artin algebras
R. Schiffler, Quiver representations

1.1 Examples of algebras

(1) The group algebra $A = KG$ for a group G. Then A-modules are the same as representations of the group G, so group homomorphisms $G \to \mathrm{GL}(V)$.

(2) Let $V = K^n$ and let $q : V \to K$ be a quadratic form. The associated *Clifford algebra* $C_K(V, q)$ is TV/I, where TV is the tensor algebra on V over K and I is the ideal generated by $v^2 - q(v)1$ for all $v \in V$. For example if V has basis $e_1, \ldots, e_n$ and $q(\sum_{i=1}^n \lambda_i e_i) = \sum_{i=1}^n a_i \lambda_i^2$ for all $\lambda_1, \ldots, \lambda_n \in K$, then $C(V, q)$ is generated by $e_1, \ldots, e_n$ subject to the relations $e_i^2 = a_i 1$

and $e_ie_j = -e_je_i$ for $i \neq j$. It has basis the products $e_{i_1}e_{i_2}\dots e_{i_k}$ with $i_1 < i_2 < \cdots < i_k$.

In particular the *exterior algebra* is the case when $q = 0$.

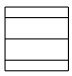
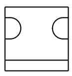
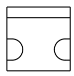
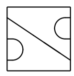
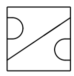
For example $C_{\mathbb{R}}(\mathbb{R}^2, q)$ with $q(x_1, x_2) = -x_1^2 - x_2^2$ is the quaternions.

(3) The *Temperley-Lieb algebra* $TL_n(\delta)$ for $n \geq 1$ and $\delta \in K$ has basis the diagrams with two vertical rows of n dots, connected by n nonintersecting curves. Two are considered equal if the same vertices are connected. The product ab is defined by

$$ab = \delta^r \cdot (\text{diagram obtained by joining } a \text{ and } b \text{ and removing any loops}),$$

where r is the number of loops removed.

Picture for $n = 3$. Basis elements 1, u_1, u_2, p, q.

$1 =$ , $u_1 =$ , $u_2 =$ , $p =$ , $q =$ 

One can show that $\mathrm{TL}_n(\delta)$ is generated by $u_1 \dots, u_{n-1}$ subject to the relations $u_i^2 = \delta u_i$, $u_iu_{i\pm1}u_i = u_i$ and $u_iu_j = u_ju_i$ if $|i - j| > 1$.

The Temperley-Lieb algebra was invented to study Statistical Mechanics. It is now also important in Knot Theory. (Fields medal for Vaughan Jones in 1990.)

(4) $A = KQ$, the path algebra of a finite quiver Q. We are interested in classification of configurations of vector spaces and linear maps. It is finite-dimensional provided Q has no oriented cycles.

If Q is one vertex and a loop, a representation is given by a vector space and an endomorphism. If finite-dimensional, choosing a basis, an n-dimensional representation is given by an $n \times n$ matrix. Two representations are isomorphic if the matrices are similar. If the field K is algebraically closed, any matrix is similar to one in Jordan normal form. Thus any representation is isomorphic to a direct sum of Jordan block representations.

If Q is two vertices with an arrow between them, a representation is given by two vector spaces and a linear map $a : X \to Y$. A finite-dimensional representation is given by an $m \times n$ matrix M. Two representations given by M, M' are isomorphic if there are invertible matrices P, Q with $PM = M'Q$. Equivalently if M, M' are related by row and column operations.

Can always transform to block matrix of 1s and 0s. Thus finite-dimensional representations are determined by the dimension vector and the rank.

(5) If I is an ideal in KQ then representations of $A = KQ/I$ correspond to configurations of vector spaces and linear maps satisfying the relations in I. For example if Q is

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} n$$

and $I = (a_2a_1, a_3a_2, \dots a_{n-1}a_{n-2})$ then A-modules correspond to complexes of vector spaces with n terms.

The quiver with one vertex and loop x with relation $x^n = 0$. This is the algebra $K[x]/(x^n)$. Any module is a direct sum of modules given by Jordan blocks with eigenvalue 0 and size up to n.

1.2 Radical and socle

Theorem/Definition. If R is a ring and $x \in R$, the following are equivalent
(i) $xS = 0$ for any simple left module S.
(ii) $x \in I$ for every maximal left ideal I
(iii) $1 - ax$ has a left inverse for all $a \in R$.
(iv) $1 - ax$ is invertible for all $a \in R$.
(i')-(iv') The right-hand analogues of (i)-(iv).
The set of such elements is called the *(Jacobson) radical* $J(R)$ of R. It is an ideal in R.

Proof (i) implies (ii). If I is a maximal left ideal in R, then R/I is a simple left module, so $x(R/I) = 0$, so $x(I + 1) = I + 0$, so $x \in I$.

(ii) implies (iii). If there is no left inverse, then $R(1 - ax)$ is a proper left ideal in R, so contained in a maximal left ideal I by Zorn's Lemma. Now $x \in I$, and $1 - ax \in I$, so $1 \in I$, so $I = R$, a contradiction.

(iii) implies (iv) $1 - ax$ has a left inverse u, and $1 + uax$ has a left inverse v. Then $u(1 - ax) = 1$, so $u = 1 + uax$, so $vu = 1$. Thus u has a left and right inverse, so it is invertible and these inverses are equal, and are themselves invertible. Thus $1 - ax$ is invertible.

(iv) implies (i). If $s \in S$ and $xs \neq 0$, then $Rxs = S$ since S is simple, so $s = axs$ for some $a \in R$. Then $(1 - ax)s = 0$, but then $s = 0$ by (iv).

(iv) implies (iv'). If b is an inverse for $1-ax$, then $1+xba$ is an inverse for $1-xa$. Namely $(1-ax)b = b(1-ax) = 1$, so $axb = b-1 = bax$, and then $(1+xba)(1-xa) = 1+xba-xa-xbaxa = 1$, and $(1-xa)(1+xba) = 1-xaxba-xa+xba = 1$.

Examples.
(a) For the ring $\mathbb{Z}$, the simple modules are $\mathbb{Z}/p\mathbb{Z}$. Then $J(\mathbb{Z}) = \bigcap_p p\mathbb{Z} = 0$.
(b) Similarly, for the ring $K[x]$, the simple modules are $K[x]/(f(x))$, $f(x)$ irreducible. Then $J(K[x]) = 0$.
(c) If K is a field and Q a finite quiver, then $J(KQ)$ has as basis the paths from i to j such that there is no path from j to i.

Lemma 1. If I is an ideal in which every element is nilpotent (a 'nil ideal'), then $I \subseteq J(R)$.

Proof. If $x \in I$ and $a \in R$ then $ax \in I$, so $(ax)^n = 0$ for some n. Then $1-ax$ is invertible with inverse $1+ax+(ax)^2+\cdots+(ax)^{n-1}$. Thus $x \in J(R)$.

If M is an R-module and I an ideal in R, we write IM fot the set of sums of products im.

Nakayama's Lemma. Suppose M is a finitely generated R-module.
(i) If $J(R)M = M$, then $M = 0$.
(ii) If $N \subseteq M$ is a submodule with $N + J(R)M = M$, then $N = M$.

Proof. (i) Suppose $M \neq 0$. Let $m_1, \dots, m_n$ be generators with n minimal. Since $J(R)M = M$ we can write $m_n = \sum_{i=1}^n r_i m_i$ with $r_i \in J(R)$. This writes $(1-r_n)m_n$ in terms of the others. But $1-r_n$ is invertible, so it writes m_n in terms of the others. Contradiction.

(ii) Apply (i) to M/N.

Definition. The *socle* of a module M is the sum of its simple submodules,

$$\operatorname{soc} M = \sum_{S \subseteq M \text{ simple}} S$$

The *radical* of a module M is the intersection of its maximal submodules.

$$\operatorname{rad} M = \bigcap_{N \subseteq M,\ M/N \text{ simple}} N$$

Thus $J(R) = \operatorname{rad}({}_R R)$. The quotient $M/\operatorname{rad} M$ is called the *top* of M.

Lemma 2.
(i) $\operatorname{soc} M$ is the unique largest semisimple submodule of M.
(ii) If $N \subseteq M$ and M/N is semisimple, then $\operatorname{rad} M \subseteq N$.
Note that in general M/N is not semisimple, eg $M = {}_{\mathbb{Z}}\mathbb{Z}$.

Proof. (i) Use that a module is semisimple if and only if it is a sum of simple submodules.

(ii) Suppose $M/N = \bigoplus S_i$. For each j let U_j be the kernel of the map $M \to M/N \to \bigoplus S_i \to S_j$. Then $M/U_j \cong S_j$, so U_j is a maximal submodule. If $m \in \operatorname{rad} M$, then $m \in U_j$ for all j, so m is sent to zero in $M/N \cong \bigoplus S_i$, so $m \in N$.

1.3 Radical and socle (finite-dimensional case)

Let K be a field and let A be a K-algebra. Any A-module M becomes a vector space. Often in this section M or A is finite-dimensional. The results in this section generalize to finite length or artinian modules and artinian rings.

Lemma 1. If M is a finite-dimensional A-module, then $M/\operatorname{rad} M$ is semisimple, so $\operatorname{rad} M$ is the unique smallest submodule N of M with M/N semisimple.

Proof. Since M is finite-dimensional, it has DCC on submodules, so we can write $\operatorname{rad} M$ as a finite intersection of maximal submodules $U_1 \cap \cdots \cap U_n$. Then $M/\operatorname{rad} M$ embeds in $(M/U_1) \oplus \cdots \oplus (M/U_n)$, so it is semisimple.

Wedderburn's Theorem. For a finite-dimensional algebra A the following are equivalent
(i) ${}_A A$ is a semisimple module
(ii) Every A-module is semisimple
(iii) $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ with the D_i division algebras.
(iv) $J(A) = 0$.

The Artin-Wedderburn Theorem generalizes this to arbitrary rings R. Part (iv) becomes that the ring is left artinian and $J(R) = 0$.

Proof. We already had the equivalence of (i) to (iii), and (i) if and only if (iv) follows from Lemma 1.

Example. If G is a finite group and K has characteristic 0 (or not dividing the order of G), then KG is semisimple. (Maschke's Theorem). Also Clifford algebras associated to non-degenerate quadratic forms are semismple and the Temperley-Lieb algebras over $\mathbb{C}$ are semisimple for generic δ.

The product of two ideals is $IJ =$ sums of products ij with $i \in I$ and $j \in J$. The powers of an ideal are defined inductively by $I^n = I^{n-1}I$. An ideal is *nilpotent* if $I^n = 0$ for some n. If so, then every element of I is nilpotent.

Proposition. Suppose that A is a finite-dimensional algebra.
(i) $J(A)$ is a nilpotent ideal in A and $A/J(A)$ is a semisimple algebra.
(iii) If I is an ideal in which every element is nilpotent and A/I is semisimple, then $I = J(A)$.

Proof. (i) Since A is finite-dimensional, we have

$$J(A) \supseteq J(A)^2 \supseteq \cdots \supseteq J(A)^n = J(A)^{n+1} = \dots$$

for some n. Then $J(A)J(A)^n = J(A)^n$, so $J(A)^n = 0$ by Nakayama's lemma. Now use Lemma 1.

(ii) Use §1.2 Lemmas 1 and 2.

Example. If A is the algebra of matrices of shape

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

then $A = S \oplus I$ where S and I consist of matrices of shape

$$S = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that I is an ideal in A, and it consists of nilpotent elements. Also one can check that S is a subalgebra in A, and it is clearly isomorphic to $M_2(K) \times K$, so semisimple. Thus $A/I \cong S$ is semisimple. Thus $J(A) = I$.

Lemma 2. If A is a finite-dimensional algebra and M is an A-module, then
(i) $\operatorname{rad} M = J(A)M$.
(ii) $\operatorname{soc} M = \{m \in M : J(A)m = 0\}$.

Proof (i) If N is a maximal submodule of M, then M/N is simple, and so $J(A)(M/N) = 0$. Thus $J(A)M \subseteq N$, and hence $J(A)M \subseteq \operatorname{rad} M$. On the other hand $M/J(A)M$ is an $A/J(A)$-module, so semisimple as an $A/J(A)$-module, so also semisimple as an A-module, so $\operatorname{rad} M \subseteq J(A)M$ by §1.2 Lemma 2.

(ii) Any simple submodule S of M satisfies $J(A)S = 0$, so $J(A)m = 0$ for all $m \in \operatorname{soc} M$, so $\operatorname{soc} M$ is contained in the RHS. Now the RHS is an $A/J(A)$-module, so semisimple as an $A/J(A)$-module, so also semisimple as an A-module, so the RHS is contained in $\operatorname{soc} M$.

Let Q be a finite quiver and write $(KQ)_+$ for the ideal in KQ spanned by the non-trivial paths. Clearly $(KQ)_+^n$ is the ideal spanned by paths of length $\geq n$, and $KQ/(KQ)_+ \cong K \times \cdots \times K$.

Definition. An ideal $I \subseteq KQ$ is *admissible* if
(1) $I \subseteq (KQ)_+^2$
(2) $(KQ)_+^n \subseteq I$ for some n.

Lemma 3. If I is an admissible ideal in KQ then $A = KQ/I$ is finite-dimensional, $J(A) = (KQ)_+/I$, and $A/J(A) \cong K \times \cdots \times K$.

Proof. Clear.

Example. Let Q be the quiver with one vertex and one loop x, so $KQ = K[x]$. Then the admissible ideals in KQ are (x^n) for $n \geq 2$, and the radical of $KQ/(x^n)$ is $(x)/(x^n)$.

1.4 Local rings

Lemma/Definition. A ring R is called a *local ring* if it satisfies the following equivalent conditions.
(i) $R/J(R)$ is a division ring.
(ii) The non-invertible elements of R form an ideal.
(iii) There is a unique maximal left ideal in R.

Proof. (i) implies (ii). The elements of $J(R)$ are not invertible, so it suffices to show that any $x \notin J(R)$ is invertible. Now $J(R) + x$ is an invertible element in $R/J(R)$, say with inverse $J(R) + a$. Then $1 - ax, 1 - xa \in J(R)$. But this implies ax and xa are invertible, hence so is x.

(ii) implies (iii). Clear.

(iii) implies (i). Assuming (iii), $J(R)$ is the unique maximal left ideal, so $\overline{R} = R/J(R)$ is a simple R-module, and so a simple $\overline{R}$-module. Then $\overline{R} \cong \mathrm{End}_{\overline{R}}(\overline{R})$, which is a division ring by Schur's Lemma.

Lemma. A finite-dimensional algebra is local if and only if every element is invertible or nilpotent.

Proof. If local, then every element is invertible or nilpotent, since elements of $J(A)$ are nilpotent.

Conversely, suppose every element of A is nilpotent or invertible. The nilpotent elements form an ideal, for if ax is invertible, with $x^n = 0$, then $0 = [(ax)^{-1}a]^n x^n = 1$, and if there are x, y nilpotent with $x + y$ invertible, then we may assume that $x + y = 1$. But then $x = 1 - y$ is invertible.

1.5 Indecomposable modules

Recall that an A-module M is *indecomposable* if there is no direct sum decomposition $M = X \oplus Y$ with X and Y non-zero submodules of M. It is equivalent that $\mathrm{End}(M)$ contains no idempotents except 0,1.

Fitting's Lemma. If M is a finite-dimensional A-module (or more generally of finite length) and $\theta \in \mathrm{End}_A(M)$, then there is a decomposition

$$M = M_0 \oplus M_1$$

such that $\theta|_{M_0}$ is a nilpotent endomorphism of M_0 and $\theta|_{M_1}$ is an invertible endomorphism of M_1. In particular, if M is indecomposable, then any endomorphism is invertible or nilpotent or invertible, so $\mathrm{End}_A(M)$ is local.

Proof. There are chains of submodules

$$\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta^2) \supseteq \mathrm{Im}(\theta^3) \supseteq \dots$$

$$\mathrm{Ker}(\theta) \subseteq \mathrm{Ker}(\theta^2) \subseteq \mathrm{Ker}(\theta^3) \subseteq \dots$$

which must stabilize since M is finite dimensional. Thus there is some n with $\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$ and $\mathrm{Ker}(\theta^n) = \mathrm{Ker}(\theta^{2n})$. We show that

$$M = \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n).$$

If $m \in \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n)$ then $m = \theta^n(m')$ and $\theta^{2n}(m') = \theta^n(m) = 0$, so $m' \in \mathrm{Ker}(\theta^{2n}) = \mathrm{Ker}(\theta^n)$, so $m = \theta^n(m') = 0$. If $m \in M$ then $\theta^n(m) \in$

$\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$, so $\theta^n(m) = \theta^{2n}(m'')$ for some m''. Then $m = (m - \theta^n(m'')) + \theta^n(m'') \in \mathrm{Ker}(\theta^n) + \mathrm{Im}(\theta^n)$.

Now it is easy to see that the restriction of θ to $\mathrm{Ker}(\theta^n)$ is nilpotent, and its restriction to $\mathrm{Im}(\theta^n)$ is invertible.

We now apply the idea of the Jacobson radical to the module category.

Lemma/Definition. If X and Y are A-modules, we define $\mathrm{rad}_A(X, Y)$ to be the set of all $\theta \in \mathrm{Hom}_A(X, Y)$ satisfying the following equivalent conditions.
(i) $1_X - \phi\theta$ is invertible for all $\phi \in \mathrm{Hom}_A(Y, X)$.
(ii) $1_Y - \theta\phi$ is invertible for all $\phi \in \mathrm{Hom}_A(Y, X)$.
Thus by definition $\mathrm{rad}(X, X) = J(\mathrm{End}(X))$.

Proof of (i) implies (ii). If u is an inverse for $1_X - \phi\theta$ then $1_Y + \theta u\phi$ is an inverse for $1_Y - \theta\phi$.

Lemma 1.
(a) $\mathrm{rad}_A(X, Y)$ is a subspace of $\mathrm{Hom}_A(X, Y)$.
(b) Given maps $X \to Y \to Z$, if one is in the radical, so is the composition.
(c) $\mathrm{rad}_A(X \oplus X', Y) = \mathrm{rad}_A(X, Y) \oplus \mathrm{rad}_A(X', Y)$ and $\mathrm{rad}_A(X, Y \oplus Y') = \mathrm{rad}_A(X, Y) \oplus \mathrm{rad}_A(X, Y')$.

Proof. (a) For a sum $\theta + \theta'$, let f be an inverse for $1 - \phi\theta$. Then $1 - \phi(\theta + \theta') = (1 - \phi\theta)(1 - f\phi\theta')$, a product of invertible maps.

(b) Clear.

(c) Straightforward.

Definition. A module map $\theta : X \to Y$ is a *split mono* if there is a map $\phi : Y \to X$ with $\phi\theta = 1_X$, Equivalently if θ is an isomorphism of X with a direct summand of Y.

A module map $\theta : X \to Y$ is a *split epi* if there is a map $\psi : Y \to X$ with $\theta\psi = 1_Y$. Equivalently if θ identifies Y with a direct summand of X.

Lemma 2.
(i) If X is indecomposable, then $\mathrm{rad}_A(X, Y)$ is the set of maps which are not split monos.
(ii) If Y is indecomposable, then $\mathrm{rad}_A(X, Y)$ is the set of maps which are not split epis.
(iii) If X and Y are indecomposable, then $\mathrm{rad}_A(X, Y)$ is the set of non-

isomorphisms.

Proof. (i) Suppose $\theta \in \operatorname{Hom}(X, Y)$. If θ is a split mono there is $\phi \in \operatorname{Hom}(Y, X)$ with $\phi\theta = 1_X$, so $1 - \phi\theta$ is not invertible. Conversely if there is some ϕ with $f = 1 - \phi\theta$ not invertible, then f is nilpotent, and so $\phi\theta = 1 - f$ is invertible. Then $(\phi\theta)^{-1}\phi\theta = 1_X$, so θ is split mono.

(ii) is dual and (iii) follows.

Krull-Remak-Schmidt Theorem. Every finite-dimensional A-module is isomorphic to a direct sum of indecomposable modules,

$$M \cong X_1 \oplus \cdots \oplus X_n.$$

Moreover if $M \cong Y_1 \oplus \cdots \oplus Y_m$ is another decomposition into indecomposables, then $m = n$ and the X_i and Y_j can be paired off so that corresponding modules are isomorphic.

Proof. Given any two modules X and Y, we can define a vector space

$$t(X, Y) = \operatorname{Hom}_A(X, Y)/\operatorname{rad}_A(X, Y).$$

If X is indecomposable, then $D = \operatorname{End}(X)/J(\operatorname{End}(X))$ is a division algebra, and we define

$$\mu_X(Y) = \frac{\dim t(X, Y)}{\dim D}.$$

In fact $t(X, Y)$ is an $\operatorname{End}(Y)$-$\operatorname{End}(X)$-bimodule, and in fact an $\operatorname{End}(Y)/J(\operatorname{End}(Y))$-$\operatorname{End}(X)/J(\operatorname{End}(X))$-bimodule. In particular it is a right D-module, and we are taking its dimension as a vector space over D.

Now

$$t(X, M) = t(X, X_1 \oplus \cdots \oplus X_n) \cong t(X, X_1) \oplus \cdots \oplus t(X, X_n)$$

so $\mu_X(M) = \mu_X(X_1 \oplus \cdots \oplus X_n) = \mu_X(X_1) + \cdots + \mu_X(X_n)$. Also

$$\mu_X(X_i) = \begin{cases} 1 & (X_i \cong X) \\ 0 & (X_i \not\cong X) \end{cases}.$$

Thus $\mu_X(M)$ is the number of the X_i which are isomorphic to X. Similarly, it is the number of Y_j which are isomorphic to X. Thus these numbers are equal.

Notation. If M is a f.d. module we write $\operatorname{add} M$ for the subcategory of A-modules consisting of the direct summands of M^n. Equivalently it is the modules which are isomorphic to a direct sum of copies of the indecomposable direct summands of M.

For example $\operatorname{add} A$ is exactly the f.d. projective left A-modules.

We say that a module is *basic* if in the decomposition into indecomposables, the summands are non-isomorphic.

Thus up to isomorphism, given M there is a unique basic module B with $\operatorname{add} M = \operatorname{add} B$.

1.6 Projectives and injectives for f.d. algebras

Let A be a finite-dimensional algebra. Henceforth, unless explicitly stated otherwise, we only consider finite-dimensional modules.

The functor $D(-) = \operatorname{Hom}_K(-, K)$ gives an antiequivalence between the categories of finite-dimensional left and right A-modules. It restricts to give an equivalence between projective modules on one side and injective modules on the other side.

The functor $\operatorname{Hom}_A(-, A)$ gives an antiequivalence between the categories of finite-dimensional left and right projective A-modules.

Definition. The Nakayama functor is

$$\nu(-) = D \operatorname{Hom}_A(-, A) : A - mod \to A - mod.$$

Lemma 1. (i) ν is naturally isomorphic to $DA \otimes_A -$.
(ii) ν has right adjoint

$$\nu^-(-) = \operatorname{Hom}_A(D(-), A) \cong \operatorname{Hom}_A(DA, -) : A - mod \to A - mod.$$

(iii) ν restricts to an equivalence from projective left modules to injective left modules, with inverse equivalence given by ν^-.
(iv) $\operatorname{Hom}(X, \nu P) \cong D \operatorname{Hom}(P, X)$ for X, P left A-modules, P projective.

Proof.

(i) $D(DA \otimes_A X) \cong \operatorname{Hom}_A(X, DDA) \cong \operatorname{Hom}(X, A)$. Now apply D.

(ii), (iii) Clear.

(iv) The composition

$$\operatorname{Hom}(P,A)\otimes_A X \cong \operatorname{Hom}(P,A)\otimes_A \operatorname{Hom}(A,X) \to \operatorname{Hom}(P,X)$$

is an isomorphism, since it is for $P = A$. Thus

$$D\operatorname{Hom}(P,X) \cong \operatorname{Hom}_K(\operatorname{Hom}(P,A)\otimes_A X, K)$$

$$\cong \operatorname{Hom}(X, \operatorname{Hom}_K(\operatorname{Hom}(P,A),K)) = \operatorname{Hom}(X,\nu P).$$

Definition. Let $\theta : X \to Y$ be a map of A-modules. We say that θ is *left minimal* if for $\alpha \in \operatorname{End}(Y)$, if $\alpha\theta = \theta$, then α is invertible. We say that θ is *right minimal* if for $\beta \in \operatorname{End}(X)$, if $\theta\beta = \theta$, then β is invertible.

