Noncommutative algebra 1

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William Crawley-Boevey

Introduction

My aim in this course is to cover the following topics:
(1) Basics of rings and modules (for students of differing backgrounds)
(2) Examples and constructions of algebras
(3) Module categories and related properties of modules
(4) Homological algebra: Ext and Tor, global dimension

This is the first course in a master sequence, which continues with:
Noncommutative algebra 2. Representations of finite-dimensional algebras
Noncommutative algebra 3. Geometric methods.

It is also the first part of a sequence to be given by Henning Krause, which will continue with quasi-hereditary algebras and derived categories.

Examples class by Andrew Hubery.

Why study noncommutative algebra?

- Representation theory: to study groups, Lie algebras, algebraic groups, etc., one needs to understand their representations, and for this one should study the group algebra, universal enveloping algebra, Schur algebra, etc.

- Physics: many algebras arise, e.g. for spin in quantum mechanic (Clifford algebras), statistical mechanics (Temperley-Lieb algebras), dimer models (dimer algebras), etc.

- Differential equations: linear differential equations correpond to modules for the ring of differential operators. The notion of a quantum group (which is an algebra, not a group!) arose in the study of integrable systems.

- Topology: The cohomology of a topological space gives a ring. The Jones polynomial for knots came from the representation theory of Hecke algebras.

- Number Theory: a basic object is the Brauer group, classifying central simple algebras. The final step in Wiles and Taylor's proof of Fermat's last theorem involved a different type of Hecke algebra.

- Functional analysis is all about noncommutative algebras, such as C^*-algebras and von Neumann algebras; but it is a different story.

- Linear algebra: Jordan normal form is the classification of f.d. modules for $K[x]$. If you know the Jordan normal form of two $n \times n$ matrices, what can you say about the Jordan normal form of their sum? There is a partial solution using deformed preprojective algebras and representations of quivers.

References. There are many good books on this topic. Some suggestions.
F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, 2nd edition Springer 1992 .
A. J. Berrick and M. E. Keating, Categories and Modules with K-Theory in View, Cambridge University Press 2000.
P. M. Cohn, Algebra 2, 2nd edition Wiley 1989. c.f. also P. M. Cohn, Basic Algebra, 2005.
P. M. Cohn, Algebra 3, 2nd edition Wiley 1991.
B. Farb and R. K. Dennis, Noncommutative Algebra, Springer 1993.
M. Hazewinkel, N. Gubareni and V. V. Kirichenko, Algebras, Rings and Modules, Kluwer 2005.
T.-Y. Lam, A First Course in Noncommutative Rings , Springer 1991.
T.-Y. Lam, Lectures on Modules and Rings, Springer 1999.
J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, 2nd edition American Math. Soc. 2001.
M. S. Osborne, Basic Homological Algebra, Springer 2000.
R. S. Pierce, Associative Algebras, Springer 1982.
J. J. Rotman, An Introduction to Homological Algebra, 2nd edition Springer 2009.
J. J. Rotman, Advanced Modern Algebra, American Math. Soc. 2010, or Parts I and 2, 2015 and 2017.
L. Rowen, Ring Theory, Student edition, Acdemic Press 1991.
B. Stenström, Rings of Quotients, Springer 1975.
C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press 1994.

1 Basics of rings and modules

1.1 Rings

We consider rings R which are *unital*, so there is $1 \in R$ with $r1 = 1r = r$ for all $r \in R$. Examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, $R[x]$ of ring of polynomials in an indeterminate x with coefficients in a ring R, $M_n(R)$ the ring of $n \times n$ matrices with entries in a ring R.

A *subring* of a ring is a subset $S \subseteq R$ which is ring under the same operations, with the same unity as R. A *ring homomorphism* is a mapping $\theta : R \to S$ preserving addition and multiplication and such that $\theta(1) = 1$.