Lemma 2. Given a map $\theta : X \to Y$ of finite-dimensional A-modules
(i) There is a decomposition $Y = Y_0 \oplus Y_1$ such that $\operatorname{Im}(\theta) \subseteq Y_1$ and $X \to Y_1$ is left minimal.
(ii) There is a decomposition $X = X_0 \oplus X_1$ such that $\theta(X_0) = 0$ and $X_1 \to Y$ is right minimal.

Proof. (i) Of all decompositions $Y = Y_0 \oplus Y_1$ with $\operatorname{Im}(\theta) \subseteq Y_1$ choose one with Y_1 of minimal dimension. Let θ_1 be the map $X \to Y_1$. Let $\alpha \in \operatorname{End}(Y_1)$ with $\alpha\theta_1 = \theta_1$. By the Fitting decomposition, $Y_1 = \operatorname{Im}(\alpha^n) \oplus \operatorname{Ker}(\alpha^n)$ for $n \gg 0$. Now $\alpha^n\theta_1 = \theta_1$, so $\operatorname{Im}(\theta_1) \subseteq \operatorname{Im}(\alpha^n)$, and we have another decomposition $Y = [Y_0 \oplus \operatorname{Ker}(\alpha^n)] \oplus \operatorname{Im}(\alpha^n)$. By minimality, $\operatorname{Ker}(\alpha^n) = 0$, so α is injective, and hence an isomorphism.

Proposition 1. Let M be a module and let $\theta : M \to I$ be a map with I injective. The following are equivalent
(i) Any map from M to an injective factors through θ, and θ is left minimal (i.e. θ is an injective envelope).
(ii) θ is a monomorphism and left minimal.
(iii) The induced map $\theta|_{\operatorname{soc} M} : \operatorname{soc} M \to \operatorname{soc} I$ is an isomorphism.
For every module M there exists a map θ with these properties. It is unique up to a (non-unique) isomorphism.

Proof. (i) iff (ii). Clear, using that any module can be embedded in an injective module.

(ii) implies (iii). Identify M as a submodule of I. We use that the inclusion $M \to I$ is an essential extension. Clearly $\theta|_{\operatorname{soc} M}$ is injective. Suppose not onto. Then by semisimplicity $\operatorname{soc} I = \operatorname{soc} M \oplus X$ for some $X \neq 0$. But then $X \cap M = 0$, for any simple submodule of it is in $\operatorname{soc} M \cap X$. Contradiction.

(iii) implies (ii). Any simple submodule of $\operatorname{Ker}\theta$ is in $\operatorname{Ker}(\theta|_{\operatorname{soc} M})$, so zero. Thus θ is injective. If α is an endomorphism of I with $\alpha\theta = \theta$, then $\alpha|_{\operatorname{soc} I} \in \operatorname{End}(\operatorname{soc} I)$, and $\alpha|_{\operatorname{soc} I}\theta|_{\operatorname{soc} M} = \theta|_{\operatorname{soc} M}$. This forces $\alpha_{\operatorname{soc} I}$ to be injective, so α is injective, so an isomorphism by dimensions.

Using the duality D to turn statement about projectives into injectives, we get

Proposition 2. Let M be a module and let $\theta : P \to M$ be a map with P projective. The following are equivalent
(i') Any map from a projective to M factors through θ, and θ is right minimal (i.e. θ is a projective cover).
(ii') θ is surjective and right minimal.
(iii') The induced map $\overline{\theta} : P/\operatorname{rad} P \to M/\operatorname{rad} M$ is an isomorphism.
For every module M there exists a map θ with these properties. It is unique up to a (non-unique) isomorphism.

1.7 Structure of f.d. algebras

Let A be a finite-dimensional algebra over K. We decompose ${}_AA$ into indecomposables, and collect isomorphic terms, so

$$A \cong P[1]^{r_1} \oplus \cdots \oplus P[n]^{r_n}$$

with the $P[i]$ non-isomorphic modules.

Properties.
(1) The modules $P[1], \ldots, P[n]$ are a complete set of non-isomorphic indecomposable projective modules. They are called *principal indecomposable modules* (pims). The finite-dimensional projective A-modules are exactly the direct sums of copies of the $P[i]$.

This is just the Krull-Remak-Schmidt Theorem,

(2) Let $S[i] = P[i]/\operatorname{rad} P[i]$. The modules $S[1], \ldots, S[n]$ are a complete set of non-isomorphic simple A-modules.

Namely, $S[i]$ is semisimple, but any endomorphism θ of it lifts to $P[i]$, so is nilpotent or invertible, so θ is nilpotent or invertible, so $S[i]$ is indecomposable, so simple. Inverse isomorphisms between $S[i]$ and $S[j]$ lift to maps $P[i] \to P[j] \to P[i]$ whose composition can't be nilpotent, so must be invertible, so $P[i] \cong P[j]$, so $i = j$. Any simple module S has a non-zero map from some $P[i]$, but then the map $P[i] \to S$ must give a non-zero map $S[i] \to S$, and this must be an isomorphism.

(3) Let $D_i = \mathrm{End}_A(P[i])/J(\mathrm{End}_A(P[i]))^{op} \cong \mathrm{End}_A(S[i])^{op}$. It is a division algebra. If K is algebraically closed, then $D_i = K$.

Use that $P[i]$ is indecomposable, or Schur's Lemma. If K is algebraically closed and $0 \neq x \in D$, then the map $D \to D$ of multiplication by x must have an eigenvalue $\lambda \in K$. Then $(x - \lambda)y = 0$ for some $0 \neq y \in D$. Since D is a division algebra, $x = \lambda \in K$.

(4) $A/J(A) \cong M_{r_1}(D_1) \times \cdots \times M_{r_n}(D_n)$. Under this isomorphism, the simple module $S[i]$ corresponds to the module given by $D_i^{r_i}$, so $\dim S[i] = r_i \dim D_i$.

Namely, $A/J(A) \cong \mathrm{End}_A(A/J(A))^{op} \cong \mathrm{End}_A\left(\bigoplus_i (P[i]/\operatorname{rad} P[i])^{r_i}\right)^{op}$.

Then $S[i] \cong \mathrm{Hom}(A, S[i]) \cong \bigoplus_j \mathrm{Hom}(P[j], S[i])^{r_j} \cong \mathrm{End}(S[i])^{r_i}$.

(5) Define $I[i] = \nu P[i]$. It is an injective module with $\operatorname{soc} I[i] \cong S[i]$. The modules $I[1], \dots, I[n]$ are a complete set of non-isomorphic indecomposable injective modules, and any injective module is a direct sum of copies of them.

We check $\mathrm{Hom}(S[j], I[i]) \cong D\,\mathrm{Hom}(P[i], S[j])$ which is D_i if $j = i$ and otherwise 0.

Examples. (1) For the algebra

$$A = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \qquad J(A) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

we get $A = Ae^{11} \oplus Ae^{22} \oplus Ae^{33}$,

$$P[1] = Ae^{11} = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \cong Ae^{22}, \quad P[2] = Ae^{33} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

$$\operatorname{rad} P[1] = J(A)P[1] = 0 \quad \operatorname{rad} P[2] = J(A)P[2] = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \cong P[1].$$

Then $D_i = K$, $S[1] = P[1]$ is 2-dimensional and $S[2] = P[2]/\operatorname{rad} P[2]$ is 1-dimensional.

(2) If an algebra is given as $A = KQ/I$ with I admissible and $Q_0 = \{1, \dots, n\}$, then we get

$P[i] = Ae_i$. As a representation of Q, the vector space at vertex j is $e_j(KQ/I)e_i$ so has as basis the paths from i to j modulo the relations.

$\operatorname{rad} P[i] = J(A)e_i$. The vector space at j has as basis the non-trivial paths from i to j modulo the relations.

$S[i] = Ae_i/\operatorname{rad} Ae_i$. As a representation of Q the vector space at i is K, the other vector spaces are zero, and the linear maps corresponding to arrows are zero.

$I[i] = D(e_iA)$. As a representation of Q, the vector space at vertex j is $D(e_i(KQ/I)e_j)$, is has as basis the dual basis vectors corresponding to paths from j to i modulo the relations.

(3) For the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$ with relations $I = (ba)$ we have

$$P[1] = K \to K \to 0 \to 0, P[2] = 0 \to K \to K \to K,$$

$$P[3] = 0 \to 0 \to K \to K, P[4] = 0 \to 0 \to 0 \to K.$$

for example $P[1] = Ae_1$ has basis e_1, a (but not ba or cba as they are in the ideal I). Now $e_1 \in e_1Ae_1$ and $a \in e_2Ae_1$ so they belong to the vector spaces at vertices 1 and 2 for $P[1]$.

$$S[1] = K \to 0 \to 0 \to 0, S[2] = 0 \to K \to 0 \to 0,$$

$$S[3] = 0 \to 0 \to K \to 0, S[4] = 0 \to 0 \to 0 \to K.$$

Also

$$I[1] = K \to 0 \to 0 \to 0, I[2] = K \to K \to 0 \to 0,$$
$$I[3] = 0 \to K \to K \to 0, I[4] = 0 \to K \to K \to K.$$

(4) The commutative square quiver $ba = dc$, vertices 1(source),2,3,4(sink). The representation $P[1] \cong I[4]$ is 1-dimensional at each vertex.

(5) A cyclically oriented square

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[d, "b"] \\ 4 \arrow[u, "d"] & 3 \arrow[l, "c"] \end{tikzcd}$$

with relations cba and dc. For example $P[1]$ has basis the paths starting 1, modulo the relations, so e_1, a, ba. These basis elements belong to the vector spaces at vertices 1,2,3. Moreover $ae_1 = a$, so the arrow a sends the basis element corresponding to e_1 to the basis element corresponding to a. Thus

$$P[1] = \begin{tikzcd} K \arrow[r, "1"] & K \arrow[d, "1"] \\ 0 \arrow[u] & K \arrow[l] \end{tikzcd}$$

Similarly

$$P[2] = \begin{tikzcd} 0 \arrow[r] & K \arrow[d, "1"] \\ K \arrow[u] & K \arrow[l, "1"'] \end{tikzcd} \quad P[3] = \begin{tikzcd} 0 \arrow[r] & 0 \arrow[d] \\ K \arrow[u] & K \arrow[l, "1"'] \end{tikzcd} \quad P[4] = \begin{tikzcd} K \arrow[r, "1"] & K \arrow[d, "1"] \\ K \arrow[u, "1"] & K \arrow[l, "0"'] \end{tikzcd}$$

For example in $P[4]$ the arrow c sends the basis element bad in the vector space at vertex 3 to $cbad = 0$, and not to e_4, which is the basis element of the vector space at vertex 4.

Definition. We say that a finite-dimensional algebra A is basic if ${}_A A$ is basic. It is equivalent that $A/J(A) \cong D_1 \times \cdots \times D_n$ with the D_i division algebras.

Theorem. Any f.d. algebra is Morita equivalent to a basic one.

Proof. Let $P = P[1] \oplus \cdots \oplus P[n]$ be the basic module with $\operatorname{add} P = \operatorname{add} A$. The P is a finitely generated projective generator for $A-mod$, so A is Morita equivalent to $B = \operatorname{End}_A(P)^{op}$. Now

$$B/J(B) \cong \operatorname{End}_A(P/\operatorname{rad} P)^{op} \cong \operatorname{End}_A(S[1] \oplus \cdots \oplus S[n])^{op} \cong D_1 \times \cdots \times D_n.$$

Theorem (Gabriel's less famous theorem about quivers). If A is a f.d. K-algebra, and $A/\operatorname{rad} A \cong K \times \cdots \times K$ (for example if A is basic and K is algebraically closed), then $A \cong KQ/I$ for some quiver Q and admissible ideal I.

Proof. We have a decomposition $A = P[1] \oplus \cdots \oplus P[n]$ without multiplicities. Using the isomorphism $A \cong \operatorname{End}(A)^{op}$, the projections onto the $P[i]$ give a complete set of inequivalent primitive orthogonal idempotents $e_1, \ldots, e_n$. Let $J = J(A)$. We have

$$J = \bigoplus_{i,j} e_j J e_i.$$

and

$$J^2 = \bigoplus_{i,j} e_j J^2 e_i$$

so

$$J/J^2 \cong \bigoplus_{i,j} (e_j J e_i)/(e_j J^2 e_i).$$

Let Q be the quiver with $Q_0 = \{1, \dots, n\}$ and with

$$\dim(e_j J e_i)/(e_j J^2 e_i)$$

arrows from i to j, for all i, j. Define an algebra homomorphism

$$\theta : KQ \to A$$

sending e_i to e_i, and sending the arrows from i to j to elements in $e_j J e_i$ inducing a basis of the quotient. Let $U = \theta(KQ_+)$. We have $U \subseteq J$ and $U + J^2 = J$, using that

$$J = \bigoplus_{i,j} e_j J e_i.$$

Thus by Nakayama's Lemma, $U = J$. It follows that θ is surjective. Let $I = \operatorname{Ker}\theta$. If m is sufficiently large that $J^m = 0$, then $\theta(KQ_+^m) = U^m = 0$, so $KQ_+^m \subseteq I$. Suppose $x \in I$. Write it as $x = u + v + w$ where u is a linear combination of e_i's, v is a linear combination of arrows, and w is in KQ_+^2. Since $\theta(e_i) = e_i$ and $\theta(v), \theta(w) \in J$, we must have $u = 0$. Now $\theta(v) = -\theta(w) \in J^2$, so that $\theta(v)$ induces the zero element of J/J^2. Thus $v = 0$. Thus $x = w \in KQ_+^2$.

1.8 Homological algebra for finite-dimensional algebras

Let A be a finite-dimensional algebra.

Definition. A projective resolution

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

is *minimal* if at each stage, the map $\epsilon : P_0 \to M$, $d_1 : P_1 \to \operatorname{Ker}(\epsilon)$, $d_2 : P_2 \to \operatorname{Ker}(d_1)$ and so on, is a projective cover. Dually for an injective resolution

$$0 \to M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d_1} I^2 \to \dots,$$

the maps $\epsilon : M \to I^0$, $I^0/\operatorname{Im}(\epsilon) \to I^1$, $I^1/\operatorname{Im}(d^0) \to I^2$ and so on must be injective envelopes.

The minimal projective and injective resolutions of M are unique up to (non-unique) isomorphism.

Lemma 1. $\dim \operatorname{Ext}^k(S[i], M)$ is equal to $\dim D_i$ times the multiplicity of $I[i]$ as a summand of I^k in the minimal injective resolution of M.

$\dim \operatorname{Ext}^k(M, S[j])$ is equal to $\dim D_j$ times the multiplicity of $P[j]$ as a summand of P_k in the minimal projective resolution of M.

Proof. By minimality, any element of $\operatorname{soc} I^i$ is in the image of the map $I^{i-1} \to I^i$, so is killed by the map $I^i \to I^{i+1}$. Thus in the complex $\operatorname{Hom}(S[i], I^*)$, the differential is zero.

Example. For the oriented cycle of example (5) in the last section, the simple modules have minimal projective resolutions

$$\begin{aligned}
0 \to P[1] \to P[4] \to P[2] \to P[1] \to & S[1] \to 0, \\
0 \to P[3] \to P[2] \to & S[2] \to 0, \\
0 \to P[1] \to P[4] \to P[3] \to & S[3] \to 0, \\
0 \to P[1] \to P[4] \to & S[4] \to 0.
\end{aligned}$$

For example the projective cover of $S[1]$ is $P[1]$, giving an exact sequence

$$0 \to \Omega^1 S[1] \to P[1] \to S[1] \to 0$$

which is

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & K \arrow[r] \arrow[d, "1"] & K \arrow[r, "1"] & K \arrow[r] \arrow[d, "1"] & K \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[u] & K \arrow[l] & 0 \arrow[u] & K \arrow[l] & 0 \arrow[u] & 0 \arrow[l] &
\end{tikzcd}$$

and the projective cover of $\Omega^1 S[1]$ is $P[2]$, giving an exact sequence

$$0 \to \Omega^2 S[1] \to P[2] \to \Omega^1 S[1] \to 0$$

which is

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & K \arrow[r] \arrow[d, "1"] & 0 \arrow[r] & K \arrow[r] \arrow[d, "1"] & 0 \\
& K \arrow[u] & 0 \arrow[l] & K \arrow[u] & K \arrow[l, "1"] & 0 \arrow[u] & K \arrow[l] &
\end{tikzcd}$$

so $\Omega^2 S[1] \cong S[3]$, etc.

Lemma 2. If $A = KQ/I$ with I admissible, then the number of arrows from i to j is $\dim \mathrm{Ext}^1(S[i], S[j])$.

Proof. Since I is admissible, $I \subseteq (KQ)_+^2$. Now $P[i] = (KQ/I)e_i$, so $\mathrm{rad}\, P[i] = ((KQ)_+/I)e_i$, and $\mathrm{rad}\,\mathrm{rad}\, P[i] = ((KQ)_+^2/I)e_i$. Thus the

$$\text{top of } \mathrm{rad}\, P[i] \cong ((KQ)_+/(KQ)_+^2))e_i \cong \bigoplus_j S[j]^{n_{ij}}$$

where n_{ij} is the number of arrows from i to j. Then in the minimal projective resolution of $S[i]$,

$$\cdots \to P_1 \to P[i] \to S[i] \to 0$$

P_1 is the projective cover of $\mathrm{rad}\, P[i]$, so also of top of $\mathrm{rad}\, P[i]$, so the multiplicity of $P[j]$ is n_{ij}. Thus $\dim \mathrm{Ext}^1(S[i], S[j]) = n_{ij}$.

Lemma 3. The following are equivalent for a module M
(i) $\mathrm{proj.\,dim}\, M \leq n$
(ii) $\mathrm{Ext}^{n+1}(M, S) = 0$ for all simples S.
(iii) the minimal projective resolution of M has $P_k = 0$ for $k > n$.
Similarly for the injective dimension.

Proof. (i) implies (ii) is clear.

(ii) implies (iii). By the lemma above, the minimal projective resolution of M has $P_{n+1} = 0$.

(iii) implies (i). Trivial.

Proposition. The global dimension of a f.d. algebra is the supremum of the projective dimensions of its simple modules.

Proof. If every simple S has a projective resolution of length $\leq n$, then every semisimple module has a projective resolution of length $\leq n$, so every semisimple module has projective dimension $\leq n$. Now every module X has a filtration $X \supseteq J(A)X \supseteq \cdots \supseteq J(A)^N X = 0$ in which the quotients are semisimple, and the long exact sequence shows that an extension of modules of projective dimension $\leq n$ again has projective dimension $\leq n$.

Corollary. For a f.d. algebra, the left and right global dimensions are the same.

Proof. If the right global dimension is $\leq n$, then the simple right modules have injective resolutions of length $\leq n$. Dualizing, the simple left modules have projective resolutions of length $\leq n$. Thus the left global dimension is $\leq n$.

Theorem. If A is a f.d. hereditary algebra and $A/J(A) \cong K \times \cdots \times K$ (for example if A is basic and K is algebraically closed), then A is isomorphic to a path algebra KQ.

Proof. The algebra can be given as $A = KQ/I$ with I admissible. Consider the exact sequence of KQ-modules

$$0 \to I/(I.KQ_+) \to KQ_+/(I.KQ_+) \to KQ_+/I \to 0.$$

The middle module is annihilated by I, so this is a sequence of A-modules. The RH module is a submodule of $A = KQ/I$, so it is projective as an A-module. Thus the sequence splits. Letting

$$M = KQ_+/(I.KQ_+), \quad N = I/(I.KQ_+) \oplus KQ_+/I.$$

we deduce that $M \cong N$. Thus $M/(KQ_+)M \cong N/(KQ_+)N$, which gives

$$KQ_+/KQ_+^2 \cong (I/(KQ_+.I + I.KQ_+)) \oplus (KQ_+/KQ_+^2).$$

Thus by dimensions, $I = KQ_+.I + I.KQ_+$. Now if $I \neq 0$ there is a maximal k such that $I \subseteq (KQ)_+^k$. But then $I = KQ_+.I + I.KQ_+ \subseteq (KQ)_+^{k+1}$, a contradiction.

1.9 Some homological properties and conjectures for f.d. algebras

Definitions. (i) An algebra is *self-injective* if ${}_AA$ is an injective module. Equivalently the modules $P[i]$ and $I[j]$ are the same, up to a permutation. This is left-right symmetric.

(ii) A *Frobenius* algebra is an algebra A with a bilinear form

$$(-,-) : A \times A \to K$$

which is non-degenerate and associative, that is, $(ab, c) = (a, bc)$ for all $a, b, c \in A$. The form defines an isomorphism ${}_AA \to {}_ADA$, $a \mapsto (-, a)$, so A is injective. Conversely, any such isomorphism gives a suitable bilinear form.

(iii) A is a *symmetric* algebra if also $(a, b) = (b, a)$. In this case the form defines an isomorphism ${}_AA_A \to {}_ADA_A$, and conversely. For a symmetric algebra we have $I[i] = \nu(P[i]) = DA \otimes_A P[i] \cong A \otimes_A P[i] \cong P[i]$.

Examples. (1) The group algebra $A = KG$ of a finite group is symmetric with $(a, b) = \lambda_1$ where $ab = \sum_{g \in G} \lambda_g g$.

(2) The path algebra of a cyclic quiver with relation that all paths of length k equal to 0 is Frobenius. For example the oriented cycle with vertices 1,2 and arrows $a : 1 \to 2$ and $b : 2 \to 1$ with relations ab, ba. The indecomposable projectives and injectives all have dimension 2.

(3) For a commutative algebra the three concepts are the same (for (i) and (ii), since the algebra is basic). Commutative Frobenius algebras appear in topological quantum field theory.

Lemma 1. If $\operatorname{inj.dim} {}_AA = n$, any A-module has $\operatorname{proj.dim} M \leq n$ or ∞.

For example, every non-projective module for a self-injective algebra has infinite projective dimension.

Proof. Say $\operatorname{proj.dim} M = i < \infty$. There is some N with $\operatorname{Ext}^i(M, N) \neq 0$. Choose $0 \to L \to P \to N \to 0$ with P projective. The long exact sequence for $\operatorname{Hom}(M, -)$ gives

$$\cdots \to \operatorname{Ext}^i(M, P) \to \operatorname{Ext}^i(M, N) \to \operatorname{Ext}^{i+1}(M, L) \to \ldots$$

Now $\operatorname{Ext}^{i+1}(M, L) = 0$, so $\operatorname{Ext}^i(M, P) \neq 0$, so $\operatorname{Ext}^i(M, A) \neq 0$, so $i \leq n$.

Definition. An algebra A is *(Iwanaga) Gorenstein* if $\operatorname{inj.dim} {}_AA < \infty$ and $\operatorname{inj.dim} A_A < \infty$.

Conjecture (see Auslander and Reiten, Applications of contravariantly finite subcategories, Adv. Math 1991). If one is finite, so is the other.

Theorem. If $\operatorname{inj.dim} {}_AA = r$ and $\operatorname{inj.dim} A_A = s$ are both finite, they are equal.

Proof. $\operatorname{proj.dim} {}_ADA = \operatorname{inj.dim} A_A = s$, so $s \leq r$ by Lemma 1. Dually $s \geq r$.

Also true for noetherian rings (Zaks, Injective dimension of semi-primary rings, J. Alg. 1969).

Definition. A module M is *faithful* if $am = 0$ for all $m \in M$ implies $a = 0$, that is, if the map $A \to \mathrm{End}_K(M)$ is injective.

A f.d. A-module M is faithful if and only if there is an embedding $A \to M^n$ for some n. Namely, if $A \hookrightarrow M^n$, $a \in A$ and $am = 0$ for all $m \in M$, then $ax = 0$ for all $x \in M^n$, so $a1 = 0$ for $1 \in A$. Thus $a = 0$. Conversely, if M is faithful, choose a basis $m_1, \dots, m_n$ of M. This gives a map $A \to M^n$, $a \mapsto (am_1, \dots, am_n)$. If $a \mapsto 0$, then $am_i = 0$ for all i, so $am = 0$ for all $m \in M$.

Definition. An algebra is *QF-3* (in the sense of Thrall) if it has a faithful projective-injective module.

Examples. Any self-injective algebra is QF-3.

The commutative square algebra is QF-3, because the socle of any indecomposable projective is the simple projective module S, so the algebra embeds in a direct sum of copies of the injective envelope I of S, but this is also projective.