A *(two-sided) ideal* in a ring R is a subgroup $I \subseteq R$ such that $rx \in I$ and $xr \in I$ for all $r \in R$ and $x \in I$. The ideal *generated by* a subset $S \subseteq R$ is

$$(S) = \{\sum_{i=1}^{n} r_i s_i r_i' : n \geq 0, r_i, r_i' \in R, s_i \in S\}.$$

If I is an ideal in R, then R/I is a ring.
Examples: $\mathbb{F}_p = \mathbb{Z}/(p) = \mathbb{Z}/p\mathbb{Z}$, $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$.

The isomorphism theorems (see for example, P.M.Cohn, Algebra, vol. 1).
(1) A homomorphism $\theta : R \to S$ induces an isomorphism $R/\operatorname{Ker}\theta \cong \operatorname{Im}\theta$.
(2) If I is an ideal in R and S is a subring of R then $S/(S \cap I) \cong (S + I)/I$.
(3) If I is an ideal in R, then the ideals in R/I are of the form J/I with J an ideal in R containing I, and $(R/I)/(J/I) \cong R/J$.

The opposite ring R^{op} is obtained from R by using the multiplication $\cdot$, where $r \cdot s = sr$. The transpose defines an isomorphism $M_n(R)^{op} \to M_n(R^{op})$.

A product of rings $\prod_{i \in I} R_i$ is naturally a ring, e.g. $R^n = R \times R \times \cdots \times R$ or $R^I = \prod_{i \in I} R$, the set of functions $I \to R$.

1.2 Modules

Let R be a ring. A *(left) R-module* consists of an additive group M equipped with a mapping $R \times M \to M$ which is an *action*, meaning
- $(rr')m = r(r'm)$ for $r, r' \in R$ and $m \in M$,

- it is distributive over addition, and
- it is unital: $1m = m$ for all m.

An R-module *homomorphism* $\theta : M \to N$ is a map of additive groups with $\theta(rm) = r\theta(m)$ for $r \in R$ and $m \in M$.

A *submodule* of a R-module M is a subgroup $N \subseteq M$ with $rn \in N$ for all $r \in R$, $n \in N$. Given a submodule N of M one gets a quotient module M/N.

The isomorphism theorems for R-modules (see for example P.M.Cohn, Algebra, vol. 2).
(1) A homomorphism $\theta : M \to N$ induces an isomorphism $M/\operatorname{Ker}\theta \cong \operatorname{Im}\theta$.
(2) If L and N are submodules of a module M, then $L/(L \cap N) \cong (L+N)/N$.
(3) If N is a submodule of M, then the submodules of M/N are of the form L/N where L is a submodule of M containing N, and $(M/N)/(L/N) \cong M/L$.

If $\theta : R \to S$ is a ring homomorphism, any S-module ${}_SM$ becomes an R-module denoted ${}_RM$ or ${}_\theta M$ by *restriction*: $r.m = \theta(r)m$.

Dually there is the notion of a *right R-module* with an action $M \times R \to R$. Apart from notation, it is the same thing as a left R^{op}-module. If R is commutative, the notions coincide.

If R and S are rings, then an *R-S-bimodule* is given by left R-module and right S-module structures on the same additive group M, satisfying $r(ms) = (rm)s$ for $r \in R$, $s \in S$ and $m \in M$.

A ring R is naturally an R-R-bimodule. A (two-sided) ideal of R is a sub-bimodule of R. A *left* or *right ideal* of R is a submodule of R as a left or right module.

A product of R-modules $\prod_{i \in I} X_i$ is naturally an R-module. We write X^I for the product of copies of X indexed by a set I, so the set of functions $I \to X$.

The *(external) direct sum* or *coproduct* of modules is:

$$\bigoplus_{i \in I} X_i \left(\text{or } \coprod_{i \in I} X_i \right) = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i \colon x_i = 0 \text{ for all but finitely many } i \right\}.$$

One writes $X^{(I)} = \bigoplus_{i \in I} X$.

If the X_i ($i \in I$) are submodules of an R-module X, then addition gives a

homomorphism
$$\bigoplus_{i\in I} X_i \to X, \quad (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i.$$
The image is the *sum* of the X_i, denoted $\sum_{i\in I} X_i$. If this homomorphism is an isomorphism, then the sum is called an *(internal) direct sum*, and also denoted $\bigoplus_{i\in I} X_i$.