Remark. If A is QF-3 then the faithful projective-injective module M is unique (up to multiplicities). It must be a direct sum of indecomposable projective-injective modules, and every indecomposable projective-injective module I must occur, because $I \hookrightarrow A \hookrightarrow M^n$. Since I is injective, it is a direct summand of M^n, hence by Krull-Remak-Schmidt, I is a summand of M.

Definition. Let $0 \to A \to I^0 \to I^1 \to \dots$ be the minimal injective resolution of a f.d. algebra A. One says that A has *dominant dimension* $\geq n$ if $I^0, \dots, I^{n-1}$ are all projective.

Special case. $\mathrm{dom.\,dim}\, A \geq 1$ iff A is QF-3. For example if it is QF-3, with faithful projective-injective module M, then there is an embedding $A \to M^n$, and then the injective envelope of A is a direct summand of M^n, so it is projective.

Nakayama conjecture (1958). If all I^n are projective, i.e. $\mathrm{dom.\,dim}\, A = \infty$, then A is self-injective.

Generalized Nakayama conjecture (Auslander and Reiten 1975). For any f.d. algebra A, every indecomposable injective occur as a summand of some I^n.

It clearly implies the Nakayama conjecture, for if the I^n are projective, and each indecomposable injective occurs as a summand of some I^n, then the indecomposable injectives are projective.

Example. For the commutative square, vertices 1(source),2,3,4(sink). There are injective resolutions

$$\begin{aligned}
0 \to & P[1] \to I[4] \to 0, \\
0 \to & P[2] \to I[4] \to I[3] \to 0, \\
0 \to & P[3] \to I[4] \to I[2] \to 0, \\
0 \to & P[4] \to I[4] \to I[2] \oplus I[3] \to I[1] \to 0,
\end{aligned}$$

so

$$0 \to A \to I[4]^4 \to I[2]^2 \oplus I[3]^2 \to I[1] \to 0,$$

so all indecomposable injectives occur.

Finitistic Dimension Conjecture (see H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 1960) For any f.d. algebra A,

$$\operatorname{fin.dim} A = \sup\{\operatorname{proj.dim} M \mid \operatorname{proj.dim} M < \infty\}$$

is finite.

Note that $\operatorname{fin.dim} A$ is not necessarily the same as the maximum of the projective dimensions of the simple modules of finite projective dimension.

Lemma 2. If $\operatorname{inj.dim} A_A = n < \infty$ then $\operatorname{fin.dim} A < \infty$ implies A Gorenstein implies $\operatorname{fin.dim} A = n$.

Proof. We have $\operatorname{proj.dim}{}_A DA = n < \infty$. Thus any injective module has projective dimension $< \infty$. Take a minimal injective resolution of $0 \to {}_A A \to I^0 \to \dots$. We show by induction on i that $\operatorname{proj.dim} \Omega^{-i} A < \infty$. There is an exact sequence

$$0 \to \Omega^{-(i-1)} A \to I^{i-1} \to \Omega^{-i} A \to 0.$$

Applying $\operatorname{Hom}_A(-, X)$ for a module X gives a long exact sequence

$$\dots \to \operatorname{Ext}^m(\Omega^{-(i-1)} A, X) \to \operatorname{Ext}^{m+1}(\Omega^{-i} A, X) \to \operatorname{Ext}^{m+1}(I^{i-1}, X) \to \dots$$

For m sufficiently large the outside terms are zero, hence so is the middle.

Suppose $\Omega^{-i} A \neq 0$. Let $f : \Omega^{-i} A \to I^i$ be the inclusion. Then f belongs to the middle term in the complex

$$\operatorname{Hom}(\Omega^{-i} A, I^{i-1}) \to \operatorname{Hom}(\Omega^{-i} A, I^i) \to \operatorname{Hom}(\Omega^{-i} A, I^{i+1})$$

and it is sent to zero in the third term. Now f is not in the image of the map from the first term, for otherwise the map $I^{i-1} \to \Omega^{-i}A$ is a split epimorphism, so the inclusion $\Omega^{-(i-1)}A \to I^i$ is a split monomorphism. But the resolution was minimal, so the inclusion is an isomorphism, and $\Omega^{-i}A = 0$. Thus $\mathrm{Ext}^i(\Omega^{-i}AA) \neq 0$. Thus $\mathrm{proj.dim}\,\Omega^{-i}A \geq i$. Thus $\mathrm{fin.dim}\,A = \infty$.

For the second implication, the theorem and Lemma 1 gives $\mathrm{fin.dim}\,A \leq n$, and consideration of DA gives equality.

Lemma 3. The finitistic dimension conjecture implies the Nakayama conjecture.

Proof. Suppose all I^n are projective. Then the syzygies $\Omega^{-n}A$ have finite projective dimension. In particular if the finitistic dimension is f, then $M = \Omega^{-(f+1)}A$ has projective dimension $\leq f$, so $\Omega^f M \cong \Omega^{-1}A$ is projective, so the sequence $0 \to A \to I^0 \to \Omega^{-1}A \to 0$ splits, so A is a summand of I^0, so injective.

1.10 Generator correspondence

I discuss what I call 'Generator correspondence'. I learnt it from Sauter and Pressland (work in progress). They attribute it to Kato and Tachikawa. It generalizes Morita-Tachikawa correspondence, see Ringel. It is essentially equivalent to Auslander's Wedderburn correspondence.

T. Kato, Rings of U-dominant dimension ≥ 1, Tohoku Math. J. 1969.
H. Tachikawa, On splitting of module categories, Math. Z. 1969
M. Auslander, Representation theory of Artin algebras. I, Comm. Alg. 1974.
C. M. Ringel, Artin algebras of dominant dimension at least 2, manuscript 2007, available from his Bielefeld homepage.

We consider pairs $(A, {}_AM)$ consisting of an algebra and a module (both f.d.), up to an equivalance which identifies $(A, {}_A, M)$ with $(A', {}_{A'}M')$ whenever there is an equivalence of categories $A - mod \to A' - mod$ under which $\mathrm{add}\,M$ and $\mathrm{add}\,M'$ correspond.

Given a pair (A, M) we construct a new pair $(\mathrm{End}_A(M)^{op}, DM)$. We call it the *dual pair*.

Lemma. If $(A, {}_AM)$ and $(A', {}_{A'}M')$ are equivalent, then so are $(\mathrm{End}_A(M)^{op}, DM)$ and $(\mathrm{End}_{A'}(M')^{op}, D(M'))$.

The proof is an exercise in Morita equivalence. I couldn't find an elegant argument, so I omit it.

Applying duality twice you recover the same pair (up to equivalence) provided that the map $A \to \mathrm{End}_{\mathrm{End}_A(M)^{op}}(DM)$ is an isomorphism.

Recall that an A-module M is a *generator* if for every module X there is a surjection $M^{(I)} \to X$. It is equivalent that $A \in \mathrm{add}\, M$. Dually M is a *cogenerator* if this is always an injection $X \to M^I$. For A, M finite-dimensional it is equivalent that $DA \in \mathrm{add}(M)$.

Given an injective B-module U, we say that a B-module Z has $U-\mathrm{dom.\,dim}\, Z \geq n$ provided there is an exact sequence

$$0 \to Z \to U^0 \to U^1 \to \cdots \to U^{n-1}$$

with $U^i \in add(U)$.

Theorem (Generator correspondence). The construction of dual pairs gives a 1-1 correspondence between

(a) equivalence classes of pairs $(A, {}_AM)$ where M is a generator, and

(b) equivalence classes of pairs $(B, {}_BU)$ where U is injective and $U-\mathrm{dom.\,dim}\, B \geq 2$.

Moreover, in this case there are inverse equivalences

$$A-\mathrm{mod} \underset{M\otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \text{Category of } B\text{-modules with } U-\mathrm{dom.\,dim} \geq 2.$$

Proof. (1). Suppose (A, M) satisfies (a) and $B = \mathrm{End}_A(M)^{op}$ and $U = {}_BDM$ are the dual pair. We need to show that U is injective, $U - \mathrm{dom.\,dim}\, B \geq 2$ and the natural map $A \to \mathrm{End}_B(U)^{op}$ is an isomorphism.

Now $A \in \mathrm{add}(M)$, so $M_B \cong \mathrm{Hom}_A(A, M) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}(B)$, so $DM \in \mathrm{add}(DB)$ is injective.

Now M has an injective resolution starting $0 \to M \to I^0 \to I^1$. Apply $\mathrm{Hom}_A(M, -)$ to get an exact sequence of left B-modules $0 \to B \to \mathrm{Hom}_A(M, I^0) \to \mathrm{Hom}_A(M, I^1)$. Now $I^i \in \mathrm{add}(DA)$, so $\mathrm{Hom}_A(M, I^i) \in \mathrm{add}(\mathrm{Hom}_A(M, DA)) = \mathrm{add}(DM)$.

For X, Y A-modules there is a map

$$\mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_B(\mathrm{Hom}_A(Y,M), \mathrm{Hom}_A(X,M))$$

This is a functorial in X and Y and an isomorphism for $Y = M$, so also for $Y \in \mathrm{add}(M)$, so for $Y = A$. Taking also $X = A$ this gives an isomorphism $A \to \mathrm{Hom}_B(M,M) = \mathrm{End}_B(M) \cong \mathrm{End}_B(U)^{op}$.

(2) Suppose (B,U) satisfies (b) and $A = \mathrm{End}_B(U)^{op}$ and $M = {}_ADU$ are the dual pair. We need to show that M is a generator and the natural map $B \to \mathrm{End}_A(M)^{op}$ is an isomorphism.

We first show the inverse equivalences. Clearly M is an A-B-bimodule and it is projective as a right B-module.

As a right A-module

$$D(M \otimes_B U) = \mathrm{Hom}_K(M \otimes_B U, K) \cong \mathrm{Hom}_B(U, \mathrm{Hom}_K(M,K)) \cong \mathrm{Hom}_B(U,U) = A$$

so $M \otimes U \cong DA$.

For any B-module X there is a natural transformation $\phi_X : X \to \mathrm{Hom}_A(M, M \otimes_B X)$, $\phi_X(x)(m) = m \otimes x$. This map is an isomorphism for $X = U$ since

$$\mathrm{Hom}_A(M, M \otimes_B U) \cong \mathrm{Hom}_A(M, DA) \cong DM \cong U.$$

Thus ϕ_X is an isomorphism for $X \in \mathrm{add}(U)$. Given a B-module ${}_BX$ with $U - \mathrm{dom.\,dim}\, X \geq 2$, we have an exact sequence

$$0 \to X \to U^0 \to U^1$$

with $U^i \in \mathrm{add}(U)$. Since M_B is projective, this gives an exact sequence

$$0 \to M \otimes_B X \to M \otimes_B U^0 \to M \otimes_B U^1.$$

This is a sequence of A-modules, and applying $\mathrm{Hom}_A(M, -)$ it gives a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\phi_X"'] & U^0 \arrow[r] \arrow[d, "\phi_{U^0}"'] & U^1 \arrow[d, "\phi_{U^1}"'] \\
0 \arrow[r] & \mathrm{Hom}_A(M, M \otimes_B X) \arrow[r] & \mathrm{Hom}_A(M, M \otimes_B U^0) \arrow[r] & \mathrm{Hom}_A(M, M \otimes_B U^1).
\end{tikzcd}$$

Since ϕ_{U^0} and ϕ_{U^1} are isomorphisms, so is ϕ_X. Thus starting from X, the composition of the two functors recovers X.

Given any A-module Y, there is a natural transformation $\psi_Y : M \otimes_B \mathrm{Hom}_A(M,Y) \to Y$, $\psi_Y(m \otimes \theta) = \theta(m)$. This map is an isomorphism for $Y = DA$ since $M \otimes_B U \cong DA$. Thus ψ_Y is an isomorphism for Y an injective A-module. For general Y, choose an injective resolution

$$0 \to Y \to I^0 \to I^1.$$

Applying $\mathrm{Hom}_A(M,-)$ gives an exact sequence

$$0 \to \mathrm{Hom}_A(M,Y) \to \mathrm{Hom}_A(M,I^0) \to \mathrm{Hom}_A(M,I^1)$$

and $\mathrm{Hom}(A,I^i) \in \mathrm{add}(\mathrm{Hom}_A(M,DA)) = \mathrm{add}(DM) = \mathrm{add}(U)$, so $U-\mathrm{dom.dim}\,\mathrm{Hom}_A(M,Y) \geq 2$. Also one gets a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_B \mathrm{Hom}_A(M,Y) & \longrightarrow & M \otimes_B \mathrm{Hom}_A(M,I^0) & \longrightarrow & M \otimes_B \mathrm{Hom}_A(M,I^1) \\
 & & {\scriptstyle \psi_Y}\big\downarrow & & {\scriptstyle \psi_{I^0}}\big\downarrow & & {\scriptstyle \psi_{I^1}}\big\downarrow \\
0 & \longrightarrow & Y & \longrightarrow & I^0 & \longrightarrow & I^1
\end{array}$$

Since ψ_{I^0} and ψ_{I^1} are isomorphisms, so is ψ_Y. Thus starting from Y, the composition of the two functors recovers Y.

Finally, since $U - \mathrm{dom.dim}\, B \geq 2$ we can take $Y = B$, and we get

$$B \cong \mathrm{End}({}_B B)^{op} \cong \mathrm{End}_A(M \otimes_B B)^{op} \cong \mathrm{End}_A(M)^{op}.$$

Also $U \in \mathrm{add}({}_B DB)$, so

$$A = \mathrm{Hom}_B(U,U) \in \mathrm{add}(\mathrm{Hom}_B(U,DB)) = \mathrm{add}(DU) = \mathrm{add}(M)$$

so M is a generator.

Special cases. Suppose (A,M) corresponds to (B,U) under the generator correspondence.

(i) (Morita equivalence) M is also projective iff U is also a cogenerator

If M is projective then $B = \mathrm{Hom}_A(M,M) \in \mathrm{add}(\mathrm{Hom}_A(A,M)) = \mathrm{add}(M_B)$.

If U is a cogenerator then $B_B \in \mathrm{add}(DU) = \mathrm{add}(M)$, so $M = \mathrm{Hom}_B(B,M) \in \mathrm{add}(\mathrm{Hom}_B(M,M)) = \mathrm{add}(A)$.

(ii) (Morita-Tachikawa correspondence) M is also a cogenerator iff U is also projective.

If $DA \in \text{add}(M)$, then $U \cong \text{Hom}_A(M, DA) \in \text{add}(\text{Hom}_A(M, M)) = \text{add}(B)$.

If $U \in \text{add}(B)$ then $A_A = \text{Hom}_B(U, U) \in \text{add}(\text{Hom}_B(B, U)) = \text{add}(U) = \text{add}(DM)$, so $DA \in \text{add}(M)$

Note that in this case B is QF-3, and U is uniquely determined up to equivalence as the direct sum of all indecomposable projective-injective modules (with nonzero multiplicities). Thus in this case we get a 1-1 correspondence

$$\frac{\text{Pairs } (A, M) \text{ with } M \text{ gen-cogen}}{\text{equiv}} \leftrightarrow \frac{\text{Algebras } B \text{ with dom.dim } B \geq 2}{\text{Morita equiv}}.$$

(iii) (Müller, The classification of algebras by dominant dimension, Canad. J. Math 1968) $\text{Ext}^i_A(M, M) = 0$ for $1 \leq i < n$ iff $U - \text{dom.dim } B \geq n + 1$.

If the condition on Exts holds, then an injective resolution $0 \to M \to I^0 \to \dots$ gives a complex

$$0 \to \text{Hom}_A(M, M) \to \text{Hom}_A(M, I^0) \to \dots \to \text{Hom}_A(M, I^n)$$

which is in fact exact. Now as a left B-module we have $\text{Hom}_A(M, I^i) \in \text{add}(\text{Hom}_A(M, DA)) = \text{add}(U)$, so this sequence shows that $B = \text{End}_A(M)$ has $U = \text{dom.dim} \geq n + 1$.

Conversely suppose $0 \to B \to U^0 \to \dots \to U^n$ is an exact sequence of B-modules with $U^i \in \text{add}(U)$. Since M_B is projective we get an exact sequence

$$0 \to M \to M \otimes_B U^0 \to \dots \to M \otimes_B U^n.$$

Now $M \otimes_B U^i \in \text{add}(M \otimes_B U) = \text{add}(DA)$, so this is the start of an injective resolution of M. Thus the cohomology of the complex

$$\text{Hom}_A(M, M \otimes_B U^0) \to \dots \to \text{Hom}_A(M, M \otimes_B U^n)$$

at the term with U^i is $\text{Ext}^i_A(M, M)$. But, as before, we have $U^i \cong \text{Hom}_A(M, M \otimes_B U^i)$, so this complex is exact. Thus $\text{Ext}^i_A(M, M) = 0$ for $1 \leq i < n$.

Proposition 1. The following are equivalent.
(i) The Nakayama conjecture (if $\text{dom.dim } B = \infty$ then B is self-injective).
(ii) If ${}_AM$ is a generator-cogenerator and $\text{Ext}^i_A(M, M) = 0$ for all $i > 0$ then M is projective.

Proof (i) implies (ii). Say M satisfies the hypotheses. There is a corresponding (B, U) with U faithful projective-injective and $\text{dom.dim } B = \infty$.

Thus B is self-injective, so $\mathrm{add}(U) = \mathrm{add}(B)$, so U is a cogenerator, so B is projective.

(ii) implies (i). Say $\mathrm{dom.dim}\, B = \infty$. Thus B is QF-3 and let U be the faithful projective-injective module. Then there is corresponding (A, M) with M a generator-cogenerator and Exts vanish. Thus M is projective, so U is a cogenerator, so involves all indecomposable injectives, so they are all projective.

Proposition 2. The following are equivalent.
(i) The Generalized Nakayama Conjecture (every indecomposable injective occurs as a summand of some I^i in the minimal injective resolution of B).
(ii) If ${}_AM$ is a generator and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i > 0$ then M is projective.

Proof. (i) implies (ii). Suppose M satisfies the conditions, then there is corresponding (B, U) and $U - \mathrm{dom.dim}\, B = \infty$. Thus by (i), U must be a cogenerator, so M is projective.

(ii) implies (i). Let U be the sum of all indecomposable injectives occuring in the I^i. Then there is corresponding (A, M) and Exts vanish for M. Thus M is projective, so U is a cogenerator, so all indecomposable injectives occur as a summand of U.

Boundedness Conjecture (Happel, Selforthogonal modules, 1995). If M is an A-module with $\mathrm{Ext}^n_A(M, M) = 0$ for all $n > 0$ then $\#M \leq \#A$, where $\#M$ denotes the number of non-isomorphic indecomposable summands of M.

This implies the GNC.

Special cases continued.

(iv) (Auslander, 1974) $\mathrm{add}(M) = A - \mathrm{mod}$ iff $\mathrm{gl.dim}\, B \leq 2 \leq \mathrm{dom.dim}\, B$ and U is the faithful projective-injective.

For the condition on the left to be possible A must have *finite representation type*, that is, only finitely many indecomposables. Then M is uniquely determined up to multiplicities. Thus one gets a 1-1 correspondence

$$\frac{\text{Algebras } A \text{ of finite representation type}}{\text{Morita equiv}} \leftrightarrow \frac{\text{Algebras } B \text{ with gl.dim}\, B \leq 2 \leq \text{dom.dim}\, B}{\text{Morita equiv}}.$$

The algebra B is called the *Auslander algebra* of A.

Suppose $\mathrm{add}(M) = A - \mathrm{mod}$. Given a B-module Z, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Since M_B is projective, tensoring with M gives an exact sequence

$$0 \to M \otimes_B \mathrm{Ker}(f) \to M \otimes_B P_1 \to M \times_B P_0 \to M \otimes_B Z \to 0.$$

Applying $\mathrm{Hom}_A(M, -)$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & P_1 & \longrightarrow & P_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(M, M \otimes \mathrm{Ker}(f)) & \longrightarrow & \mathrm{Hom}(M, M \otimes P_1) & \longrightarrow & \mathrm{Hom}(M, M \otimes P_0)
\end{array}$$

The two vertical maps on the right are isomorphisms, hence so is the first. Now $M \otimes \mathrm{Ker}(f)$ is an A-module, so in $\mathrm{add}(M)$, so $\mathrm{Ker}(f) \cong \mathrm{Hom}_A(M, M \otimes \mathrm{Ker}(f)) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}({}_B B)$, so it is projective. Thus $\mathrm{proj.\,dim}\, Z \leq 2$. Thus $\mathrm{gl.\,dim}\, B \leq 2$.

Conversely suppose $\mathrm{gl.\,dim}\, B \leq 2 \leq \mathrm{dom.\,dim}\, B$. If Y is an A-module, it has an injective coresolution starting

$$0 \to Y \to I^0 \to I^1.$$

Applying $\mathrm{Hom}_A(M, -)$ we get an exact sequence of B-modules

$$0 \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, I^0) \xrightarrow{g} \mathrm{Hom}_A(M, I^1) \to \mathrm{Coker}(g) \to 0$$

and $\mathrm{Hom}_A(M, I^i) \in \mathrm{add}(\mathrm{Hom}_A(M, DA)) = \mathrm{add}(DM) = \mathrm{add}(U)$. Now ${}_B U$ is projective and $\mathrm{proj.\,dim}\, \mathrm{Coker}(g) \leq 2$, so $\mathrm{Hom}_A(M, Y)$ is a projective B-module. Then $Y \cong M \otimes_B \mathrm{Hom}_A(M, Y) \in \mathrm{add}(M)$.

(v) (Iyama, 2007) M is a 'n-cluster tilting module' iff $\mathrm{gl.\,dim}\, B \leq n + 1 \leq \mathrm{dom.\,dim}\, B$.

Details omitted.

1.11 No loops conjecture

No Loops Conjecture (Proved by Igusa 1990, based on Lenzing 1969). If $A = KQ/I$, I admissible, and A has finite global dimension then Q has no loops (that is, $\mathrm{Ext}^1(S[i], S[i]) = 0$ for all i).

Proof. We use the trace function of Hattori and Stallings. I only sketch the proof of its properties.

(1) For any matrix $\theta \in M_n(A)$ we consider its trace $\operatorname{tr}(\theta) \in A/[A,A]$, where $[A,A]$ is the subspace of A spanned by the commutators $ab-ba$. This ensures that $\operatorname{tr}(\theta\phi) = \operatorname{tr}(\phi\theta)$. This works also for matrices of size $m \times n$ and $n \times m$.

(2) If P is a f.g. projective A-module it is a direct summand of a f.g. free module $F = A^n$. Let $p : F \to P$ and $i : P \to F$ be the projection and inclusion. One defines $\operatorname{tr}(\theta)$ for $\theta \in \operatorname{End}(P)$ to be $\operatorname{tr}(i\theta p)$. This is well defined, for if

$$A^n = F' \underset{i}{\overset{p}{\rightleftarrows}} P \underset{p'}{\overset{i'}{\rightleftarrows}} F' = A^m$$

then $\operatorname{tr}(i\theta p) = \operatorname{tr}((ip')(i'\theta p)) = \operatorname{tr}((i'\theta p)(ip')) = \operatorname{tr}(i'\theta p')$.

(3) Any module M has a finite projective resolution $P_* \to M$, and an endomorphism θ of M lifts to a map between the projective resolutions

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle\theta_n}\big\downarrow & & & & {\scriptstyle\theta_1}\big\downarrow & & {\scriptstyle\theta_0}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & \\
0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Define $\operatorname{tr}(\theta) = \sum_i (-1)^i \operatorname{tr}(\theta_i)$. One can show that does not depend on the projective resolution or the lift of θ.

(4) One can show that given a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle\theta'}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & {\scriptstyle\theta''}\big\downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

one has $\operatorname{tr}(\theta) = \operatorname{tr}(\theta') + \operatorname{tr}(\theta'')$.

(5) It follows that any nilpotent endomorphism has trace 0, since

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}\theta & \longrightarrow & M & \longrightarrow & M/\operatorname{Im}\theta & \longrightarrow & 0 \\
 & & {\scriptstyle\theta|_{\operatorname{Im}\theta}}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & \\
0 & \longrightarrow & \operatorname{Im}\theta & \longrightarrow & M & \longrightarrow & M/\operatorname{Im}\theta & \longrightarrow & 0
\end{array}$$

so $\operatorname{tr}(\theta) = \operatorname{tr}(\theta|_{\operatorname{Im}\theta}) = \operatorname{tr}(\theta|_{\operatorname{Im}(\theta^2)}) = \cdots = 0$.

(6) Thus any element of $J(A)$ as a map $A \to A$ has trace 0, so $J(A) \subseteq [A, A]$. Thus $(KQ)_+ \subseteq I + [KQ, KQ]$.

(7) Any loop of Q gives an element of $(KQ)_+$. But it is easy to see that

$$I + [KQ, KQ] \subseteq \text{span of arrows which are not loops} + (KQ)_+^2,$$

for example if p, q are paths then $[p, q] \in (KQ)_+^2$ unless they are trivial paths or one is trivial and the other is an arrow. Thus there are no loops.

Strong no loops conjecture (proved by Igusa, Liu, Paquette 2011). If S is a 1-dimensional simple module for a f.d. algebra and S has finite injective or projective dimension, then $\operatorname{Ext}^1(S, S) = 0$.

Extension Conjecture (Liu, Morin). If S is simple module for a f.d. algebra and $\operatorname{Ext}^1(S, S) \neq 0$ then $\operatorname{Ext}^n(S, S) \neq 0$ for infinitely many n.

2 Auslander-Reiten Theory

From now on we work only with f.d. algebras, over an alg. closed field, and all modules are f.d.

2.1 The transpose

We write $M^\vee$ for $\mathrm{Hom}_A(M, A)$. This defines a functor from modules on one side to modules on the other side. It gives an antiequivalence between the categories of finitely generated projective left and right A-modules.

Given a left (or right) module M, we fix a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0.$$

That is, $g : P_0 \to M$ and $f : P_1 \to \mathrm{Ker}(g)$ are projective covers. The *transpose* $\mathrm{Tr}\, M$ is the cokernel of the map $f^\vee : P_0^\vee \to P_1^\vee$. It is a module on the other side. Thus there is an exact sequence

$$0 \to M^\vee \to P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$$

Note that Tr doesn't define a functor on the module categories.

Theorem.
(i) Up to isomorphism, $\mathrm{Tr}\, M$ doesn't depend on the choice of minimal projective presentation of M.
(ii) If P is projective, then $\mathrm{Tr}\, P = 0$.
(iii) $\mathrm{Tr}(M \oplus N) \cong \mathrm{Tr}\, M \oplus \mathrm{Tr}\, N$.
(iv) If M has no nonzero projective summand, the same is true for $\mathrm{Tr}\, M$, and $P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$ is a minimal projective presentation.
(v) If M has no nonzero projective summand then $\mathrm{Tr}\,\mathrm{Tr}\, M \cong M$.

Proof. Two different minimal projective presentations of M fit in a commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d] & P_0 \arrow[r, "g"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
P_1' \arrow[r, "f'"] & P_0' \arrow[r, "g'"] & M \arrow[r] & 0
\end{tikzcd}$$

and the minimality ensures that the vertical maps are isomorphisms. Applying $(-)^\vee$, one sees that the two different constructions of $\mathrm{tr}\, M$ are isomorphic.

(ii) is clear.

(iii) Straightforward since the direct sum of minimal projective presentations of M and N gives a minimal projective presentation of $M \oplus N$.

(iv) Suppose Q is a non-zero projective summand of $\operatorname{Tr} M$. Then there is a split epi $P_1^\vee \to Q$ whose composition with $f^\vee$ is zero. Thus there is a split mono $Q^\vee \to P_1$ whose composition with f is zero. Contradicts that $P_1 \to \operatorname{Ker}(g)$ is a projective cover.

Suppose $P_1^\vee \to \operatorname{Tr} M$ is not a projective cover. Then there is a non-zero summand Q of $P_1^\vee$ with image zero in $\operatorname{Tr} M$. This gives a map $Q \to \operatorname{Im}(f^\vee)$. Since Q is projective and $P_0^\vee \to \operatorname{Im}(f^\vee)$ is onto, we get a map $Q \to P_0^\vee$ whose composition with $f^\vee$ is the inclusion of Q in $P_1^\vee$. Thus f composed with the map $P_0 \to Q^\vee$ is the projection $P_1 \to Q^\vee$. Thus $\operatorname{Ker}(g) = \operatorname{Im}(f)$ is not contained in $\operatorname{rad} P_1$. Contradicts that $g : P_0 \to M$ is a projective cover.

Suppose that $P_0^\vee \to \operatorname{Im}(f^\vee)$ is not a projective cover. Then there is a non-zero summand Q of $P_0^\vee$ whose composition with $f^\vee$ is zero. Then there is a split epimorphism $P_0 \to Q^\vee$ whose composition with f is zero. This induces a split epimorphism $M \to Q^\vee$, contradicting the fact that M has no non-zero projective summand.

(v). $\operatorname{Tr}\operatorname{Tr} M$ is the cokernel of the map $P_1^{\vee\vee} \to P_0^{\vee\vee}$, that is, $P_1 \to P_0$.

Corollary. Tr induces a bijection between isomorphism classes of indecomposable non-projective left and right A-modules.

Definition. Given modules M, N, we denote by $\operatorname{Hom}^{\mathrm{proj}}(M, N)$ the set of all maps $M \to N$ which can be factorized through a projective module $M \to P \to N$.

Clearly $\operatorname{Hom}^{\mathrm{proj}}(M, N)$ is a subspace of $\operatorname{Hom}(M, N)$, for example if θ factors through P and θ' factors throught P' then $\theta + \theta'$ factors through $P \oplus P'$. Moreover $\operatorname{Hom}^{\mathrm{proj}}$ is an ideal in the module category.

We define $\underline{\operatorname{Hom}}(M, N) = \operatorname{Hom}(M, N)/\operatorname{Hom}^{\mathrm{proj}}(M, N)$. These form the Hom spaces in a category, the *stable module category*, denoted A-$\underline{\mathrm{mod}}$.

Theorem. The transpose defines inverse anti-equivalences

$$A\text{-}\underline{\mathrm{mod}} \rightleftarrows \underline{\mathrm{mod}}\text{-}A.$$

Proof. First we show that Tr defines a contravariant functor from A-mod to $\underline{\text{mod}}$-A. Any map $\theta : M \to M'$ can be lifted to a map of projective presentations

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "\theta_1"'] & P_0 \arrow[r, "g"] \arrow[d, "\theta_0"'] & M \arrow[r] \arrow[d, "\theta"'] & 0 \\ P_1' \arrow[r, "f'"] & P_0' \arrow[r, "g'"] & M' \arrow[r] & 0 \end{tikzcd}$$

Applying $()^\vee$ there is an induced map ϕ.

$$\begin{tikzcd} P_0'^\vee \arrow[r, "f'^\vee"] \arrow[d, "\theta_0^\vee"'] & P_1'^\vee \arrow[r, "p'"] \arrow[d, "\theta_1^\vee"'] & \operatorname{Tr} M' \arrow[r] \arrow[d, "\phi"'] & 0 \\ P_0^\vee \arrow[r, "f^\vee"] & P_1^\vee \arrow[r, "p"] & \operatorname{Tr} M \arrow[r] & 0 \end{tikzcd}$$

The map ϕ depends on θ_0 and θ_1, which are not uniquely determined. We show that any choices lead to the same element of $\underline{\text{Hom}}(\operatorname{Tr} M', \operatorname{Tr} M)$. For this we may assume that $\theta = 0$, and need to show that ϕ factors through a projective.

Thus assume that θ is zero. Then $g'\theta_0 = 0$. Thus there is $h : P_0 \to P_1'$ with $\theta_0 = f'h$. This gives $h^\vee : P_1'^\vee \to P_0^\vee$ with $\theta_0^\vee = h^\vee f'^\vee$. Now we have a commutative diagram

$$\begin{tikzcd} P_0'^\vee \arrow[r, "f'^\vee"] \arrow[d, "\theta_0^\vee"'] & P_1'^\vee \arrow[r, "p'"] \arrow[d, "f^\vee h^\vee"'] & \operatorname{Tr} M' \arrow[r] \arrow[d, "0"'] & 0 \\ P_0^\vee \arrow[r, "f^\vee"] & P_1^\vee \arrow[r, "p"] & \operatorname{Tr} M \arrow[r] & 0. \end{tikzcd}$$

Taking the difference of the vertical maps, there is also a commutative diagram

$$\begin{tikzcd} P_0'^\vee \arrow[r, "f'^\vee"] \arrow[d, "0"'] & P_1'^\vee \arrow[r, "p'"] \arrow[d, "\theta_1^\vee - f^\vee h^\vee"'] & \operatorname{Tr} M' \arrow[r] \arrow[d, "\phi"'] & 0 \\ P_0^\vee \arrow[r, "f^\vee"] & P_1^\vee \arrow[r, "p"] & \operatorname{Tr} M \arrow[r] & 0. \end{tikzcd}$$

But then $(\theta_1^\vee - f^\vee h^\vee)f'^\vee = 0$. Thus there is a map $s : \operatorname{Tr} M' \to P_1^\vee$ with $\theta_1^\vee - f^\vee h^\vee = sp'$. It follows that $psp' = \phi p'$, so since p' is surjective, $\phi = ps$, so ϕ factors through a projective.

Thus a morphism $g : M \to M'$ gives a well-defined morphism $\operatorname{Tr} g = [\phi] \in \underline{\text{Hom}}(\operatorname{Tr} M', \operatorname{Tr} M)$. It is straightforward that this construction behaves well on compositions of morphsms, so that the transpose defines a contravariant functor A-mod to $\underline{\text{mod}}$-A.

Now clearly the transpose sends any projective module to 0, so it sends any morphism factoring through a projective to 0, so it descends to a contravariant functor A-$\underline{\text{mod}}$ to $\underline{\text{mod}}$-A. Now it is straightforward that it is an antiequivalence.

2.2 Auslander-Reiten formula

Definition. We define A-$\overline{\text{mod}}$ as the category with Hom spaces

$$\overline{\mathrm{Hom}}(M,N) = \mathrm{Hom}(M,N)/\,\mathrm{Hom}^{\mathrm{inj}}(M,N)$$

where $\mathrm{Hom}^{\mathrm{inj}}(M,N)$ is the maps factoring through an injective module.

Lemma 1. $\underline{\mathrm{Hom}}(M,N) \cong \overline{\mathrm{Hom}}(DN,DM)$, so D gives an antiequivalence between $\underline{\text{mod}}$-A and A-$\overline{\text{mod}}$.

Proof. Straightforward.

Definition. The *Auslander-Reiten translate* is $\tau = D\,\mathrm{Tr}$ and the inverse construction is $\tau^- = \mathrm{Tr}\,D$.

By the results of the previous section we have inverse bijections

$$\text{non-projective indec mods/iso} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{non-injective indec mods/iso}$$

and inverse equivalences

$$A\text{-}\underline{\text{mod}} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} A\text{-}\overline{\text{mod}}.$$

Applying D to the exact sequence defining $\mathrm{Tr}\,M$, we see that there is an exact sequence

$$0 \to \tau M \to \nu(P_1) \to \nu(P_0) \to \nu(M) \to 0.$$

Thus τ con be computed by taking a minimal projective presentation of M, applying the Nakayama functor (which turns each $P[i]$ into $I[i]$) and taking the kernel.

Example. For the commutative square with source 1 and sink 4, the simple $S[2]$ has minimal projective presentation

$$P[4] \to P[2] \to S[2] \to 0$$

so we get

$$0 \to \tau S[2] \to I[4] \to I[2]$$

so $\tau S[2] \cong P[3]$.

Lemma 2. If M is an A-module, then
(i) $\operatorname{proj.dim} M \leq 1 \Leftrightarrow \operatorname{Hom}(DA, \tau M) = 0 \Leftrightarrow$ there is no non-zero map from an injective module to τM.
(ii) $\operatorname{inj.dim} M \leq 1 \Leftrightarrow \operatorname{Hom}(\tau^- M, A) = 0 \Leftrightarrow$ there is no non-zero map from $\tau^- M$ to a projective module.

Proof. (i) Recall that $\nu^-(-) = \operatorname{Hom}(DA, -)$, and that $\nu^-(\nu(P)) \cong P$. Thus we get $0 \to \nu^-(\tau M) \to \nu^-(\nu(P_1)) \to \nu^-(\nu(P_0))$ exact, so $0 \to \nu^-(\tau M) \to P_1 \to P_0$. Thus $\operatorname{proj.dim} M \leq 1$ iff $P_1 \to P_0$ is injective iff $\nu^-(\tau M) = 0$ iff $\operatorname{Hom}(DA, \tau M) = 0$.

(ii) Dual.

Lemma 3. Given a right A-module M, a left A-module N, $m \in M$ and $n \in N$ let $f_{mn} : M^\vee \to N$ be the map defined by $f_{mn}(\alpha) = \alpha(m)n$. It is a left A-module map. There is a natural transformation

$$\theta_{MN} : D\operatorname{Hom}(M^\vee, N) \to \operatorname{Hom}(M, DN), \quad \theta_{MN}(\xi) = (m \mapsto (n \mapsto \xi(f_{mn}))).$$

Then θ_{MN} is an isomorphism for M projective. And in general the image of θ_{MN} is $\operatorname{Hom}^{\mathrm{proj}}(M, DN)$.

Proof. The first part is clear. Clearly θ_{MN} is well-defined. Both $D\operatorname{Hom}(M^\vee, N)$ and $\operatorname{Hom}(M, DN)$ define functors which are contravariant in M and N, and it is straightforward that θ_{MN} is natural in M and N.

For M projective, the map is an isomorphism, since it is for $M = A$. Thus given a map $f : M \to P$ with P projective, we get a commutative diagram

$$\begin{tikzcd}
D\operatorname{Hom}(P^\vee, N) \arrow[r, equal] \arrow[d, "b"'] & \operatorname{Hom}(P, DN) \arrow[d, "a"'] \\
D\operatorname{Hom}(M^\vee, N) \arrow[r, "\theta_{MN}"] & \operatorname{Hom}(M, DN)
\end{tikzcd}$$

Any map $M \to DN$ factoring through P is in the image of a, so in $\operatorname{Im}(\theta)$.

Varying P, we get $\mathrm{Hom}^{\mathrm{proj}}(M, DN) \subseteq \mathrm{Im}(\theta)$.

Now take a basis of $M^\vee$. This defines a map $M \to P$, where $P = A^n$. Then $P^\vee \to M^\vee$ is onto. Thus $\mathrm{Hom}(M^\vee, N) \to \mathrm{Hom}(P^\vee, N)$ is 1-1. Thus b is onto. Thus $\mathrm{Im}(\theta) = \mathrm{Im}(a) \subseteq \mathrm{Hom}^{\mathrm{proj}}(M, DN)$.

Theorem. There are isomorphisms

$$\underline{\mathrm{Hom}}(\tau^- N, M) \cong D\,\mathrm{Ext}^1(M, N) \cong \overline{\mathrm{Hom}}(N, \tau M).$$

Proof. Given a minimal projective presentation $P_1 \to P_0 \to M \to 0$, write $\Omega^1 M$ for the image of $P_1 \to P_0$, so there is

$$0 \to \Omega^1 M \to P_0 \to M \to 0$$

and hence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(\Omega^1 M, N) \to \mathrm{Ext}^1(M, N) \to 0.$$

Also we have

$$0 \to M^\vee \to P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$$

so

$$0 \to (\mathrm{Tr}\, M)^\vee \to P_1 \to P_0$$

so

$$0 \to (\mathrm{Tr}\, M)^\vee \to P_1 \to \Omega^1 M \to 0.$$

and hence

$$0 \to \mathrm{Hom}(\Omega^1 M, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}((\mathrm{Tr}\, M)^\vee, N).$$

Thus we have a commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & D\operatorname{Ext}^1(M,N) \arrow[d] & \\
 & D\operatorname{Hom}((\operatorname{Tr} M)^\vee, N) \arrow[r] \arrow[d] & D\operatorname{Hom}(P_1, N) \arrow[r] \arrow[d, equal] & D\operatorname{Hom}(\Omega^1 M, N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Hom}(\operatorname{Tr} M, DN) \arrow[r] \arrow[d] & \operatorname{Hom}(P_1^\vee, DN) \arrow[r] & \operatorname{Hom}(P_0^\vee, DN) & \\
 & \underline{\operatorname{Hom}}(\operatorname{Tr} M, DN) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

The argument of the snake lemma then gives an isomorphism $D\operatorname{Ext}^1(M,N) \to \underline{\operatorname{Hom}}(\operatorname{Tr} M, DN)$.

Now use that Lemma 1 to rewrite this as $\overline{\operatorname{Hom}}(N, D\operatorname{Tr} M)$, or use that Tr gives inverse anti-equivalences between A-$\underline{\text{mod}}$ and $\underline{\text{mod}}$-A to rewrite it as $\underline{\operatorname{Hom}}(M, \operatorname{Tr} DN)$.

Corollary. If A is hereditary, we get

$$\operatorname{Hom}(\tau^- N, M) \cong D\operatorname{Ext}^1(M,N) \cong \operatorname{Hom}(N, \tau M).$$

Proof. Use Lemma 2. We have $\operatorname{Hom}(\tau^- N, M) \cong \underline{\operatorname{Hom}}(\tau^- N, M)$ if $\operatorname{inj.dim} N \leq 1$, and $\operatorname{Hom}(N, \tau M) \cong \overline{\operatorname{Hom}}(N, \tau M)$ if $\operatorname{proj.dim} M \leq 1$.

2.3 Auslander-Reiten sequences

Definition. By an *Auslander-Reiten sequence* or *almost split sequence* we mean an exact sequence

$$0 \to X \to Y \to Z \to 0$$

satisfying
(i) It is not split.

(ii) X and Z are indecomposable
(iii) Every map $M \to Z$, which is not a split epi, factors through $Y \to Z$.
(iv) Every map $X \to N$, which is not a split mono, factors through $X \to Y$.

Proposition. An Auslander-Reiten sequence, if it exists, is determined up to isomorphism by either of the end terms. That is, if

$$0 \to X' \to Y' \to Z \to 0$$

is another Auslander-Reiten sequence ending at Z, there there is a commutative diagram in which the vertical maps are isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

and dually for another Auslander-Reiten sequence starting with X.

Proof. By assumption there is a map $Y \to Y'$. Dually there is a map $Y' \to Y$. These induce maps $X \to X'$ and $X' \to X$. If the composition $X \to X$ isn't an isomorphism, then it is nilpotent, so some power is zero. But then the sequence is split. But then taking θ to be the corresponding power of the map $Y \to Y$ we get

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "0"'] & Y \arrow[r, "g"] \arrow[d, "\theta"'] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

and this can only happen if the sequence is split, for θ factors as hg for some $h : Z \to Y$, so $ghg = g\theta = g1$, so since g is onto, $gh = 1$.

Theorem. Let Z be an non-projective indecomposable A-module and let $X = \tau Z$ be the corresponding non-injective indecomposable module. (Or equivalently let X be non-injective indecomposable and let $Z = \tau^{-}X$.) Then there exists an Auslander-Reiten sequence

$$0 \to X \to Y \to Z \to 0.$$

Proof. $\mathrm{Ext}^1(Z, X)$ is an $\mathrm{End}(X)$-$\mathrm{End}(Z)$-bimodule.

First we consider it as a right $\mathrm{End}(Z)$-module. Since Z is indecomposable, $\mathrm{End}(Z)$ is a local ring. Since Z is not projective $\mathrm{End}^{proj}(Z)$ is contained in the maximal ideal of $\mathrm{End}(Z)$. Thus $\underline{\mathrm{End}}(Z)$ has simple top as a left $\mathrm{End}(Z)$-module. Thus $D\underline{\mathrm{End}}(Z)$ has simple socle as a right $\mathrm{End}(Z)$-module. Thus by the AR formula $\mathrm{Ext}^1(Z,X)$ has simple socle S as a right $\mathrm{End}(Z)$-module. Since K is algebraically closed, $\dim S = 1$.

Now we consider $\mathrm{Ext}^1(Z,X)$ as a left $\mathrm{End}(X)$-module. By the same argument it has simple socle T as a left $\mathrm{End}(X)$-module.

But if U is an A-B-bimodule, then any endomorphism of U_B sends $\mathrm{soc}(U_B)$ into itself. Thus $\mathrm{soc}(U_B)$ is an A-submodule of U. Thus S is 1-dimensional $\mathrm{End}(X)$-submodule of $\mathrm{Ext}^1(Z,X)$, so $S = T$.

Let
$$\xi : 0 \to X \to Y \to Z \to 0$$
be an exact sequence corresponding to a non-zero element of S.

(i) Since $\xi \neq 0$. (ii) Trivial.

(iii) Suppose $M \to Z$ not a split epi.

The map $\mathrm{Hom}(Z,M) \to \mathrm{End}(Z)$ has image contained in the radical of $\mathrm{End}(Z)$.

Thus the map $\underline{\mathrm{Hom}}(Z,M) \to \underline{\mathrm{End}}(Z)$ has image contained in the radical of $\underline{\mathrm{End}}(Z)$.

Thus the map $D\underline{\mathrm{End}}(Z) \to D\underline{\mathrm{Hom}}(Z,M)$ kills the socle of $D\underline{\mathrm{End}}(Z)$ as a $\mathrm{End}(Z)$-module.

Thus the map $\mathrm{Ext}^1(Z,X) \to \mathrm{Ext}^1(M,X)$ kills ξ. Thus the pullback of ξ by $M \to Z$ splits.

(iv) By duality.

2.4 Irreducible maps

Recall that we have defined $\mathrm{rad}(M,N) \subseteq \mathrm{Hom}(M,N)$. If M is indecomposable it is the set of maps which are not split monos. If N is indecomposable it is the set of maps which are not split epis. If M and N are indecomposable it is the set of non-isomorphisms.

Definition. Given modules M, N we define $\mathrm{rad}^2(M,N)$ to be the set of all homomorphisms $M \to N$ which can be written as a composition

$$M \xrightarrow{f} X \xrightarrow{g} N$$

with $f \in \operatorname{rad}(M, X)$ and $g \in \operatorname{rad}(X, N)$. This is a subspace of $\operatorname{rad}(M, N)$.

Suppose that M and N are indecomposable. We say that a map $M \to N$ is *irreducible* if it is in $\operatorname{rad}(M, N)$, but not in $\operatorname{rad}^2(M, N)$. It is equivalent that it is not an isomorphism, and whenever it factorizes as gf, either f is a split mono, or g is a split epi.

We define the *multiplicity of irreducible maps from M to N* to be

$$\operatorname{irr}(M, N) = \dim[\operatorname{rad}(M, N)/\operatorname{rad}^2(M, N)].$$

Recall from the section on the Krull-Remak-Schmidt Theorem that the multiplicity of M as a direct summand of a module U is

$$\mu_M(U) = \dim \frac{\operatorname{Hom}(M, U)}{\operatorname{rad}(M, U)}.$$

Recall that the indecomposable projective $P[i]$ has submodule $\operatorname{rad} P[i]$ with $P[i]/\operatorname{rad} P[i] \cong S[i]$, and that the indecomposable injective $I[i]$ has socle $S[i]$.

Theorem. Let M be indecomposable and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an AR sequence.
(i) $\operatorname{irr}(M, P[i]) = \mu_M(\operatorname{rad} P[i])$, the number of copies of M in the decomposition of $\operatorname{rad} P[i]$.
(ii) $\operatorname{irr}(M, Z) = \mu_M(Y)$, the number of copies of M in the decomposition of the middle term of the AR sequence ending at Z.
(i') $\operatorname{irr}(I[i], M) = \mu_M(I[i]/S[i])$, the number of copies of M in the decomposition of $I[i]/S[i]$.
(ii') $\operatorname{irr}(X, M) = \mu_M(Y)$, the number of copies of M in the decomposition of the middle term of the AR sequence starting at X.