If $(m_i)_{i\in I}$ is a family of elements of an R-module M, the submodule *generated by* (m_i) is
$$\sum_{i\in I} Rm_i = \{\sum_{i\in I} r_i m_i : r_i \in R, \text{ all but finitely many zero}\},$$
or equivalently the image of the map $R^{(I)} \to M$, $(r_i) \mapsto \sum_{i\in I} r_i m_i$.

Every module M has a generating set, for example M itself. A module M is *finitely generated* if it has a finite generating set. Equivalently if there is a map from R^n onto M for some $n \in \mathbb{N}$.

A family $(m_i)_{i\in I}$ is an *(R-)basis* for M if it generates M and is R-linearly independent, that is, if
$$\sum_{i\in I} r_i m_i = 0$$
with all but finitely many $r_i = 0$, implies all $r_i = 0$. That is, the map $R^{(I)} \to M$ is bijective. A module M is *free* if it has a basis; equivalently $M \cong R^{(I)}$ for some I.

Example. $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Q}$ are not free $\mathbb{Z}$-modules.

Lemma. Any proper submodule of a finitely generated module is contained in a maximal proper submodule.

Proof. Apply Zorn's Lemma to the set of proper submodules containing the submodule. Finite generation ensures that the union of a chain of proper submodules is a proper submodule.

1.3 Algebras

Fix a commutative ring K (often a field). An *(unital associative) algebra* over K, or K*-algebra* consists of a ring which is at the same time a K-module, with the same addition, and such that multiplication is a K-module homomorphism in each variable.

To turn a ring R into a K-algebra is the same as giving a homomophism from K to the centre of R, $Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}$. Given the K-module structure on R, we have the map $K \to Z(R)$, $\lambda \mapsto \lambda 1$. Given a map $f : K \to Z(R)$ we have the K-module structure $\lambda.m = f(\lambda)m$.

A ring is the same thing as a $\mathbb{Z}$-algebra.

Any module for a K-algebra R becomes naturally a K-module via $\lambda.m = (\lambda 1)m$. It can also be considered as a R-K-bimodule.

If R and S are K-algebras, then unless otherwise stated, one only considers R-S-bimodules for which the left and right actions of K are the same.

A *K-algebra homomorphism* is a ring homomorphism which is also a K-module homomorphism, or equivalenty a ring homomorphism which is compatible with the ring homomorphisms from K.

Example 1. Hamilton's quaternions $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$.

If M, N are R-modules, the set of R-module homomorphisms $\mathrm{Hom}_R(M, N)$ becomes a K-module via

$$(\theta + \phi)(m) = \theta(m) + \phi(m), \quad (\lambda\theta)(m) = \lambda\theta(m)(= \theta(\lambda m).$$

But it is not necessarily an R-module, unless R is commutative. For example if we define $r\theta$ for $r \in R$ by $(r\theta)(m) = r\theta(m)$, then for $s \in R$ we have $(r\theta)(sm) = rs\theta(m)$ and $s((r\theta)(m)) = sr\theta(m)$.

Bimodule structures on M or N give module structures on $\mathrm{Hom}_R(M, N)$. For example if M is an R-S-bimodule and N is an R-T-bimodule then $\mathrm{Hom}_R(M, N)$ becomes an S-T-bimodule via $(s\theta t)(m) = \theta(ms)t$.

Example 2. $\mathrm{End}_R(M)$ the set of endomorphisms of an R-module M is a K-algebra.

If R is any K-algebra, then the R-module structures on a K-module M are in 1:1 correspondence with K-algebra homomorphisms $R \to \mathrm{End}_K(M)$.

Example 3. If G is a group, written multiplicatively, the *group algebra KG* is the free K-module with basis the elements of G, and with multiplication given by $g \cdot h = gh$ for $g, h \in G$. Thus a typical element of KG can be written as $\sum_{g \in G} a_g g$ with $a_g \in K$, almost all zero, and

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{k \in G}(\sum_{g \in G} a_g b_{g^{-1}k})k.$$

A *representation* of G over K consists of a K-vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. There is a 1-1 correspondence between representations of G and KG-modules via $\rho(g)(v) = gv$.

Example 4. Given a set X, the *free (associative) algebra* $K\langle X\rangle$ is the free K-module on the set of all words in the letters of X, including the trivial word 1. It becomes a K-algebra by concatenation of words. For example for $X = \{x, y\}$ we write $K\langle x, y\rangle$, and it has basis

$$1, x, y, xx, xy, yx, yy, xxx, xxy, \dots$$

In case $X = \{x\}$ one recovers the polynomial ring $K[x]$.

If R is any K-algebra, there is a 1:1 correspondence between maps of sets $X \to R$ and K-algebra maps $K\langle X\rangle \to R$.

Thus there is a 1:1 correspondence between $K\langle X\rangle$-module structures on a K-module M and maps of sets $X \to \mathrm{End}_K(M)$.

If X is a subset of R, the K-subalgebra of R *generated* by X is the image of the natural homomorphism $K\langle X\rangle \to R$.

1.4 Exact sequences

Let R be a ring or an algebra. A sequence of modules and homomorphisms

$$\dots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is said to be *exact* at M if $\operatorname{Im} f = \operatorname{Ker} g$. It is *exact* if it is exact at every module. A *short exact sequence* is one of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

so f is injective, g is surjective and $\operatorname{Im} f = \operatorname{Ker} g$.

Any map $f : M \to N$ gives an exact sequence

$$0 \to \operatorname{Ker} f \to M \to N \to \operatorname{Coker} f \to 0$$

where $\operatorname{Coker} f := M/\operatorname{Im} f$, and short exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0, \quad 0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0.$$

Snake Lemma. Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
(0 & \longrightarrow &)L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N'(& \longrightarrow & 0)
\end{array}$$

there is an induced exact sequence

$$(0 \to) \operatorname{Ker} \alpha \to \operatorname{Ker} \beta \to \operatorname{Ker} \gamma \xrightarrow{c} \operatorname{Coker} \alpha \to \operatorname{Coker} \beta \to \operatorname{Coker} \gamma (\to 0).$$

The maps, including the connecting homomorphism c, are given by diagram chasing.

There is also the Five Lemma, and many variations. Maybe we only need:

Corollary. If α and γ are isomorphisms, so is β.

If L and N are modules, one gets an exact sequence

$$0 \to L \xrightarrow{i_L} L \oplus N \xrightarrow{p_N} N \to 0$$

where i_L and p_N are the inclusion and projection maps.

Lemma/Definition. A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, is a *split* if it satisfies the following equivalent conditions
(i) f has a *retraction*, a morphism $r : M \to L$ with $rf = 1_L$.
(ii) g has a *section*, a morphism $s : N \to M$ with $gs = 1_N$.
(iii) There is an isomorphism $\theta : M \to L \oplus N$ giving a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\
0 & \longrightarrow & L & \xrightarrow{i_L} & L \oplus N & \xrightarrow{p_N} & N & \longrightarrow & 0.
\end{array}$$

Proof of equivalence. (i)$\Rightarrow$(iii). Define $\theta(m) = (r(m), g(m))$. The diagram commutes and θ is an isomorphism by the Snake lemma.

(ii)$\Rightarrow$(iii). Define $\phi : L \oplus N \to M$ by $\phi(\ell, n) = f(\ell) + s(n)$. It gives a commutative diagram the other way up, so ϕ is an isomorphism by the Snake lemma, and then take $\theta = \phi^{-1}$.

(iii)$\Rightarrow$(i) and (ii). Define $r = p_L\theta$ and $s = \theta^{-1} i_N$.

1.5 Idempotents

Let K be a commutative ring and let R be a K-algebra (including the case of a ring, with $K = \mathbb{Z}$).