Proof. (i) If $f : M \to P[i]$ is not an isomorphism, then it can't be onto, so it maps into $\operatorname{rad} P[i]$. Thus composition with the inclusion $\operatorname{rad} P[i] \to P[i]$ induces an isomorphism $\operatorname{Hom}(M, \operatorname{rad} P[i]) \to \operatorname{rad}(M, P[i])$. This restricts to an isomorphism $\operatorname{rad}(M, \operatorname{rad} P[i]) \to \operatorname{rad}^2(M, P[i])$. Thus

$$\operatorname{irr}(M, P[i]) = \dim[\operatorname{Hom}(M, \operatorname{rad} P[i])/\operatorname{rad}(M, \operatorname{rad} P[i])] = \mu_M(\operatorname{rad} P[i]).$$

(ii) Using that f and g are radical homomorphisms, one gets left exact sequences

$$0 \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,Y) \to \mathrm{rad}(M,Z)$$

and

$$0 \to \mathrm{Hom}(M,X) \to \mathrm{rad}(M,Y) \to \mathrm{rad}^2(M,Z).$$

Now both of these sequences are actually right exact by the AR property. For example any map $\theta \in \mathrm{rad}^2(M,Z)$ factorizes as $\theta = \psi\phi$ with $\phi \in \mathrm{rad}(M,U)$ and $\psi \in \mathrm{rad}(U,Z)$. But then ϕ factors as $g\chi$ for some $\chi \in \mathrm{Hom}(U,Y)$, and then $\theta = g(\chi\phi)$, and $\chi\phi \in \mathrm{rad}(M,Y)$. Thus

$$\mathrm{irr}(M,Z) = \dim[\mathrm{Hom}(M,Y)/\,\mathrm{rad}(M,Y)] = \mu_M(Y).$$

Corollary If X is indecomposable, then

$$-\dim X + \sum_M \mathrm{irr}(X,M)\dim M = \begin{cases} -\dim S[i] & (M \cong I[i]) \\ \dim \tau^- X & (M \text{ not injective}) \end{cases}$$

where the sum is over all indecomposable modules up to isomorphism. Moreover if $A = KQ/I$ then the same applies for dimension vectors.

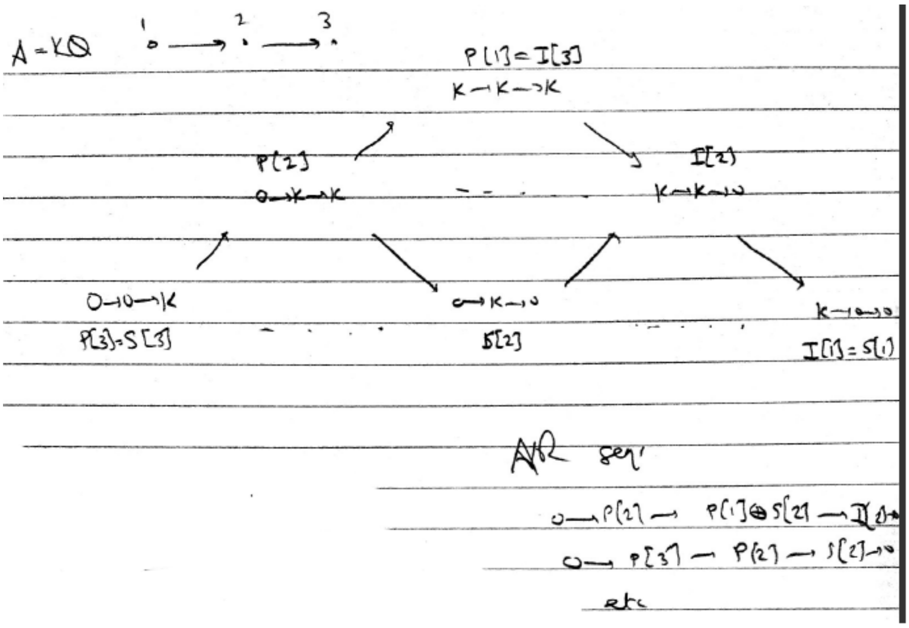
2.5 Auslander-Reiten quiver

Definition. Given a f.d. algebra A, the *Auslander-Reiten quiver* of A has vertices corresponding to the isomorphism classes of indecomposable A-modules, and the number of arrows $M \to N$ is $\mathrm{irr}(M,N)$.

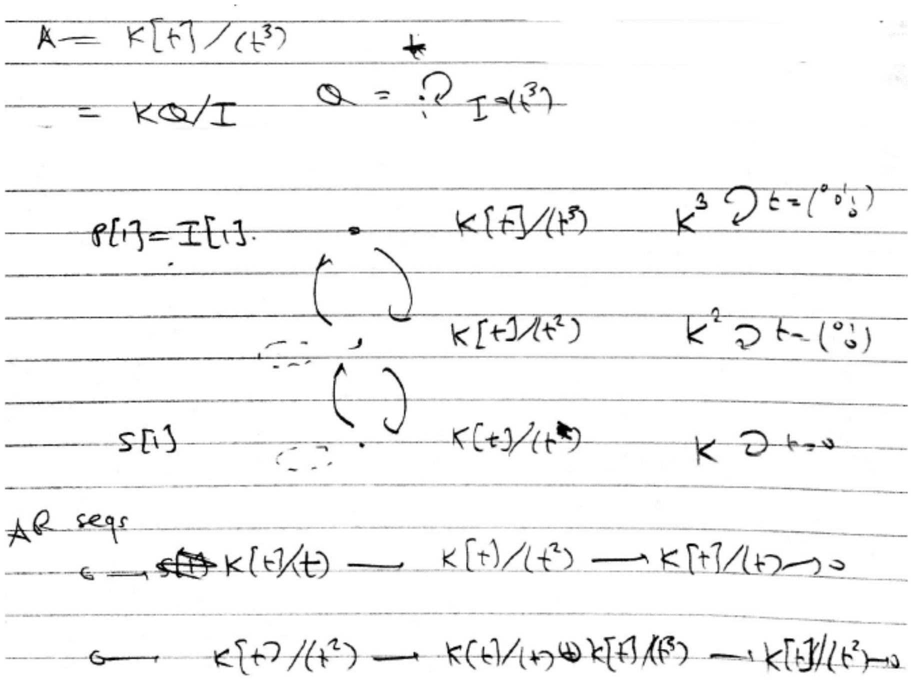
It is often useful to indicate the AR translate τ by a dotted line joining Z and $\tau Z = D\,\mathrm{Tr}\,Z$.

In general the AR quiver is not connected. It is finite iff the algebra has only finitely many indecomposable modules, that is, it has finite representation type.

Examples.



A3 with linear orientation.



$K[t]/(t^3)$.

Harada-Sai Lemma. A composition of $2^n - 1$ non-isomorphisms between indecomposables of dimension $\leq n$ must be zero.

Proof. We show for $m \leq n$ that a composition of $2^m - 1$ non-isomorphisms between indecomposables of dimension $\leq n$ has rank $\leq n - m$.

If $m = 1$ this is clear. If $m > 1$, a composition of $2^m - 1$ non-isomorphisms can be written as a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

where f and h are compositions of $2^{m-1} - 1$ non-isomorphisms. By induction $\operatorname{rank} f, \operatorname{rank} h \leq n - m + 1$. If either has strictly smaller rank, we're done. Thus suppose that $\operatorname{rank} f = \operatorname{rank} h = \operatorname{rank} hgf = n - m + 1$.

This implies that $\operatorname{Ker} f = \operatorname{Ker} hgf$ and $\operatorname{Im} hgf = \operatorname{Im} h$. It follows that $Y = \operatorname{Ker} hg \oplus \operatorname{Im} f$ and $Z = \operatorname{Ker} h \oplus \operatorname{Im} gf$. For example if $y \in Y$ then $hg(y) = hgf(x)$, so $y = f(x) + (y - f(x)) \in \operatorname{Im} f + \operatorname{Ker} hg$, and if $y \in \operatorname{Im} f \cap \operatorname{Ker} hg$ then $y = f(x)$ and $hgf(x) = 0$, so $x \in \operatorname{Ker} hgf = \operatorname{Ker} f$, so $y = 0$.

By indecomposability f is onto and h is 1-1, but then g is an isomorphism. Contradiction.

Theorem (Auslander). If C is a connected component of the AR quiver containing modules of bounded dimension, then there are no nonzero maps between indecomposables in C and indecomposables not in C.

If in addition $A = KQ/I$ with I admissible and Q connected, then C is the whole of the AR quiver of A.

Proof. By duality, suppose that $\theta : M \to Z$ is a nonzero map with M not in C and Z in C.

If Z is projective, then θ maps into $\operatorname{rad} Z$, and so there is an indecomposable summand Z' of $\operatorname{rad} Z$ and maps $M \to Z' \to Z$ with nonzero composition.

If Z is non-projective, with AR sequence $0 \to X \to Y \to Z \to 0$ then θ factors through Y, and there is an indecomposable summand Z' of Y and maps $M \to Z' \to Z$ with nonzero composition.

Now repeat with Z' to get maps $M \to Z'' \to Z' \to Z$ with non-zero composition.

Get contradiction.

Now suppose $A = KQ/I$ with Q connected. Let $X \in C$.

Choose any vertex i with $e_i X \neq 0$. Then there is a nonzero map $P[i] \to X$, so $P[i] \in C$.

Now every $P[j] \in C$, for if there is an arrow between i and j then there is a nonzero map between $P[i]$ and $P[j]$.

Finally, if Y is any indecomposable there is a map from some $P[j]$ to Y, so $Y \in C$.

2.6 Knitting construction

We work with an algebra $A = KQ/I$.

Preparation. Compute the modules $\operatorname{rad} P[i]$ and decompose into indecomposable summands.

Iterative construction. We suppose we have drawn a subquiver of the AR quiver with the property that if an indecomposable module X is in the subquiver, then so are all arrows ending at X, and suppose we know the dimension vectors of the modules we have drawn.

We start with the empty subquiver.

If we have drawn all the summands of $\operatorname{rad} P[i]$, but haven't yet drawn $P[i]$, we can now draw $P[i]$ and fill in the arrows ending at $P[i]$ with their multiplicities. In particular we can start by drawing the simple projective modules.

Suppose we have drawn an indecomposable module X. If we have drawn all projectives $P[i]$ such that X is a summand of $\operatorname{rad} P[i]$, and if we have drawn $\tau^- U$ for all non-injective undecomposables U with an arrow $U \to X$, then we can be sure that we have drawn all arrows starting at X.

If we have drawn all arrows starting at X then

$$-\underline{\dim} X + \sum_M \operatorname{irr}(X, M)\, \underline{\dim} M = \begin{cases} -\underline{\dim} S[i] & (X \cong I[i]) \\ \underline{\dim} \tau^- X & (X \text{ not injective}) \end{cases}$$

so we know whether or not X is injective by the sign of the left hand side. If it is not injective, and we haven't yet drawn $\tau^- X$, we can now do so, and draw $\operatorname{irr}(M, \tau^- X) = \operatorname{irr}(X, M)$ arrows from M to X, for all M.

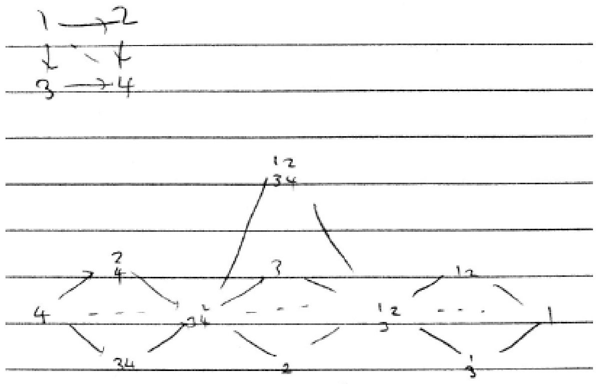
Repeat.

Several possibilities. (a) Get stuck, because either there is no simple projective, or we have written down some summands of rad $P[i]$, for some projective P, but can't write down all summands, so can't write down $P[i]$.

(b) Terminate after a finite number of steps. By Auslander's Theorem we have the whole AR quiver.

(c) Go on forever. In this case we have constructed one or more connected components of the AR quiver, called 'preprojective' components.

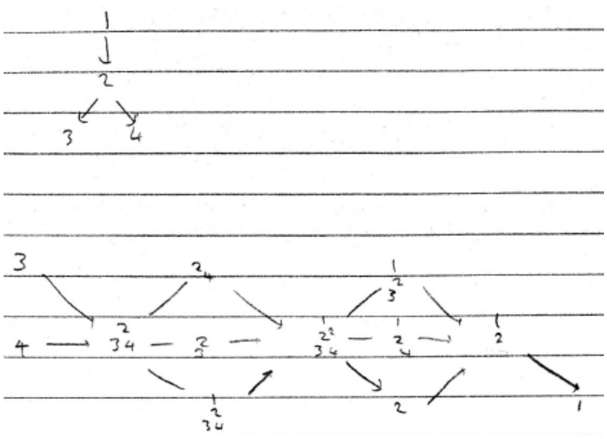
Examples.



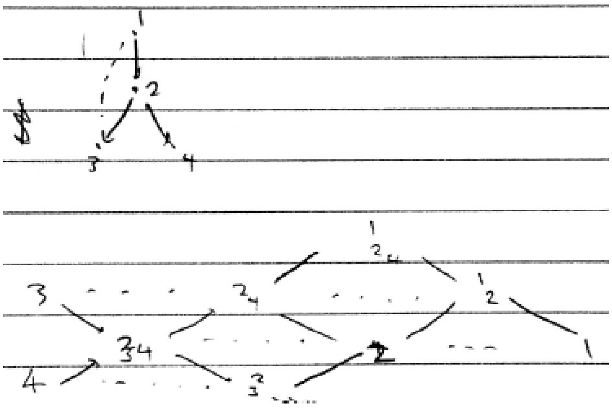
Commutative square

A3

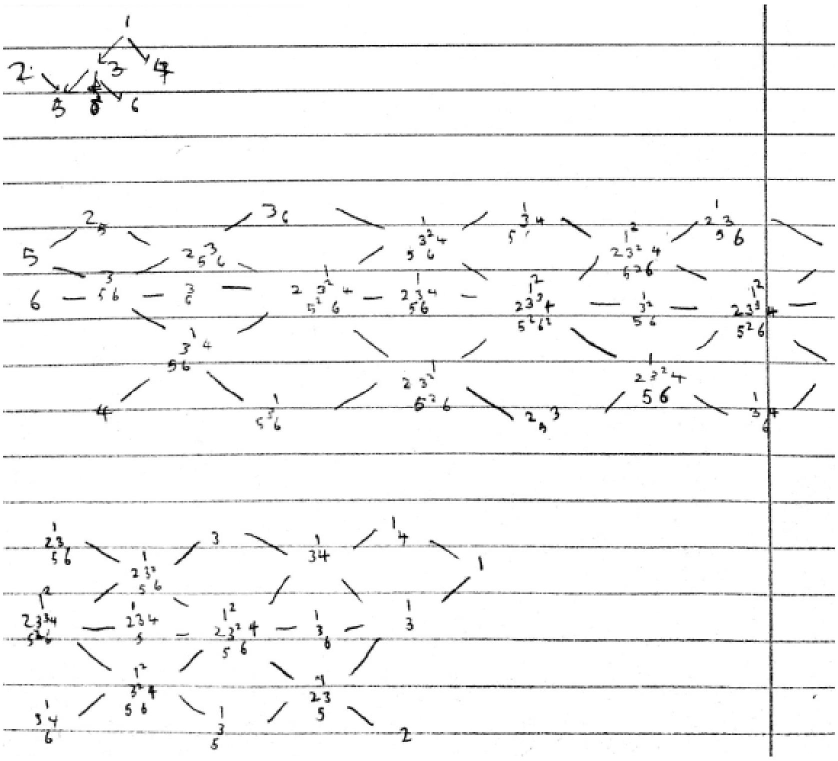
A3 with another orientation



D4



D4 with zero relation

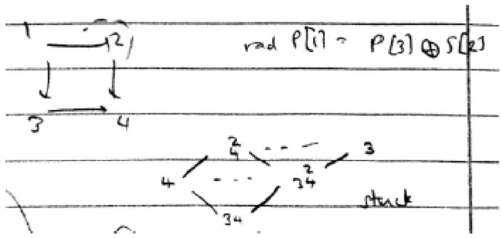


E6

4-subspace,

Kronecker quiver,

Example with decomposable radical, etc.



Maybe one gets stuck.

Dually construct preinjective components starting with simple injective.

2.7 Graded modules

The knitting procedure fails for many algebras. But a tool called 'covering theory' can often be used to make it work. By Gordon and Green (1982) it is essentially equivalent to study graded modules.

Recall that A is a $\mathbb{Z}$-graded algebra if

$$A = \bigoplus_{n\in\mathbb{Z}} A_n, \qquad A_n.A_m \subseteq A_{n+m}.$$

It follows that $1 \in A_0$. We assume that A is f.d.. Thus only finitely many A_n are nonzero.

Theorem 1. A is local iff A_0 is local.

Proof. Suppose A is local. If I is a proper left ideal in A_0 then $AI \subseteq J(A)$ since it is a left ideal in A, and if $a \in A$ and $i \in I$, then i is not invertible in A_0, so not in A, so ai is not invertible, so it is in $J(A)$. Thus $I \subseteq J(A) \cap A_0$. Thus $J(A) \cap A_0$ is the unique maximal left ideal in A_0.

Now suppose that A_0 is local.

Let $I = \sum_{n\neq 0} A_n A_{-n} \subseteq A_0$.

If $a \in A_n$ and $b \in A_{-n}$ with $n \neq 0$, then a is nilpotent, so not invertible, so ab is not invertible in A, so it is not invertible in A_0, so $ab \in J(A_0)$. Thus $I \subseteq J(A_0)$.

Thus I is nilpotent. Say $I^N = 0$.

Let L be the ideal in A generated by all A_n ($n \neq 0$). Clearly $L = I \oplus \bigoplus_{n\neq 0} A_n$.

It suffices to show that L is nilpotent, for then $L \subseteq J(A)$, so that $A/J(A)$ is a quotient of $A/L \cong A_0/I$, which is local.

Suppose that A lives in d different degrees.

It suffices to show that any product $\ell_1 \ell_2 \dots \ell_{dN}$ of homogeneous elements of L is zero.

Suppose not. Let d_i be the degree of $\ell_1 \ell_2 \dots \ell_i$.

We have $dN + 1$ numbers $d_0, d_1, \dots, d_{dN}$ taking at most d different values, so some value must occur at least $N + 1$ times. Say

$$d_{i_1} = d_{i_2} = \cdots = d_{i_{N+1}}$$

with $i_1 < i_2 < \cdots < i_{N+1}$. Then we can write the product as

$$\ell_1 \dots \ell_{i_1} (\ell_{i_1+1} \dots \ell_{i_2})(\ell_{i_2+1} \dots \ell_{i_3}) \dots (\ell_{i_N+1} \dots \ell_{i_{N+1}}) \ell_{i_{N+1}+1} \dots \ell_{dN}$$

But each of the bracketed terms has degree 0, so is in I, so their product is zero.

Definition. Recall that a $\mathbb{Z}$-graded A-module is an A-module

$$M = \bigoplus_{n\in\mathbb{Z}} M_n, \qquad A_n . M_m \subseteq M_{n+m}.$$

We only consider f.d. graded modules, and write A-grmod for the category of f.d. $\mathbb{Z}$-graded A-modules, with

$$\mathrm{Hom}_{A\text{-grmod}}(M, N) = \{\theta \in \mathrm{Hom}_A(M, N) \mid \theta(M_n) \subseteq N_n \text{ for all } n \in \mathbb{Z}\}.$$

Given a graded module M and $i \in \mathbb{Z}$ we write $M(i)$ for the module with shifted grading $M(i)_n = M_{i+n}$. There is a functor F from A-grmod to A-mod which forgets the grading.

Lemma 1. If M, N are graded A-modules, then $\mathrm{Hom}_A(FM, FN)$ can be graded,

$$\mathrm{Hom}_A(FM, FN) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(M, N(n)).$$

In this way

$$\mathrm{End}_A(FM) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(M, M(n))$$

becomes a graded algebra.

Proof. Given a homomorphism $\theta : FM \to FN$, we get linear maps $\theta_n : M \to N$ defined by

$$\theta_n(m) = \sum_{i \in \mathbb{Z}} \theta(m_i)_{i+n}$$

where a subscript k applied to an element of a graded module picks out the degree k component of the element.

Now if $a \in A$ is homogeneous of degree d, then $(am)_i = a.m_{i-d}$, so

$$\theta_n(am) = \sum_i \theta((am)_i)_{i+n} = \sum_i \theta(a.m_{i-d})_{i+n} = \sum_i (a\theta(m_{i-d}))_{i+n}$$

$$= \sum_i a.\theta(m_{i-d})_{i+n-d} = \sum_j a.\theta(m_j)_{j+n} = a\theta_n(m).$$

Thus $\theta_n \in \mathrm{Hom}_{A\text{-grmod}}(M, N(n))$. Clearly θ is the sum of the θ_n. The rest is clear.

Corollary. (i) A graded module M is indecomposable iff the ungraded module FM is indecomposable.

(ii) If M and N are indecomposable graded modules with $FM \cong FN$ then M is isomorphic to $N(i)$ for some i.

Proof. (i) By Theorem 1, $\mathrm{End}_A(FM)$ is local iff its degree zero part is local. This is $\mathrm{End}_A(FM)_0 = \mathrm{End}_{A\text{-grmod}}(M)$. Now the ungraded module FM is indecomposable iff its endomorphism algebra $\mathrm{End}_A(FM)$ is local. The graded module M is indecomposable iff its endomorphism algebra $\mathrm{End}_{A\text{-grmod}}(M)$ has no non-trivial idempotents, and since it is f.d., it is equivalent that it is local.

(ii) Suppose $\theta : FM \to FN$ is an isomorphism. Then $\theta^{-1}\theta = 1_{FM}$, so $(\theta^{-1}\theta)_0 = 1_M$, so $\sum_i (\theta^{-1})_{-i}\theta_i = 1_M$. Since $\mathrm{End}(M)$ is local, some $(\theta^{-1})_i\theta_i$ is invertible, so $\theta_i : M \to N(i)$ is a split mono of graded modules, and hence also an isomorphism.

Setup. Let $A = KQ/I$ be a f.d. algebra. Suppose that A is graded in such a way that the trivial paths have degree 0 and the arrows are homogeneous. (Recall that this defines a grading of KQ, and one just needs to check that the generators of I are homogeneous.)

Graded A-modules correspond to representations of an infinite quiver with relations. The vertex set is $Q_0 \times \mathbb{Z}$. Given a module X, the vector space at vertex (i, n) is $e_i X_n$.

We discussed this before - it is modules for an algebra with enough idempotents.

Now truncate this: given an integer range $[n, m] = \{n, n+1, \dots, m\}$, the graded modules living in degrees $[n, m]$ correspond to representations of a finite quiver with relations, with vertex set $Q_0 \times [n, m]$. Let $\tilde{A}$ be the corresponding algebra.

We write F also for the functor from $\tilde{A}$-mod to A-mod.

We suppose that A lives in non-negative degrees, so since it is f.d., it lives in degrees $[0, d]$.

Example. Vertices 1,2. Loops p, r at 1,2. Arrow $q : 1 \to 2$ drawn going down. Relations $p^2 = r^2 = 0$, $qp = rq$. $\deg p = \deg r = 1$, $\deg q = 0$. The algebra lives in degrees $[0, 1]$. etc.

Theorem 2. If $\xi : 0 \to X \to Y \to Z \to 0$ is an AR sequence of $\tilde{A}$-modules, and Z lives in degrees $[n + d, m - 2d]$, then $F(\xi)$ is an AR sequence of A-modules.

Sketch. The trivial idempotents $e_i \in A$ are homogeneous of degree 0, so the module $P_A[i] = Ae_i$ is graded, and lives in degrees $[0, d]$.

Thus the module $P_A[i](-j)$ lives in degrees $[j, j+d]$. Thus if $j \in [m, n-d]$ then $P_A[i](-j)$ corresponds to an $\tilde{A}$-module. In fact it corresponds to $P_{\tilde{A}}[(i, j)]$. Thus $F(P_{\tilde{A}}[(i, j)]) \cong P_A[i]$.

Similarly, if $j \in [m + d, n]$ then $F(I_{\tilde{A}}[(i, j)] \cong I_A[i]$.

Take a minimal projective presentation

$$P_1 \to P_0 \to Z \to 0.$$

Now P_0 only involves projective covers of simples in degrees $[n+d, m-2d]$, so P_0 lives in degrees $[n+d, m-d]$. Then P_1 only involves projective covers of simples in degrees $[n+d, m-d]$. It follows that $F(P_i)$ are projective A-modules, and that

$$F(P_1) \to F(P_0) \to F(Z) \to 0$$

is a minimal projective presentation of $F(Z)$.