Definitions
(i) An element $e \in R$ is *idempotent* if $e^2 = e$.
(ii) A family of idempotents $(e_i)_{i\in I}$ is *orthogonal* if $e_i e_j = 0$ for $i \neq j$.
(iii) A finite family of orthogonal idempotents $e_1, \dots, e_n$ is *complete* if $e_1 + \dots + e_n = 1$.

Examples.
(a) If e is idempotent, then $e, 1-e$ is a complete set of orthogonal idempotents.
(b) The diagonal unit matrices e^{ii} in $M_n(K)$ are a complete set.

Lemma 1. If M is a left R-module, then
(i) If e is idempotent, then $eM = \{m \in M : em = m\}$. This is a K-submodule of M.
(ii) If (e_i) are orthogonal idempotents, then the sum $\sum_{i\in I} e_i M$ is direct.
(iii) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then $M = e_1 M \oplus \dots \oplus e_n M$.

Proof. Straightforward. e.g. for (i), if $em = m$ then $m \in eM$, while if $m \in eM$ then $m = em' = e^2 m' = e(em') = em$.

Proposition (Peirce decomposition). If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents then $R = \bigoplus_{i,j=1}^n e_i R e_j$.

We draw the Peirce decomposition as a matrix

$$R = \begin{pmatrix} e_1Re_1 & e_1Re_2 & \dots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \dots & e_2Re_n \\ \dots & & & \\ e_nRe_1 & e_nRe_2 & \dots & e_nRe_n \end{pmatrix}$$

and multiplication in R corresponds to matrix multiplication.

Remark. If e is an idempotent, then eRe is an algebra with the same operation as R, with unit element e. Since the unit element is not the same as for R, it is not a subalgebra of R. Sometimes called a *corner algebra*.

Lemma 2. For M a left R-module, we have $\mathrm{Hom}_R(R, M) \cong M$ as R-modules, and if $e \in R$ is idempotent, then $\mathrm{Hom}_R(Re, M) \cong eM$ as K-modules.

In particular, $R \cong \mathrm{End}_R(R)^{op}$ (if we used right modules, we wouldn't need the opposite here) and $eRe \cong \mathrm{End}_R(Re)^{op}$.

Proof. Send $\theta : R \to M$ to $\theta(1)$ and $m \in M$ to $r \mapsto rm$, etc.

1.6 Hom spaces and decompositions

Let R be a K-algebra (including the case of R a ring with $K = \mathbb{Z}$).

Lemma 1. Given modules X, Y and families of modules X_i, Y_i $(i \in I)$, there are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_R(X, \prod_i Y_i) &\cong \prod_i \mathrm{Hom}_R(X, Y_i), \\ \mathrm{Hom}_R(\bigoplus_i X_i, Y) &\cong \prod_i \mathrm{Hom}_R(X_i, Y). \\ \mathrm{Hom}_R(X, \bigoplus_i Y_i) &\cong \bigoplus_i \mathrm{Hom}_R(X, Y_i) \text{ for } X \text{ finitely generated} \end{aligned}$$

Proof. Straightforward.

Lemma 2. In the algebra $\mathrm{End}_R(X_1 \oplus \cdots \oplus X_n)$, the projections onto the X_i give a complete family of orthogonal idempotents, and the Peirce decomposition is

$$\mathrm{End}_R(X_1 \oplus \cdots \oplus X_n) \cong \begin{pmatrix} \mathrm{Hom}(X_1, X_1) & \mathrm{Hom}(X_2, X_1) & \dots & \mathrm{Hom}(X_n, X_1) \\ \mathrm{Hom}(X_1, X_2) & \mathrm{Hom}(X_2, X_2) & \dots & \mathrm{Hom}(X_n, X_2) \\ \dots & & & \\ \mathrm{Hom}(X_1, X_n) & \mathrm{Hom}(X_2, X_n) & \dots & \mathrm{Hom}(X_n, X_n) \end{pmatrix}.$$

In particular, $\mathrm{End}_R(X^n) \cong M_n(\mathrm{End}_R(X))$.