Now $\tau_{\tilde{A}} Z$ is computed using the exact sequence

$$0 \to \tau_{\tilde{A}} Z \to \nu_{\tilde{A}}(P_1) \to \nu_{\tilde{A}}(P_0).$$

Since the modules $\nu_{\tilde{A}}(P_i)$ only involve injective envelopes of simples in degrees $[n+d, m-d]$, $F(\nu_{\tilde{A}}(E_i))$ is injective, and isomorphic to $\nu_A(F(E_i))$. Thus

$$0 \to F(\tau_{\tilde{A}} Z) \to F(\nu_{\tilde{A}}(P_1)) \to F(\nu_{\tilde{A}}(P_0)),$$

is identified with the sequence

$$0 \to \tau_A F(Z) \to \nu_A(F(P_1)) \to \nu_A(F(P_0)).$$

Thus $\tau_A F(Z) \cong F(\tau_{\tilde{A}} Z) \cong F(X)$.

Now $\mathrm{End}_{\tilde{A}}(Z) \to \mathrm{End}_A(F(Z))$ as the degree 0 part.

This induces a map $\underline{\mathrm{End}}_{\tilde{A}}(Z) \to \underline{\mathrm{End}}_A(F(Z))$.

This gives $D\underline{\mathrm{End}}_A(F(Z)) \to D\underline{\mathrm{End}}_{\tilde{A}}(Z)$.

Thus $\mathrm{Ext}^1_A(F(Z), F(X)) \to \mathrm{Ext}^1_{\tilde{A}}(Z, X)$.

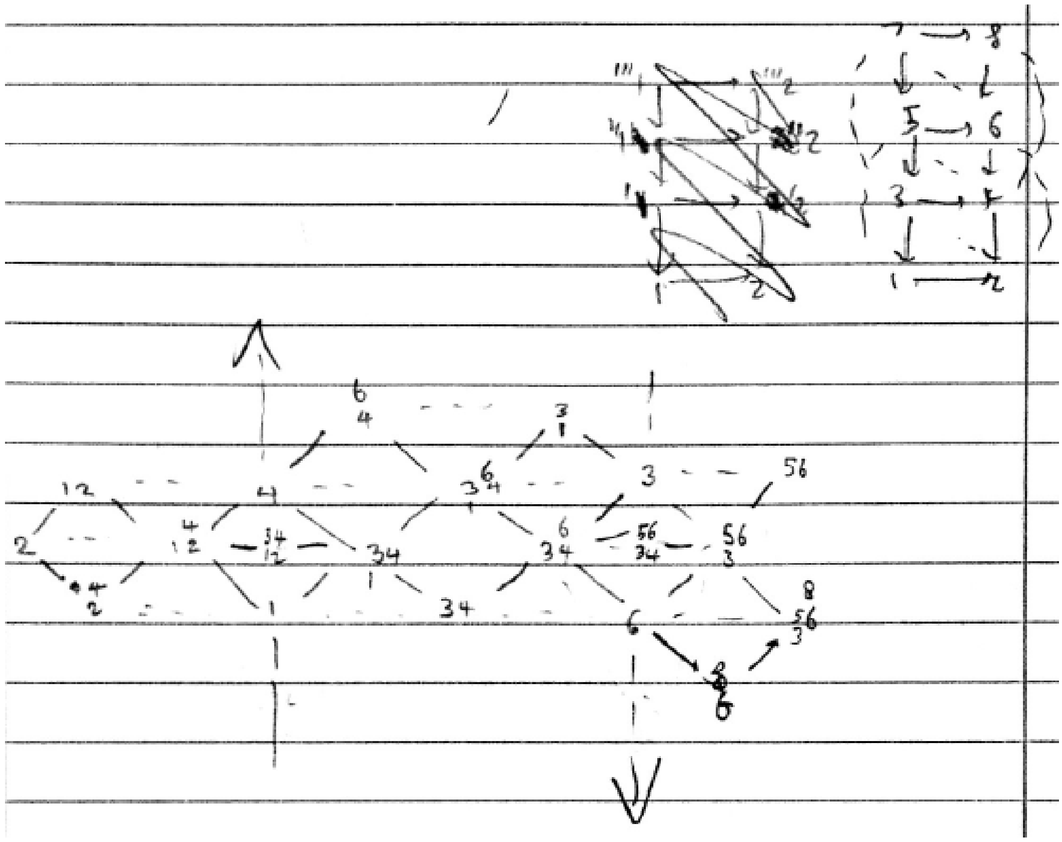
The AR sequences are defined by 1-dimensional subspaces, and one needs to check that these correspond.

Construction. Take a range of degrees $[n, m]$ with $m = 0$ and $n \ll 0$ and knit.

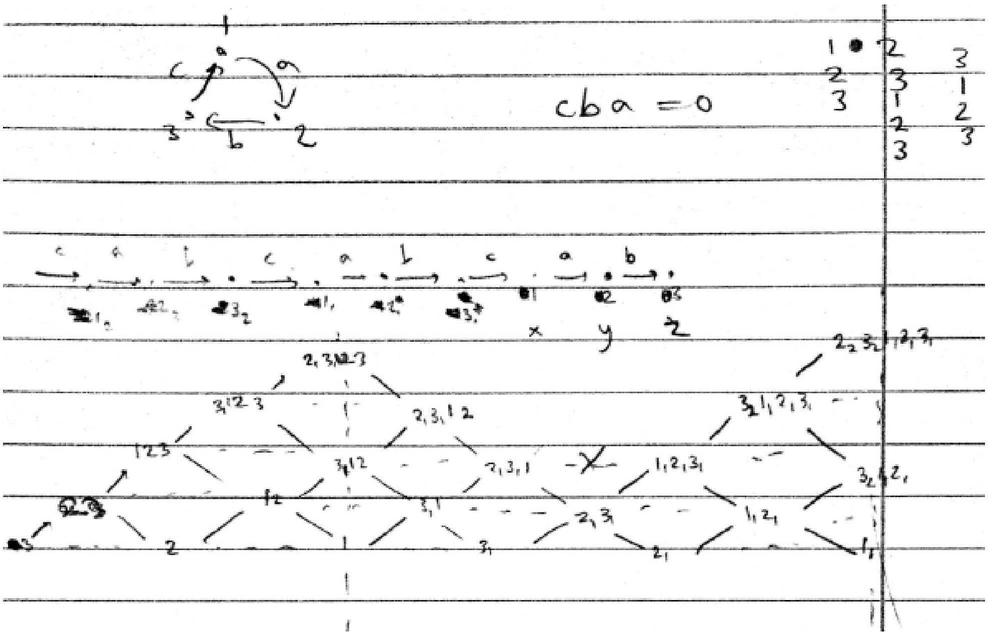
If, eventually the knitted modules live in degrees $\leq -2d$, then the subsequent AR sequences forget to AR sequences of A-modules.

If also the knitted modules are eventually all shifts of finitely many, then this gives a finite connected component of the AR quiver. By Auslander's Theorem it is the whole AR quiver.

Examples.



You only keep the piece between the vertical arrows, and identify them, to get a Möbius band. But in the lecture I drew the algebra $\tilde{A}$ with the different graded pieces going horizontally, not vertically



Here identify again. But I also drew this differently.

In the case when this process works, every module is gradeable. In general that is not true.

For example the quiver with arrows from 1 to 2 and 3, and from 2 to 3. Grade it with the arrow from 1 to 3 of degree 1 and the others of degree 0. Then the module which is K at each vertex, identity for each arrow is not gradeable.

Another example, Q with one vertex and loops p, q with relations $p^2 = qpq, q^2 = pqp, p^3 = q^3 = 0$. There is no non-trivial grading, so can't get started.

Theorem 3. If the field K has characteristic zero, and A is graded, then any A-module M with $\mathrm{Ext}^1(M, M) = 0$ is gradeable.

Proof. The result is probably folklore, but this proof comes from Keller, Murfet and van den Bergh, On two examples by Iyama and Yoshino.

Let $d : A \to A$ be the map defined by $d(a) = \deg(a)a$ for a homogeneous. It is a derivation since $d(ab) = \deg(ab)ab = (\deg(a)+\deg(b))ab = ad(b)+d(a)b$. It is called the *Euler derivation.*

Let $E = M \oplus M$ as a vector space, with A-module action given by $a(m, m') =$

$(am, d(a)m + am')$. This is an A-module structure and there is an exact sequence

$$0 \to M \xrightarrow{\binom{0}{1}} E \xrightarrow{(1 0)} M \to 0$$

By assumption this is split, so there is a map $M \to E$ of the form $m \mapsto (m, \nabla(m))$. Moreover the map $\nabla : M \to M$ satisfies

$$\nabla(am) = d(a)m + a\nabla(m)$$

so it is a *connection on M with respect to d.* Since M is f.d.,

$$M = \bigoplus_{\lambda \in K} M_\lambda$$

where M_λ is the λ-generalised eigenspace for ∇. Now for any $\lambda \in K$ and a homogeneous we have

$$(\nabla - \lambda - \deg(a))^N (am) = a(\nabla - \lambda)^N (m)$$

for all $N \geq 1$, so $a(M_\lambda) \subseteq M_{\lambda+\deg(a)}$. Thus if we let T be a set of coset representatives for $\mathbb{Z}$ as a subgroup of K under addition, and set

$$M_n = \bigoplus_{\lambda \in T+n} M_\lambda$$

then $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a graded module.

3 Representations of quivers

3.1 Bilinear and quadratic forms

Let Q be a quiver and let $A = KQ$. For simplicity throughout K is algebraically closed. We consider f.d. A-modules.

We consider $\mathbb{Z}^{Q_0}$ as column vectors, with rows indexed by Q_0. Let $\epsilon[i]$ be the coordinate vector associated to a vertex $i \in Q_0$. Thus $\epsilon[i]_j = \delta_{ij}$.

The dimension vector of a module X is $\underline{\dim}\, X \in \mathbb{Z}^{Q_0}$.

Definition. The Ringel form is the bilinear form $\langle -, - \rangle$ on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$$

The corresponding quadratic form $q(\alpha) = \langle \alpha, \alpha \rangle$ is called the Tits form. There is a corresponding symmetric bilinear form

$$(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Note that q and $(-, -)$ don't depend on the orientation of Q.

If S is an algebra and M is an S-S-bimodule, then for $n \geq 0$ the tensor power of M is

$$T^n(M) = \begin{cases} S & (n = 0) \\ \underbrace{M \otimes_S M \otimes_S \cdots \otimes_S M}_{n \text{ copies}} & (n > 0) \end{cases}$$

The tensor algebra is the graded algebra $T_S(M) = \bigoplus_{n=0}^{\infty} T^n(M)$.

Lemma 1. If $A = T_S(M)$, then there is an exact sequence of A-A-bimodules

$$0 \to A \otimes_S M \otimes_S A \xrightarrow{f} A \otimes_S A \xrightarrow{g} A \to 0$$

where $f(a \otimes m \otimes a') = am \otimes a' - a \otimes ma'$ and $g(a \otimes a') = aa'$.

Proof. For all $n \in \mathbb{N}$, the maps f and g induce a sequence

$$0 \to \bigoplus_{p+1+q=n} T^p(M) \otimes M \otimes T^q(M) \xrightarrow{f_n} \bigoplus_{i+j=n} T^i(M) \otimes T^j(M) \xrightarrow{g_n} T^n(M) \to 0$$

and it suffices to show that these sequences are exact, for the sequence we want is the direct sum of these. But this sequence can be identified with

$$0 \to T^n(M)^n \xrightarrow{f_n} T^n(M)^{n+1} \xrightarrow{g_n} T^n(M) \to 0$$

where now g_n is the summation map, and f_n sends $(t_1, \dots, t_n)$ to $(t_1, t_2 - t_1, t_3 - t_2, \dots, t_n - t_{n-1}, -t_n)$. This is clearly exact.

Lemma 2. If $A = T_S(M)$ and X is an A-module, then there is an exact sequence

$$0 \to A \otimes_S M \otimes_S X \to A \otimes_S X \to X \to 0$$

Proof. Tensor the sequence above with X, and use that A_A is free, so flat.

Lemma 3. If a ring can be written as a product $R = R_1 \times \times \cdots \times R_n$, then any left or right R-module X decomposes canonically as a direct sum of R_i-modules X_i (on the same side), and one can identify $X \otimes_R Y$ with $\bigoplus_{i=1}^n X_i \otimes_{R_i} Y_i$.

Proof. Omitted.

Theorem (Standard resolution) If X is a KQ-module (not necessarily f.d.) then it has projective resolution

$$0 \to \bigoplus_{a \in Q_1} KQe_{h(a)} \otimes_K e_{t(a)}X \to \bigoplus_{i \in Q_0} KQe_i \otimes_K e_iX \to X \to 0.$$

Proof. We can identify $KQ = T_S(M)$ where $S \cong K^{Q_0}$ is the subalgebra of KQ spanned by the trivial paths and M is the subspace of KQ spanned by the arrows. Thus $M = \bigoplus_{a \in Q_1} Ka$.

Moreover if U and V are a right and left KQ-module, then by Lemma 3 we have $U \otimes_S V \cong \bigoplus_{i \in Q_0} Ue_i \otimes_K e_iV$ and $U \otimes_S M \otimes_S V \cong \bigoplus_{a \in Q_1} Ue_{h(a)} \otimes_K e_{t(a)}V$.

This shows KQ is left hereditary (which we already knew) and

Corollary. If X and Y are (f.d.) KQ-modules, then

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y).$$

Proof. Apply $\operatorname{Hom}(-, Y)$ to the projective resolution to get an exact sequence

$$0 \to \operatorname{Hom}(X, Y) \to \bigoplus_{i \in Q_0} \operatorname{Hom}(KQe_i \otimes_K e_iX, Y) \to$$

$$\to \bigoplus_{a\in Q_1} \operatorname{Hom}(KQe_{h(a)} \otimes_K e_{t(a)}X, Y) \to \operatorname{Ext}^1(X,Y) \to 0.$$

Now $\operatorname{Hom}(KQe_j \otimes_K e_i X, Y) \cong \operatorname{Hom}_K(e_i X, \operatorname{Hom}(KQe_j, Y)) \cong \operatorname{Hom}_K(e_i X, e_j Y)$ so it has dimension $(\underline{\dim}\, X)_i (\underline{\dim}\, Y)_j$.

3.2 Classification of quivers

A quiver is Dynkin if it is obtained by orienting one of the following graphs (each with n vertices):

A_n, D_n, E_6, E_7, E_8.

A quiver is extended Dynkin if it is obtained by orienting one of the following (each with $n+1$ vertices). In each case we define $\delta \in \mathbb{N}^{Q_0}$.

$\tilde{A}_n$ (including case $n = 0$), $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

Properties. (1) Any extended Dynkin quiver has at least one extending vertex, with $\delta_i = 1$. Deleting an extending vertex one obtains the corresponding Dynkin quiver.

(2) δ is in the radical of q, that is $(\delta, \alpha) = 0$ for all α. We need to check that $(\delta, \epsilon[i]) = 0$ for all i. This is $2\delta_i - \Sigma_{j-i}\delta_j$.

Lemma 1. Every connected quiver is either Dynkin, or has an extended Dynkin subquiver.

Proof. Case-by-case analysis. If there is a loop, it contains $\tilde{A}_0$. If there is a cycle it contains $\tilde{A}_n$. If there is a vertex of valency 4 it contains $\tilde{D}_4$. If there are two vertices of valency 3 it contains $\tilde{D}_n$. Thus (unless it is A_n) it is a star with three arms. If all arms have length > 1 then contains $\tilde{E}_6$. If two arms have length 1 then Dynkin. Thus suppose one arm has length 1. If both remaining arms have length > 2 then contains $\tilde{E}_7$. Thus suppose one has length 2. If the other length is 2,3,4 then Dynkin, if > 4 it contains $\tilde{E}_8$.

Theorem. (i) If Q is Dynkin, q is positive definite, that is $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^{Q_0}$.
(ii) If Q is extended Dynkin quivers, q is positive semidefinite, that is $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^{Q_0}$. Moreover

$$\alpha \in \operatorname{rad} q \Leftrightarrow q(\alpha) = 0 \Leftrightarrow \alpha \in \mathbb{Z}\delta.$$

(iii) If Q is connected and not Dynkin or extended Dynkin then there is $\alpha \in \mathbb{N}^{Q_0}$ with $(\alpha, \epsilon[i]) \leq 0$ for all i and $q(\alpha) < 0$.

Proof. (ii) For $i \neq j$ we have $(\epsilon[i], \epsilon[j]) \leq 0$. Thus

$$0 \leq -\frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \delta_j \left(\frac{\alpha_i}{\delta_i} - \frac{\alpha_j}{\delta_j} \right)^2$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \frac{\alpha_j^2}{\delta_j} - \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_j \frac{\alpha_i^2}{\delta_i}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{j} \left(\sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \right) \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{j} \left((\delta, \epsilon[j]) - (\epsilon[j], \epsilon[j]) \delta_j \right) \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j + \sum_{j} (\epsilon[j], \epsilon[j]) \alpha_j^2$$

$$= \sum_{i,j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j = (\alpha, \alpha) = 2q(\alpha).$$

Thus q is positive semidefinite.

If $q(\alpha) = 0$ then α_i / δ_i is independent of i, so α is a multiple of δ. Since some $\delta_i = 1$, $\alpha \in \mathbb{Z}\delta$.

Trivially $\alpha \in \mathbb{Z}\delta \Rightarrow \alpha \in \operatorname{rad} q \Rightarrow q(\alpha) = 0$.

(i) Follows by embedding in the corresponding extended Dynkin diagram.

(iii) Take an extended Dynkin subquiver Q' with radical vector δ. If all vertices of Q are in Q', take $\alpha = \delta$. If i is a vertex not in Q' but connected to Q' by an arrow, take $\alpha = 2\delta + \epsilon[i]$.

Definition. We suppose that Q is Dynkin or extended Dynkin. The roots are

$$\Delta = \{\alpha \in \mathbb{Z}^{Q_0} \mid \alpha \neq 0, q(\alpha) \leq 1\}.$$

(One can define roots in general, but the definition is more complicated.)

A *root* is real if $q(\alpha) = 1$, otherwise it is *imaginary.* In the Dynkin case all roots are real. In the extended Dynkin case the imaginary roots are $r\delta$ with $r \neq 0$.

Lemma 2. Any root is positive or negative.

Proof. Write $\alpha = \alpha^+ - \alpha^-$ with $\alpha^+, \alpha^- \in \mathbb{N}^{Q_0}$ having disjoint support, then $(\alpha^+, \alpha^-) \leq 0$. But then

$$1 \geq q(\alpha) = q(\alpha^+) + q(\alpha^-) - (\alpha^+, \alpha^-) \geq q(\alpha^+) + q(\alpha^-)$$

so one of α^+, α^- is an imaginary root, hence a multiple of δ. Impossible if disjoint support.

Lemma 3. If Q is Dynkin, then Δ is finite.

Proof. Embed in an extended Dynkin quiver with radical vector δ and extending vertex i. Roots α for Q correspond to roots with $\alpha_i = 0$. Now

$$q(\alpha \pm \delta) = q(\alpha) \pm (\alpha, \delta) + q(\delta) = q(\alpha) = 1$$

so $\beta = \alpha \pm \delta$ is a root, and hence positive or negative. Now $\beta_i = \pm 1$. Thus $-\delta_j \leq \alpha_j \leq \delta_j$ for all j.

3.3 Gabriel's Theorem

In this section $A = KQ$ and we consider f.d. A-modules.

Gabriel's Theorem.
(i) KQ has finite representation type if and only if Q is Dynkin
(ii) If Q is Dynkin, then the assigment $X \rightsquigarrow \underline{\dim}\, X$ gives a 1-1 correspondence between indecomposable modules and positive roots.

In this section I give a purely homological proof of "if" in (i) and of (ii). This can also be proved using the AR translate, or using reflection functors.

Later we will classify the indecomposable representations of extended Dynkin quiver. We will see that they have infinitely many indecomposables. Since any non-Dyknin quiver contains an extended Dynkin quiver, (i) will follow.

Lemma 1. If $\xi : 0 \to X \to Y \to Z \to 0$ is a non-split exact sequence of f.d. modules for any algebra, then

$$\dim \operatorname{End}(Y) < \dim \operatorname{End}(X \oplus Z).$$

Proof. Applying $\operatorname{Hom}(-, Y)$ to the short exact sequence gives a long exact sequence

$$0 \to \operatorname{Hom}(Z, Y) \to \operatorname{Hom}(Y, Y) \to \operatorname{Hom}(X, Y) \to \dots$$

so that

$$\dim \operatorname{Hom}(Y, Y) \le \dim \operatorname{Hom}(Z, Y) + \dim \operatorname{Hom}(X, Y).$$

Similarly, applying $\operatorname{Hom}(X, -)$ gives

$$\dim \operatorname{Hom}(X, Y) \le \dim \operatorname{Hom}(X, X) + \dim \operatorname{Hom}(X, Z).$$

Now applying $\operatorname{Hom}(Z, -)$ gives the long exact sequence

$$0 \to \operatorname{Hom}(Z, X) \to \operatorname{Hom}(Z, Y) \to \operatorname{End}(Z) \xrightarrow{f} \operatorname{Ext}^1(Y, X)$$

and the connecting map f is nonzero since it sends 1_Z to the element in $\operatorname{Ext}^1(Y, X)$ represented by ξ, so

$$\dim \operatorname{Hom}(Z, Y) < \dim \operatorname{Hom}(Z, X) + \dim \operatorname{Hom}(Z, Z).$$

Combining these three inequalities we get the result.

Definition. An A-module X is a *brick* if $\operatorname{End}_A(X)$ is a division algebra. Since K is algebraically closed, it follows that $\operatorname{End}_A(X) = K$.

We say that X *has self-extensions* if $\operatorname{Ext}^1_A(X, X) \neq 0$.

Lemma 2 (Happel-Ringel Lemma). If X, Y are indecomposable and $\operatorname{Ext}^1_A(Y, X) = 0$ then any non-zero map $\theta : X \to Y$ is mono or epi. In particular, taking $X = Y$, any indecomposable module without self-extensions is a brick.

Proof. We have exact sequences

$$\xi : 0 \to \operatorname{Im} \theta \to Y \to \operatorname{Coker} \theta \to 0$$

and

$$\eta : 0 \to \operatorname{Ker} \theta \to X \to \operatorname{Im} \theta \to 0$$

From $\mathrm{Ext}^1(\mathrm{Coker}\,\theta, \eta)$ we get

$$\cdots \to \mathrm{Ext}^1(\mathrm{Coker}\,\theta, X) \xrightarrow{f} \mathrm{Ext}^1(\mathrm{Coker}\,\theta, \mathrm{Im}\,\theta) \to \mathrm{Ext}^2(\mathrm{Coker}\,\theta, \mathrm{Ker}\,\theta) = 0$$

so $\xi = f(\zeta)$ for some ζ. Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
\zeta : 0 & \longrightarrow & X & \xrightarrow{\alpha} & Z & \longrightarrow & \mathrm{Coker}\,\theta & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \big\| & & \\
\xi : 0 & \longrightarrow & \mathrm{Im}\,\theta & \xrightarrow{\delta} & Y & \longrightarrow & \mathrm{Coker}\,\theta & \longrightarrow & 0.
\end{array}$$

Now the sequence

$$0 \to X \xrightarrow{\binom{\alpha}{\beta}} Z \oplus \mathrm{Im}\,\theta \xrightarrow{(\gamma \ \ -\delta)} Y \to 0$$

is exact, so splits since $\mathrm{Ext}^1(Y, X) = 0$.

If $\mathrm{Im}\,\theta \neq 0$ then X or Y is a summand of $\mathrm{Im}\,\theta$ by Krull-Remak-Schmidt. But if θ is not mono or epi, then $\dim \mathrm{Im}\,\theta < \dim X, \dim Y$, a contradiction.

Lemma 3 (Ringel). If X is indecomposable and not a brick, then it has a submodule and a quotient which are bricks with self-extensions.

Proof. It suffices to prove that if X is indecomposable and not a brick then there is a proper submodule $U \subseteq X$ which is indecomposable with self-extensions, for if U is not a brick one can iterate, and a dual argument deals with the case of a quotient.

Pick $\theta \in \mathrm{End}(X)$ with $I = \mathrm{Im}\,\theta$ of minimal dimension $\neq 0$. We have $I \subseteq \mathrm{Ker}\,\theta$, for X is indecomposable and not a brick so θ is nilpotent. Now $\theta^2 = 0$ by minimality. Let $\mathrm{Ker}\,\theta = \bigoplus_{i=1}^r K_i$ with K_i indecomposable, and pick j such that the composition $\alpha : I \hookrightarrow \mathrm{Ker}\,\theta \twoheadrightarrow K_j$ is non-zero. Now α is mono, for the map $X \twoheadrightarrow I \xrightarrow{\alpha} K_j \hookrightarrow X$ has image $\mathrm{Im}\,\alpha \neq 0$ so α is mono by minimality.

We have $\mathrm{Ext}^1(I, K_j) \neq 0$, for otherwise the pushout

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^r K_i & \longrightarrow & X & \longrightarrow & I & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & K_j & \longrightarrow & Y & \longrightarrow & I & \longrightarrow & 0.
\end{array}$$

splits, and it follows that K_j is a summand of X, a contradiction. Now K_j has self-extensions since α induces an epi $\mathrm{Ext}^1(K_j, K_j) \to \mathrm{Ext}^1(I, K_j)$. Finally take $U = K_j$.

Lemma 4 (Riedtmann and Schofield). If $X_1, \ldots, X_k$ are indecomposable and $\mathrm{Ext}^1(X_i, X_j) = 0$ for all i, j, then there is no cycle of non-zero non-isomorphisms

$$X_1 \to X_2 \to \cdots \to X_k \to X_1.$$

Proof. Each of the maps is a mono or an epi. Thus at some stage there is an epi followed by a mono. But then the composition is not mono or epi or zero.

Theorem. Two modules X, Y without self-extensions of the same dimension vector must be isomorphic.

(Should really do this with geometry - next semester.)

Proof. We show first that the universal map $f : X \to Y^n$ where $n = \dim \mathrm{Hom}_A(X, Y)$ is mono. There are exact sequences

$$0 \to \mathrm{Ker}\, f \to X \to \mathrm{Im}\, f \to 0$$

and

$$0 \to \mathrm{Im}\, f \to Y^n \to \mathrm{Coker}\, f \to 0.$$

Since this is the universal map, the composition

$$\mathrm{Hom}(Y^n, Y) \to \mathrm{Hom}(\mathrm{Im}\, f, Y) \xrightarrow{g} \mathrm{Hom}(X, Y)$$

is onto, hence g is onto. Also

$$0 = \mathrm{Ext}^1(Y^n, Y) \to \mathrm{Ext}^1(\mathrm{Im}\, f, Y) \to \mathrm{Ext}^2(\mathrm{Coker}\, f, Y) = 0$$

so $\mathrm{Ext}^1(\mathrm{Im}\, f, Y) = 0$. Thus we get the long exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Im}\, f, Y) \twoheadrightarrow \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathrm{Ker}\, f, Y) \to \mathrm{Ext}^1(\mathrm{Im}\, f, Y) = 0$$

so $\mathrm{Hom}(\mathrm{Ker}\, f, Y) = 0$. Thus $\langle \underline{\dim}\, \mathrm{Ker}\, f, \underline{\dim}\, Y \rangle \leq 0$. Thus $\langle \underline{\dim}\, \mathrm{Ker}\, f, \underline{\dim}\, X \rangle \leq 0$. But if $\mathrm{Ker}\, f \neq 0$ then the fact that it embeds in X ensures that $\mathrm{Hom}(\mathrm{Ker}\, f, X) \neq 0$, and hence $\mathrm{Ext}^1(\mathrm{Ker}\, f, X) \neq 0$. But

$$0 = \mathrm{Ext}^1(X, X) \to \mathrm{Ext}^1(\mathrm{Ker}\, f, X) \to \mathrm{Ext}^2(\mathrm{Im}\, f, X) = 0$$

so $\mathrm{Ext}^1(\mathrm{Ker}\, f, X) = 0$, a contradiction. Thus $\mathrm{Ker}\, f = 0$.

Thus $f : X \hookrightarrow Y^n$. Applying $\mathrm{Hom}(-, Y)$ gives

$$0 = \mathrm{Ext}^1(Y^n, Y) \to \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^2(\mathrm{Coker}\, f, Y) = 0$$

so $\mathrm{Ext}^1(X,Y)=0$. Also any indecomposable summand X_i of X has a non-zero map to an indecomposable summand Y_j of Y.

Dually $\mathrm{Ext}^1(Y,X)=0$ and any indecomposable summand Y_j of Y has a non-zero map to an indecomposable summand X_i of X.

This gives a cycle of non-zero maps alternating between indecomposable summands of X and indecomposable summands of Y. By the Lemma of Riedtmann and Schofield, one of the maps must be an isomorphism. Thus X and Y have a common indecomposable summand. Thus $X \cong Z \oplus X'$ and $Y \cong Z \oplus Y'$. Then $X' \cong Y'$ by induction, and hence $X \cong Y$.

Theorem. If Q is Dynkin then the assignment $X \rightsquigarrow \underline{\dim}\, X$ gives a 1-1 correspondence between indecomposable modules and positive roots. In particular KQ has finite representation type.

Proof. If $\underline{\dim}\, X = \alpha$ then

$$q(\alpha) = \dim \mathrm{End}(X) - \dim \mathrm{Ext}^1(X,X).$$

Since q is positive definite, there are no bricks with self-extensions. Thus by Ringel's lemma, every indecomposable is a brick. Again, since q is positive definite, the dimension vector α of an indecomposable satisfies $q(\alpha)=1$, so α is a positive root, and X has no self-extensions. Thus by the theorem, two indecomposables with the same dimension vector must be isomorphic.

Suppose that α is a positive root. There are modules of dimension vector α, for example there is a semisimple module. Amongst all such modules choose one, say X, with $\dim \mathrm{End}(X)$ minimal. We show that X is indecomposable. Suppose for a contradiction that $X = U \oplus V$. By minimality, we have $\mathrm{Ext}^1(U,V) = \mathrm{Ext}^1(V,U) = 0$. Thus

$$1 = q(\alpha) = q(\underline{\dim}\, U) + q(\underline{\dim}\, V) + \dim \mathrm{Hom}(U,V) + \dim \mathrm{Hom}(V,U).$$

Since q is positive definite, this is impossible. Thus X is indecomposable.

3.4 Cartan and Coxeter matrices

Suppose that Q has no oriented cycles, so $A = KQ$ is f.d.

Definition. The Cartan matrix C has rows and column indexed by Q_0, and is defined by

$$C_{ij} = \dim \mathrm{Hom}(P[i], P[j]) = \dim e_i KQ e_j$$

$$= \text{number of paths from } j \text{ to } i.$$

Thus the jth column is $C\epsilon[j] = \underline{\dim} P[j]$, and the jth row is $C^T\epsilon[j] = \underline{\dim} I[j]$. Namely, $(C\epsilon[j])_i = C_{ij} = \dim e_i KQe_j = \dim e_i P[j]$ and $(C^T\epsilon[j])_i = C^T_{ij} = C_{ji} = \dim D(e_jKQe_i) = \dim e_i I[j]$.

Lemma 1. For any α we have $\langle \underline{\dim} P[j], \alpha\rangle = \alpha_j = \langle \alpha, \underline{\dim} I[j]\rangle$. It follows that C is invertible, with inverse $(\langle \epsilon[j], \epsilon[i]\rangle)_{ij}$.

Proof. When $\alpha = \underline{\dim} X$, we have

$$\langle \underline{\dim} P[j], \alpha\rangle = \dim \mathrm{Hom}(P[j], X) - \dim \mathrm{Ext}^1(P[j], X) = \dim e_j X$$

$$\langle \alpha, \underline{\dim} I[j]\rangle = \dim \mathrm{Hom}(X, I[j]) - \dim \mathrm{Ext}^1(X, I[j]) =$$
$$= \dim \mathrm{Hom}(P[j], X) = \dim e_j X$$

It follows for all α by additivity.

Now using that $\underline{\dim} P[j] = \sum_i C_{ij}\epsilon[i]$, the equality $\langle \underline{\dim} P[j], \epsilon[k]\rangle = \delta_{jk}$ gives that $\sum_i C_{ij}\langle \epsilon[i], \epsilon[k]\rangle = \delta_{jk}$.

Definition. The Coxeter matrix is $\Phi = -C^TC^{-1}$. That is, it is the matrix with $\Phi \underline{\dim} P[i] = -\underline{\dim} I[i]$ for all i. Thus $\Phi \underline{\dim} P = -\underline{\dim} \nu(P)$ for any projective module P.

Lemma 2. If X has no projective summand, then $\underline{\dim} \tau X = \Phi \underline{\dim} X$.

Proof. If $0 \to P_1 \to P_0 \to X \to 0$ is the minimal projective resolution, then $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation, so one gets a sequence
$$0 \to \tau X \to \nu(P_1) \to \nu(P_0) \to \nu(X) \to 0$$
Since X has no projective summand, $\mathrm{Hom}(X, A) = 0$, so $\nu(X) = 0$. Thus

$$\underline{\dim} \tau X = \underline{\dim} \nu(P_1) - \underline{\dim} \nu(P_0)$$

$$= \Phi(\underline{\dim} P_0 - \underline{\dim} P_1) = \Phi \underline{\dim} X.$$

Recall that we have $\mathrm{Hom}(\tau^- X, Y) \cong D\,\mathrm{Ext}^1(Y, X) \cong \mathrm{Hom}(X, \tau Y)$.

Lemma 3. We have $\langle \alpha, \beta\rangle = -\langle \beta, \Phi\alpha\rangle = \langle \Phi\alpha, \Phi\beta\rangle$. Moreover $\Phi\alpha = \alpha$ if and only if $(\alpha, \beta) = 0$ for all β (that is, $\alpha \in \mathrm{rad}\, q$).

Proof. $\langle \underline{\dim} P[i], \beta\rangle = \langle \beta, \underline{\dim} I[i]\rangle = -\langle \beta, \Phi \underline{\dim} P[i]\rangle$.

$\Phi\alpha = \alpha$ iff $\langle \beta, \alpha - \Phi\alpha \rangle = 0$ for all β. But this is $\langle \beta, \alpha \rangle + \langle \alpha, \beta \rangle$.

Lemma 4. If Q is Dynkin then $\Phi^N = 1$ for some $N > 0$.

Proof. $q(\Phi\alpha) = q(\alpha)$, so Φ induces a map from the set of root Δ to itself. Since Φ is invertible and the roots span $\mathbb{Z}^{Q_0}$, this map is injective, and since Δ is finite, this map is a permutation. Thus it has finite order. Since the roots span $\mathbb{Z}^{Q_0}$, it follows that Φ has finite order.

Remark. If Q is Dynkin then KQ has finite representation type, and the AR quiver may be constructed by knitting. Since KQ is hereditary, we can draw $P[i]$ once we have drawn all indecomposable projectives of strictly smaller dimension. Thus there are no obstructions to knitting. If $\Phi^N = 1$ then N gives a bound on how long knitting goes on for, since $\tau^{-(N-1)}P[i] = 0$. Namely, if not, then

$$0 \leq \underline{\dim}\, \tau^{-(N-1)}P[i] = \Phi^{-(N-1)}\, \underline{\dim}\, P[i] = \Phi\, \underline{\dim}\, P[i] = -\, \underline{\dim}\, I[i].$$

One can use this to give another proof of Gabriel's Theorem, avoiding some of the lemmas in the last section.

3.5 Preprojective, preinjective and regular modules

We set $A = KQ$ where Q is a quiver without oriented cycles of extended Dynkin type.

In this section we describe the three classes of preprojective, regular and preinjective modules.

Definitions. If X is indecomposable, then
(i) X is preprojective iff $\tau^i X = 0$ for $i \gg 0$ iff $X = \tau^{-m}P[j]$ for some $m \geq 0$ and j.
(ii) X is preinjective iff $\tau^{-i}X = 0$ for some $i \gg 0$ iff $X = \tau^m I[j]$ for some $m \geq 0$ and j.
(iii) X is regular iff $\tau^i X \neq 0$ for all $i \in \mathbb{Z}$.
We say a decomposable module X is preprojective, preinjective or regular if each indecomposable summand is.

We define the *defect* of a module X to be

$$\text{defect}(X) = \langle \delta, \underline{\dim}\, X \rangle = -\langle \underline{\dim}\, X, \delta \rangle.$$

Observe that this only depends on the dimension vector of X, so it is additive on short exact sequences.

Lemma 1. There is $N > 0$ such that $\Phi^N \underline{\dim} X = \underline{\dim} X$ for regular X.

Proof. Recall that

$$\Delta \cup \{0\} = \{\alpha \in \mathbb{Z}^{Q_0} : q(\alpha) \leq 1\}.$$

Clearly it is closed under addition of an element of $\mathbb{Z}\delta = \operatorname{rad} q \subseteq \Delta \cup \{0\}$. The set of orbits

$$(\Delta \cup \{0\})/\mathbb{Z}\delta$$

is finite, since if i is an extending vertex, then any orbit contains a vector with $\alpha_i = 0$, which is either the zero vector, or a root for the corresponding Dynkin quiver.

Recall that $\Phi\alpha = \alpha$ if and only if α is radical, and that $q(\Phi\alpha) = q(\alpha)$. Thus Φ induces a permutation of the finite set $(\Delta \cup 0)/\mathbb{Z}\delta$.

Thus there is some $N > 0$ with Φ^N the identity on $(\Delta \cup 0)/\mathbb{Z}\delta$. Since $\epsilon[i] \in \Delta$ it follows that Φ^N is the identity on $\mathbb{Z}^{Q_0}/\mathbb{Z}\delta$.

Let $\Phi^N \underline{\dim} X - \underline{\dim} X = r\delta$. An induction shows that $\Phi^{iN} \underline{\dim} X = \underline{\dim} X + ir\delta$ for all $i \in \mathbb{Z}$. If $r < 0$ this is not positive for $i \gg 0$, so X must be preprojective. If $r > 0$ this is not positive for $i \ll 0$, so X is preinjective. Thus $r = 0$.

Lemma 2. If X is indecomposable, then X is preprojective, regular or preinjective according as the defect of X is -ve, zero or +ve.

Proof. If X is preprojective then it has defect < 0, since

$$\langle \underline{\dim}\, \tau^{-m} P[j], \delta \rangle = \langle \Phi^{-m} \underline{\dim}\, P[j], \delta \rangle = \langle \underline{\dim}\, P[j], \Phi^m \delta \rangle = \langle \underline{\dim}\, P[j], \delta \rangle = \delta_j > 0.$$

Similarly preinjectives have defect > 0. If X is regular with dimension vector α, then $\Phi^N \alpha = \alpha$. Let $\beta = \alpha + \Phi\alpha + \ldots \Phi^{N-1}\alpha$. Clearly $\Phi\beta = \beta$, so $\beta = r\delta$. Now

$$0 = \langle \beta, \delta \rangle = \sum_{i=0}^{N-1} \langle \Phi^i \alpha, \delta \rangle = N \langle \alpha, \delta \rangle,$$

so $\langle \alpha, \delta \rangle = 0$, that is, X has defect 0.

Lemma 3. Let X, Y be indecomposable.
(i) If Y is preprojective and X is not, then $\operatorname{Hom}(X, Y) = 0$, $\operatorname{Ext}^1(Y, X) = 0$.
(ii) If Y is preinjective and X is not, then $\operatorname{Hom}(Y, X) = 0$, $\operatorname{Ext}^1(X, Y) = 0$.

Proof. (i) As X is not preprojective, $X \cong \tau^{-i}\tau^i X$ for $i \geq 0$. Thus

$$\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(\tau^{-i}\tau^i X, Y) \cong \mathrm{Hom}(\tau^i X, \tau^i Y) = 0$$

for $i \gg 0$. Also $\mathrm{Ext}^1(Y,X) \cong D\,\mathrm{Hom}(\tau^- X, Y) = 0$. (ii) is dual.

Remark. We draw a picture with preprojectives on left, regulars in the middle and preinjectives on the right. There are no maps going from the right to the left.

Remark. The indecomposable preprojectives and preinjectives are bricks without self-extensions. For example if $X = \tau^{-m}P[j]$ then

$$\mathrm{End}(X) = \mathrm{Hom}(\tau^{-m}P[j], \tau^{-m}P[j]) \cong \mathrm{Hom}(P[j], \tau^m\tau^{-m}P[j]) \cong \mathrm{End}(P[j])$$

and $P[j]$ is a brick since Q has no oriented cycles. Also

$$\langle \underline{\dim}\, X, \underline{\dim}\, X\rangle = \langle \Phi^{-m}\,\underline{\dim}\, P[j], \Phi^{-m}\,\underline{\dim}\, P[j]\rangle = \langle \underline{\dim}\, P[j], \underline{\dim}\, P[j]\rangle = 1.$$

from which it follows that X has no self-extensions.

Lemma 4. If α is a positive real root, and either $\langle \alpha, \delta\rangle \neq 0$ or $\alpha \leq \delta$, then there is a unique indecomposable of dimension α. It is a brick without self-extensions.

Proof. Pick a module X of dimension α with $\dim \mathrm{End}(X)$ minimal.

If X decomposes, $X = U \oplus V$. By minimality, $\mathrm{Ext}^1(U,V) = \mathrm{Ext}^1(V,U) = 0$. Then

$$1 = q(\alpha) = q(\underline{\dim}\, U) + q(\underline{\dim}\, V) + \dim \mathrm{Hom}(U,V) + \dim \mathrm{Hom}(V,U).$$

Thus, $q(\underline{\dim}\, U) = 0$, say, so $\underline{\dim}\, U \in \mathbb{Z}\delta$. Now $\underline{\dim}\, V \notin \mathbb{Z}\delta$, for otherwise $\underline{\dim}\, X \in \mathbb{Z}\delta$, but α is a real root. Thus $q(\underline{\dim}\, V) = 1$ and therefore the Hom spaces must be zero. Thus $\langle \underline{\dim}\, V, \underline{\dim}\, U\rangle = 0$, so $\langle \underline{\dim}\, V, \delta\rangle = 0$. Thus also $\langle \alpha, \delta\rangle = 0$. Now $\underline{\dim}\, U \in \mathbb{Z}\delta$, so $\delta \leq \alpha$, which contradicts the assumption on α.

Now if $\langle \alpha, \delta\rangle \neq 0$ then X is preprojective or preinjective, so it is a brick without self-extensions.

If $\alpha \leq \delta$ then it is a brick (and since α is a real root, it also no self-extensions) for otherwise, by Ringel's lemma, it has a submodule Y which is a brick with self-extensions. But then $q(\underline{\dim}\, Y) \leq 0$, so $\underline{\dim}\, Y$ is a multiple of δ, a contradiction.

Uniqueness follows.

3.6 Wide subcategories

In this section A is an arbitrary algebra, not even necessarily f.d., but we only consider f.d. modules.

Definition. A full subcategory $\mathcal{C}$ of A-mod is *wide* if it is closed under kernels, cokernels and extensions.

It follows that $\mathcal{C}$ is also closed under images. In fact $\mathcal{C}$ is an abelian category in its own right, and the inclusion functor is exact.

We say that an A-module X is *$\mathcal{C}$-simple* if it is in $\mathcal{C}$, and is simple as an object of $\mathcal{C}$. Thus there is no exact sequence $0 \to U \to X \to V \to 0$ with U, V non-zero and in $\mathcal{C}$. Clearly any $\mathcal{C}$-simple is a brick.

Example 1. If X is a brick without self-extensions, then $\operatorname{add} X$ is wide. In this case $\operatorname{add} X$ is equivalent to K-mod, and X is $\mathcal{C}$-simple.

More generally:

Example 2. If $\mathcal{B}$ is a collection of A-modules, we write $\mathcal{F}(\mathcal{B})$ for the full subcategory of A-mod consisting of the modules X with filtrations by submodules

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that each X_i/X_{i-1} is isomorphic to a module in B. Clearly $\mathcal{F}(\mathcal{B})$ is closed under extensions.

(Ringel's Simplification) Suppose that $\mathcal{B}$ is a set of orthogonal bricks, with orthogonality meaning that $\operatorname{Hom}(X, Y) = 0$ if $X, Y \in \mathcal{B}$ and $X \not\cong Y$. Then $\mathcal{F}(\mathcal{B})$ is a wide subcategory of A-mod, and the $\mathcal{F}(\mathcal{B})$-simples are the modules isomorphic to modules in $\mathcal{B}$.

Proof. Let $f : X \to Y$ be a morphism where X and Y have filtrations of lengths n and m.

We prove it by induction on n that $\operatorname{Ker} f$ has a filtration. The result for $\operatorname{Coker} f$ is dual.

We can assume $f(X_1) \neq 0$, for otherwise $X_1 \subseteq \operatorname{Ker} f$ and $\operatorname{Ker} f/X_1$ is the kernel of the induced map $f' : X/X_1 \to Y$, and by induction this kernel is in $\mathcal{F}(\mathcal{B})$, hence $\operatorname{Ker} f \in \mathcal{F}(\mathcal{B})$.

We may also assume $f(X_1) = Y_1$, for there is some i with $f(X_1) \subseteq Y_i$ but

$f(X_1) \not\subseteq Y_{i-1}$. Then

$$X_1 \to Y_i \to Y_i/Y_{i-1}$$

is a non-zero map, so an isomorphism. Thus $Y_i = Y_{i-1} \oplus f(X_1)$. Now Y has another filtration with $Y'_j = f(X_1) + Y_{j-1}$ for $1 \le j < i$ and $Y'_j = Y_j$ for $i \le j \le m$, and this filtrarion has $f(X_1) = Y'_1$.

Now $\operatorname{Ker} f$ is isomorphic to the kernel of the induced map $\overline{f} : X/X_1 \to Y/Y_1$, so by induction it is in $\mathcal{F}(\mathcal{B})$.

Remark. Any wide subcategory $\mathcal{C}$ arises as $\mathcal{F}(\mathcal{B})$ where one takes $\mathcal{B}$ to be the set of $\mathcal{C}$-simple objects.

Example 3. If $\mathcal{X}$ is a set of modules of projective dimension ≤ 1 then the perpendicular category is the full subcategory with modules

$$F^{\perp} = \{M \in A - mod : \operatorname{Hom}(X, M) = \operatorname{Ext}^1(X, M) = 0 \text{ for all } X \in \mathcal{X}\}.$$

Then $\mathcal{X}^{\perp}$ is a wide subcategory.

Proof. Say $\theta : M \to N$ is in $\mathcal{F}^{\perp}$ and $X \in \mathcal{X}$. We get

$$0 \to \operatorname{Hom}(X, \operatorname{Ker}\theta) \to \operatorname{Hom}(X, M) \to \operatorname{Hom}(X, \operatorname{Im}\theta)$$

$$\to \operatorname{Ext}^1(X, \operatorname{Ker}\theta) \to \operatorname{Ext}^1(X, M) \to \operatorname{Ext}^1(X, \operatorname{Im}\theta) \to 0.$$

Also one has

$$0 \to \operatorname{Hom}(X, \operatorname{Im}\theta) \to \operatorname{Hom}(X, N) \to \operatorname{Hom}(X, \operatorname{Coker}\theta)$$

$$\to \operatorname{Ext}^1(X, \operatorname{Im}\theta) \to \operatorname{Ext}^1(X, N) \to \operatorname{Ext}^1(X, \operatorname{Coker}\theta) \to 0.$$

Thus $\operatorname{Hom}(X, \operatorname{Ker}\theta) = 0$ and $\operatorname{Ext}^1(X, \operatorname{Ker}\theta) \cong \operatorname{Hom}(X, \operatorname{Im}\theta) = 0$, and similarly for $\operatorname{Coker}\theta$. Closure under extensions is easy.

Remark. In fact there is an epimorphism of rings, $A \to A_{\mathcal{X}}$ called the *universal localization* such that restriction induces an equivalence between moduels for $A_{\mathcal{X}}$ and $\mathcal{X}^{\perp}$.

Example 4. (Analogous to results from "Geometric invariant theory"). By a *stability function* for A we mean a function

$$\theta : \text{f.d. } A\text{-modules} \to \mathbb{R}$$

which is constant on isomorphism classes, so $\theta(X) = \theta(Y)$ if $X \cong Y$, and additive on short exact sequences, so $\theta(Y) = \theta(X) + \theta(Z)$ for a short exact sequence $0 \to X \to Y \to Z \to 0$.

Note that $\theta(0) = 0$, and that it is equivalent to fix $\theta(S) \in \mathbb{R}$ for each simple module S.

An A-module X is said to be θ-semistable if $\theta(X) = 0$ and $\theta(Y) \leq 0$ for all $Y \subseteq X$. It is θ-stable if $\theta(X) = 0$ and $\theta(Y) < 0$ for all non-zero proper submodules Y of X.

Observe that X is θ-semistable if and only if $\theta(X) = 0$ and $\theta(Z) \geq 0$ for any quotient Z of X.

The θ-semistable modules form a wide subcategory $\mathcal{C}$. The $\mathcal{C}$-simples are the θ-stables.

Proof. Let $f : X \to Y$ is a map between θ-semistable modules. We have $\theta(\operatorname{Im} f) \leq 0$ since it is a submodule of Y, and $\theta(\operatorname{Im} f) \geq 0$ since it as a quotient of X. Thus $\theta(\operatorname{Im} f) = 0$. Thus by additivity $\theta(\operatorname{Ker} f) = \theta(\operatorname{Coker} f) = 0$. Now any submodule U of $\operatorname{Ker} f$ is a submodule of X, so $\theta(U) \leq 0$. Thus $\operatorname{Ker} f$ is θ-semistable. Similarly any quotient V of $\operatorname{Coker} f$ is a quotient of Y so $\theta(V) \geq 0$. Thus $\operatorname{Coker} f$ is θ-semistable.

Suppose $X \subseteq Y$. If X and Y/X are θ-semistable, we need to show that Y is θ-semistable. By additivity we have $\theta(Y) = 0$. Now if $U \subseteq Y$ then $U \cap X \subseteq X$ and $U/(U \cap X) \cong (U + X)/X \subseteq Y/Z$, so both have $\theta \leq 0$, hence $\theta(U) \leq 0$.

Definition. Let $\mathcal{C}$ be a wide subcategory of A-mod. An module X in $\mathcal{C}$ is $\mathcal{C}$-*uniserial* if given any two submodules U, V of X with $U, V \in \mathcal{C}$, we have $U \subseteq V$ or $V \subseteq U$.

It follows that there is a chain of submodules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

with the $X_i \in \mathcal{C}$ such that each quotient X_i/X_{i-1} is $\mathcal{C}$-simple, and such that the X_i are the only submodules of X which belong to $\mathcal{C}$.

We say that X has $\mathcal{C}$-composition factors $X_1, X_2/X_1, \ldots, X_r/X_{r-1}$, $\mathcal{C}$-length r, $\mathcal{C}$-socle X_1 and $\mathcal{C}$-top X/X_{r-1}.

Definition. If $\mathcal{C}$ is a wide subcategory, we say that a functor $t : \mathcal{C} \to \mathcal{C}$ is a *1-Serre equivalence* for $\mathcal{C}$ if it is an equivalence and there is a natural isomorphism $\operatorname{Hom}(X, tY) \cong D \operatorname{Ext}^1(Y, X)$

We denote the inverse functor by t^-. Since t and t^- are equivalences they send $\mathcal{C}$-simples to $\mathcal{C}$-simples. Thus if S is $\mathcal{C}$-simple, so is $t^i S$ for all $i \in \mathbb{Z}$.

Example. Let A be the path algebra of the oriented cycle quiver with vertices $1, 2, \ldots, p$ and arrows $a_i : i \to i+1$ for all i (modulo p). Let $\mathcal{C}$ be the category of all f.d. A-modules.

(a) Let t be the functor given by rotation, so if Y is a representation of Q, then $(tY)_i = Y_{i-1}$ for all i. We show that t is a 1-Serre equivalence.

The standard resolution of Y is $0 \to P_1^Y \to P_0^Y \to Y \to 0$ where $P_0^Y = \bigoplus KQe_i \otimes_K Y_i$ and $P_1^Y = \bigoplus KQe_{i+1} \otimes_K Y_i$ and similarly for X.

Recall that for finite dimensional vector spaces V, W, the trace defines a perfect pairing $\mathrm{Hom}_K(V, W) \times \mathrm{Hom}_K(W, V) \to K$, $(\theta, \phi) \mapsto \mathrm{tr}(\theta\phi) = \mathrm{tr}(\phi\theta)$, so it defines an isomorphism between $V \to DW$.

Thus

$$D\,\mathrm{Hom}(P_1^Y, X) \cong \bigoplus_i D\,\mathrm{Hom}_K(Y_i, X_{i+1}) \cong \bigoplus_i \mathrm{Hom}_K(X_{i+1}, Y_i) \cong \mathrm{Hom}(P_0^X, tY)$$

Similarly

$$D\,\mathrm{Hom}(P_0^Y, X) \cong \bigoplus_i D\,\mathrm{Hom}_K(Y_i, X_i) \cong \bigoplus_i \mathrm{Hom}_K(X_i, Y_i) \cong \mathrm{Hom}(P_1^X, tY).$$

In fact the square in the following diagram is commutative

$$\begin{array}{ccccccc}
0 \longrightarrow & D\,\mathrm{Ext}^1(Y, X) & \longrightarrow & D\,\mathrm{Hom}(P_0^Y, X) & \longrightarrow & D\,\mathrm{Hom}(P_1^Y, X) \\
& & & \| & & \| \\
0 \longrightarrow & \mathrm{Hom}(X, tY) & \longrightarrow & \mathrm{Hom}(P_0^X, Y) & \longrightarrow & \mathrm{Hom}(P_1^X, Y)
\end{array}$$

so $D\,\mathrm{Ext}^1(Y, X) \cong \mathrm{Hom}(X, tY)$.

(b) We define modules S_{ij} (i vertex, $j \geq 1$) with basis $b_0, b_1, \ldots, b_{j-1}$ with $b_k \in M_{i-k}$ and

$$b_0 \xleftarrow{a_{i-1}} b_1 \xleftarrow{a_{i-2}} b_2 \ldots \xleftarrow{a_{i-j+1}} b_{j-1}$$

We define modules $S_{\lambda,n}$ ($0 \neq \lambda \in K$, $n \geq 1$) with vector space K^n at each vertex, with $a_i = 1$ for $i \neq p$ and $a_p = J_n(\lambda)$, a Jordan block.

The modules $S_{i1} = S[i]$ and $S_{\lambda,1}$ are clearly simple. In fact they are the only simple A-modules.

Suppose M is a simple module, i is a vertex $0 \neq m \in M_i$, and $x = a_{i+p-1} \dots a_{i+1} a_i$ is the path of length p from i to itself, then:

Either $xm = 0$ in which case there is some maximal j such that $s = a_{i+j} \dots a_{i+1} a_i m \neq 0$, and then s spans a submodule of M isomorphic to $S[i+j+1]$.

Or xm generates M, so there is some polynomial f with $m = f(x)xm$. Thus there is a non-constant polynomial g with $g(x)m = 0$. Factorizing g into linear factors, there is a non-zero element $m' \in M$ and $\lambda \in K$ with $(x - \lambda)m' = 0$. Thus the elements $a_{i+j} \dots a_{i+1} a_i m'$ span a submodule of M isomorphic to $S_{\lambda,1}$.

(c) Observe that there are exact sequences

$$0 \to S[i] \to S_{ij} \to S_{i-1,j-1} \to 0$$

and

$$0 \to S_{\lambda,1} \to S_{\lambda,n} \to S_{\lambda,n-1} \to 0.$$

One can show that these are non-split. The arguments below will then show that these modules are uniserial, so indecomposable, and that these are the only indecomposable modules.

Lemma 1. Assume $\mathcal{C}$ has a 1-Serre equivalence. If X is $\mathcal{C}$-uniserial, S is $\mathcal{C}$-simple and

$$\xi : 0 \to S \to E \xrightarrow{f} X \to 0$$

is non-split, then E is $\mathcal{C}$-uniserial.

Proof. Let T be the $\mathcal{C}$-socle of X.

It suffices to prove that if $U \subseteq E$ is in $\mathcal{C}$ and U is not contained in S, then $S \subseteq U$. Now we have $0 \neq f(U) \in \mathcal{C}$, so $T \subseteq f(U)$. Then $f^{-1}(T) = S + U \cap f^{-1}(T)$.

Since $t^- S$ is $\mathcal{C}$-simple, the inclusion $T \hookrightarrow X$ gives an isomorphism $\mathrm{Hom}(t^- S, T) \to \mathrm{Hom}(t^- S, X)$. Thus it gives an isomorphism

$$\mathrm{Ext}^1(X, S) \cong D\,\mathrm{Hom}(t^- S, X) \cong D\,\mathrm{Hom}(t^- S, T) \cong \mathrm{Ext}^1(T, S),$$

so the pullback sequence

$$\begin{tikzcd}
\zeta : 0 \arrow[r] & S \arrow[r] \arrow[d, equal] & f^{-1}(T) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
\xi : 0 \arrow[r] & S \arrow[r] & E \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

is non-split. Thus the sum $f^{-1}(T) = S + U \cap f^{-1}(T)$, cannot be a direct sum, so $S \cap U \cap f^{-1}(T) \neq 0$. Thus $S \subseteq U$.

Lemma 2. Assume t is a 1-Serre equivalence for $\mathcal{C}$. Then for each $\mathcal{C}$-simple T and $r \geq 1$ there is a unique $\mathcal{C}$-uniserial module with $\mathcal{C}$-top T and $\mathcal{C}$-length r. Its $\mathcal{C}$-composition factors are (from the top) $T, tT, \dots, t^{r-1}T$.

Proof. We work by induction on r. Suppose X is $\mathcal{C}$-uniserial of $\mathcal{C}$-length r with $\mathcal{C}$-top T and $\mathcal{C}$-socle $t^{r-1}T$. Let S be $\mathcal{C}$-simple. Now

$$\operatorname{Ext}^1(X,S) \cong D\operatorname{Hom}(t^-S,X) \cong D\operatorname{Hom}(t^-S,t^{r-1}T) \cong \begin{cases} K & (S \cong t^rT) \\ 0 & (\text{otherwise}) \end{cases}$$

so there is a non-split sequence $\xi : 0 \to S \to E \to X \to 0$ if and only if $S \cong t^{r-1}T$. Moreover in this case, since the space of extensions is 1-dimensional, any non-zero $\xi \in \operatorname{Ext}^1(X,S)$ gives rise to the same module E. It is $\mathcal{C}$-uniserial by the previous lemma.

Theorem. Assume $\mathcal{C}$ has a 1-Serre equivalence. Then every indecomposable object X in $\mathcal{C}$ is $\mathcal{C}$-uniserial.

Proof. Induction on $\dim X$. Let $S \subseteq X$ be a $\mathcal{C}$-simple submodule of X. Write the quotient as a direct sum of indecomposables

$$X/S = \bigoplus_{i=1}^{r} Y_i$$

By induction the Y_i are $\mathcal{C}$-uniserial. Now

$$\operatorname{Ext}^1(X/S,S) \cong \bigoplus_{i=1}^{r} \operatorname{Ext}^1(Y_i,S)$$

with the sequence $0 \to S \to X \to X/S \to 0$ corresponding to (ξ_i). Since X is indecomposable, all $\xi_i \neq 0$. Now

$$\operatorname{Ext}^1(Y_i,S) \cong \begin{cases} K & (\text{if } \mathcal{C}\text{-socle of } Y_i \text{ is } t^-S) \\ 0 & (\text{otherwise}) \end{cases}$$

so all Y_i have regular socle t^-S.

If $r = 1$ then X is $\mathcal{C}$-uniserial, so suppose $r \geq 2$ for contradiction. We may assume that $\dim Y_1 \leq \dim Y_2$, and then by the classification of $\mathcal{C}$-uniserials, there is an inclusion $f : Y_1 \hookrightarrow Y_2$. This map induces an isomorphism $\operatorname{Ext}^1(Y_2,S) \to \operatorname{Ext}^1(Y_1,S)$ so we can use f to adjust the decomposition of X/S to make one component ξ_i zero, a contradiction. Explicitly, we write $X/S = Y_1' \oplus Y_2 \oplus \cdots \oplus Y_r$ with $Y_1' = \{y_1 + \lambda f(y_1) : y_1 \in Y_1\}$ for some $\lambda \in K$.

3.7 Regular modules for extended Dynkin quivers

We return to the case of $A = KQ$ with Q extended Dynkin without oriented cycles.

Lemma 1. The category of regular modules is wide, and every indecomposable regular module is regular-uniserial. Moreover if S is a regular-simple of dimension vector α, then $\underline{\dim}\, S$ is a root. We have then $\tau^N S \cong S$, and $\tau S \cong S$ iff α is an imaginary root.

Proof. The regular modules are the defect-semistable modules, so form a wide subcategory. Now the restriction of τ to the regular modules is a 1-Serre equivalence.

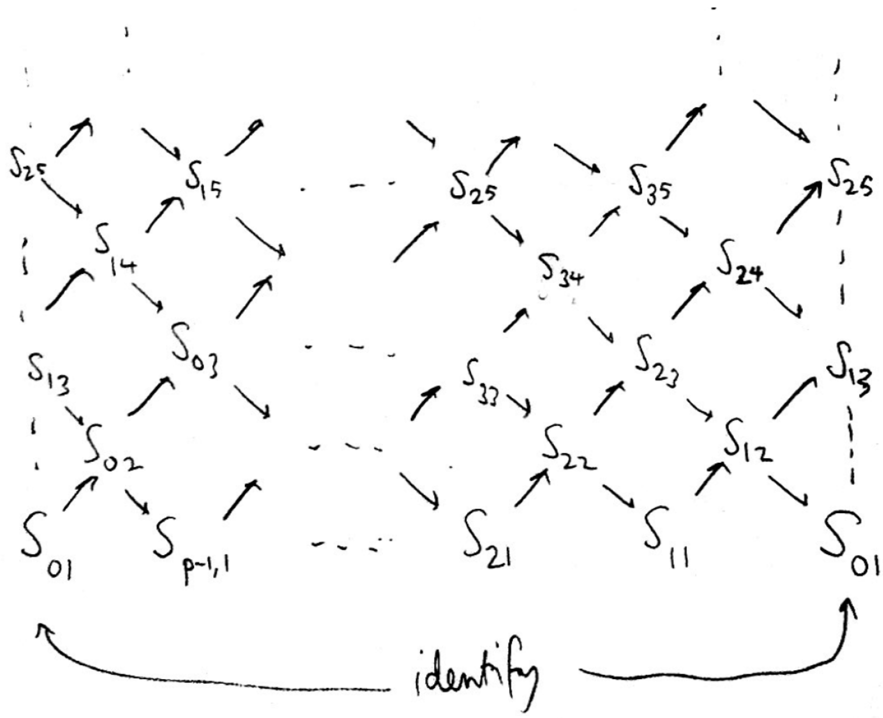
If S is regular-simple then it is a brick, so $\alpha = \underline{\dim}\, S$ is a root.

If α is a real root, then $\langle \alpha, \Phi^N \alpha \rangle = \langle \alpha, \alpha \rangle = 1$, so $\operatorname{Hom}(S, \tau^N S) \neq 0$, so $S \cong \tau^N S$.

If α is an imaginary root, then $q(\alpha) = 0$, so $\operatorname{Hom}(S, \tau S) \cong D \operatorname{Ext}^1(S, S) \neq 0$. Since S and τS are regular simples, this implies $S \cong \tau S$, and so $S \cong \tau^N S$.

Conversely if $\tau S \cong S$ then $\Phi \alpha = \alpha$, so α is radical, so α is an imaginary root.

Theorem 1. Let S be a regular simple module with period p under τ. We write S_{ij} for the regular uniserial module with regular socle $\tau^i S$ ($i \in \mathbb{Z}/p\mathbb{Z}$) and regular length $j \geq 1$. These modules all have period p under τ, and they form a connected component in the Auslander-Reiten quiver of A, of the following shape (called a tube).



Picture of tube.

Proof. Details omitted.

Next we show that the tubes are indexed by the projective line. Let e be an extending vertex, $P = P[e]$, $p = \underline{\dim}\, P$. Clearly $\langle p, p\rangle = 1 = \langle p, \delta\rangle$. Thus there is a unique indecomposable L of dimension $\delta + p$.

Now P and L are preprojective, are bricks, and have no self-extensions. $\mathrm{Hom}(L, P) = 0$, for if $\theta : L \to P$ then $\mathrm{Im}\,\theta$ is a summand of L, a contradiction.

$\mathrm{Ext}^1(L, P) = 0$ since $\langle \underline{\dim}\, L, \underline{\dim}\, P\rangle = \langle p + \delta, p\rangle = \langle p, p\rangle - \langle p, \delta\rangle = 0$.

$\dim \mathrm{Hom}(P, L) = 2$ since $\langle p, p + \delta\rangle = 2$.

Lemma 2. If $0 \neq \theta \in \mathrm{Hom}(P, L)$ then θ is mono, $\mathrm{Coker}\,\theta$ is a regular indecomposable of dimension δ, and $[reg.top(\mathrm{Coker}\,\theta)]_e \neq 0$.

Proof. Suppose θ is not mono. Now $\mathrm{Ker}\,\theta$ and $\mathrm{Im}\,\theta$ are preprojective (since they embed in P and L), and so they have defect ≤ -1. Now the sequence

$$0 \to \mathrm{Ker}\,\theta \to P \to \mathrm{Im}\,\theta \to 0$$

is exact, so

$$-1 = \text{defect}(P) = \text{defect}(\text{Ker}\,\theta) + \text{defect}(\text{Im}\,\theta) \leq -2,$$

a contradiction.

Let $X = \text{Coker}\,\theta$, and consider $\xi : 0 \to P \xrightarrow{\theta} L \to X \to 0$. Apply $\text{Hom}(-, P)$ to get $\text{Ext}^1(X, P) = K$. Apply $\text{Hom}(-, L)$ to get $\text{Hom}(X, L) = 0$. Apply $\text{Hom}(X, -)$ to get X a brick.

If X has regular top T, then

$\dim T_e = \dim \text{Hom}(P, T) = \langle p, \underline{\dim}\, T\rangle = \langle p+\delta, \underline{\dim}\, T\rangle = \dim \text{Hom}(L, T) \neq 0.$

Lemma 3. If X is regular, $X_e \neq 0$ then $\text{Hom}(Coker\theta, X) \neq 0$ for some $0 \neq \theta \in \text{Hom}(P, L)$.

Proof. $\text{Ext}^1(L, X) = 0$, so

$$\dim \text{Hom}(L, X) = \langle p + \delta, \underline{\dim}\, X\rangle = \langle p, \underline{\dim}\, X\rangle = \dim \text{Hom}(P, X) \neq 0.$$

Let α, β be a basis of $\text{Hom}(P, L)$. These give maps $a, b : \text{Hom}(L, X) \to \text{Hom}(P, X)$.

If a is an iso, let λ be an eigenvalue of $a^{-1}b$ and set $\theta = \beta - \lambda\alpha$. If a is not an iso, let $\theta = \alpha$. Either way, there is $0 \neq \phi \in \text{Hom}(L, X)$ with $\phi \circ \theta = 0$. Thus there is an induced non-zero map $\overline{\phi} : \text{Coker}\,\theta \to X$.

Lemma 4. If X is regular simple of period p, then

$$\underline{\dim}\, X + \underline{\dim}\, \tau X + \cdots + \underline{\dim}\, \tau^{p-1} X = \delta.$$

Proof. Let $\underline{\dim}\, X = \alpha$. If $\alpha_e \neq 0$ there is a map $\text{Coker}\,\theta \to X$ which must be onto. If $\alpha_e = 0$, then $\delta - \alpha$ is a root, and $(\delta - \alpha)_e = 1$, so $\delta - \alpha$ is a positive root. Either way, $\alpha \leq \delta$.

If $\alpha = \delta$, then $X \cong \tau X$, so we are done. Thus we may suppose α is a real root. Now $\delta - \alpha$ is a real root, and $\langle \delta, \delta - \alpha\rangle = 0$, so there is a regular brick Y of dimension $\delta - \alpha$.

We have $\langle \alpha, \delta-\alpha\rangle = -1$, so $0 \neq \text{Ext}^1(X, Y) \cong D\,\text{Hom}(Y, \tau X)$, so $reg.topY \cong \tau X$.

Also $\langle \delta - \alpha, \alpha \rangle = -1$, so $0 \neq \operatorname{Ext}^1(Y, X) \cong D \operatorname{Hom}(\tau^- X, Y)$, so $reg.soc Y \cong \tau^- X$.

It follows that Y must at least involve $\tau X, \tau^2 X, \ldots, \tau^{p-1} X$, so

$$\underline{\dim} X + \underline{\dim} \tau X + \cdots + \underline{\dim} \tau^{p-1} X \leq \delta.$$

Also the sum is invariant under Φ, so it is a multiple of δ.

Consequences.
(1) All but finitely many regular simples have dimension δ, so all but finitely many tubes have period 1. This is because there are only finitely many indecomposables with dimension $\leq \delta$.
(2) Each tube contains a unique module in the set

$\Omega = \{$isoclasses of indecomposables X with $\underline{\dim} X = \delta$ and $reg.top(X)_e \neq 0\}$.

(3) If X is indecomposable regular, then:
$\underline{\dim} X \in \mathbb{Z}\delta$ iff the period of X divides the regular length of X, and
$\underline{\dim} X \leq \delta$ iff the regular length of $X \leq$ period of X iff X is a brick.

Theorem 2. The assignment $\theta \mapsto \operatorname{Coker} \theta$ gives a bijection $\mathbb{P} \operatorname{Hom}(P, L) \to \Omega$, so the set of tubes is indexed by the projective line.

Proof. If U is indecomposable regular of dimension δ and $reg.top(U)_e \neq 0$, there there is a non-zero map $\operatorname{Coker} \theta \to U$, for some θ. This map must be epi, since any proper regular submodule is zero at e. Thus the map is an isomorphism.

If $\theta, \theta' \in \operatorname{Hom}(P, L)$ are non-zero and $\operatorname{Coker} \theta \cong \operatorname{Coker} \theta'$, then since $\operatorname{Ext}^1(L, P) = 0$ one gets a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "\theta'"] \arrow[d, "f"'] & L \arrow[r] \arrow[d, "g"'] & \operatorname{Coker} \theta' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r, "\theta"] & L \arrow[r] & \operatorname{Coker} \theta \arrow[r] & 0
\end{tikzcd}$$

Now f and g are non-zero multiples of the identity, so $\theta = \lambda \theta'$ with $0 \neq \lambda \in K$.

Theorem 3.
(1) If X is indecomposable then $\underline{\dim} X$ is a root.
(2) If α is a positive imaginary root there are infinitely many many indecs

with $\underline{\dim} X = \alpha$.
(3) If α is a positive real root there is a unique indec X with $\underline{\dim} X = \alpha$.

Proof. (1) If X is a brick, this is clear. If X is not a brick, it is regular. Let X have period p and regular length $rp + q$ with $1 < q \leq p$. The regular submodule Y of X with regular length q is a brick, and so $\underline{\dim} X = \underline{\dim} Y + r\delta$ is a root.

(2) We have $\alpha = r\delta$. If T is a tube of period p, then the indecomposables in T of regular length rp have dimension $r\delta$, and there are infinitely many tubes.

(3) We have proved this already in case $\langle \alpha, \delta \rangle \neq 0$, so suppose $\langle \alpha, \delta \rangle = 0$. We can write $\alpha = r\delta + \beta$ for some real root β with $0 \leq \beta \leq \delta$. We know that there is a unique regular indecomposable Y of dimension β, say of period p, and regular length q. Let X be the regular uniserial containing Y and with regular length $rp + q$. Clearly $\underline{\dim} X = r\delta + \underline{\dim} Y = \alpha$.

Finally suppose that there are two regular uniserials X, Y that have the same dimension vector α, a real root. Then $\langle \underline{\dim} X, \underline{\dim} Y \rangle = q(\alpha) = 1$, so $\mathrm{Hom}(X, Y) \neq 0$. Thus X and Y are in the same tube, say of period p. Since α is not a multiple of δ, the regular length of X is not a multiple of p. Thus if S is the regular socle of X, we have $\mathrm{Hom}(S, X) \neq 0$ and $\mathrm{Ext}^1(S, X) = 0$. Thus $\langle \underline{\dim} S, \alpha \rangle > 0$. Thus $\mathrm{Hom}(S, Y) \neq 0$. Thus Y has regular socle S. Thus since $\underline{\dim} X = \underline{\dim} Y$ we have $X \cong Y$.