Proof. Straightforward.

A module M is *indecomposable* if it is non-zero and in any decomposition into submodules $M = X \oplus Y$, either $X = 0$ or $Y = 0$.

Lemma 3. A module M is indecomposable if and only if $\mathrm{End}_R(M)$ has no non-trivial idempotents (other than 0 and 1).

Proof. An idempotent endomorphism e gives $M = \mathrm{Im}\, e \oplus \mathrm{Ker}\, e$. A decomposition $M = X \oplus Y$ gives $e =$ projection onto X.

Theorem (Specker, 1950). $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is a free $\mathbb{Z}$-module with basis $(\pi_i)_{i\in\mathbb{N}}$ where $\pi_i(a) = a_i$.

Proof. (cf. Scheja and Storch, Lehrbuch der Algebra, Teil 1, 2nd edition, Satz III.C.4, p230) It is clear that the π_i are linearly independent. Let (e_i) be the standard basis of $\mathbb{Z}^{(\mathbb{N})} \subset \mathbb{Z}^{\mathbb{N}}$. Let $h : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, and let $b_i = h(e_i)$. Let (c_n) be a sequence of positive integers such that c_{n+1} is a multiple of c_n and

$$c_{n+1} \geq n + 1 + \sum_{i=0}^{n} |c_i b_i|.$$

Let $c = h((c_n))$.

For each $m \in \mathbb{N}$ there is $y_m \in \mathbb{Z}^{\mathbb{N}}$ with

$$(c_n) = \sum_{i=0}^{m} c_i e_i + c_{m+1} y_m.$$

Applying h gives

$$c = \sum_{i=0}^{m} c_i b_i + c_{m+1} h(y_m),$$

so

$$|c - \sum_{i=0}^{m} c_i b_i| = c_{m+1} |h(y_m)|$$

is either 0 or $\geq c_{m+1}$. But if $m \geq |c|$, then

$$|c - \sum_{i=0}^{m} c_i b_i| \leq |c| + \sum_{i=0}^{m} |c_i b_i| < c_{m+1}.$$

Thus $c = \sum_{i=0}^{m} c_i b_i$ for all $m \geq |c|$. But this implies $b_i = 0$ for all $i > |c|$. Then the linear form $h - \sum_{i=0}^{|c|} b_i \pi_i$ vanishes on all of the standard basis elements e_i.

It remains to show that if $g \in \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ vanishes on all the e_i, then it is zero. Suppose given $(c_i) \in \mathbb{Z}^{\mathbb{N}}$. Expanding $c_i = c_i(3-2)^{2i}$, we can write $c_i = v_i 2^i + w_i 3^i$ for some $v_i, w_i \in \mathbb{Z}$. Then $g((c_i)) = g((v_i 2^i)) + g((w_i 3^i))$. Now for any m, $(v_i 2^i) = \sum_{i=0}^{m-1} v_i 2^i e_i + 2^m z_m$ for some $z_m \in \mathbb{Z}^{\mathbb{N}}$. Thus $g((v_i 2^i)) \in 2^m \mathbb{Z}$. Thus $g((v_i 2^i)) = 0$. Similarly for w. Thus $g((c_n)) = 0$.

Corollary. $\mathbb{Z}^{\mathbb{N}}$ is not a free $\mathbb{Z}$-module.

Proof. Say $\mathbb{Z}^{\mathbb{N}} \cong \mathbb{Z}^{(I)}$. Since $\mathbb{Z}^{\mathbb{N}}$ is uncountable, I must be. Certainly it must be infinite. Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{(I)}, \mathbb{Z}) \cong (\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}))^I \cong \mathbb{Z}^I$, which is also uncountable. But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is a free $\mathbb{Z}$-module with countable basis, so it is countable.

1.7 Simple and semisimple modules

Let R be an algebra. A module S is *simple* (or irreducible) if it has exactly two submodules, namely $\{0\}$ and S. It is equivalent that S is non-zero and any non-zero element is a generator. In particular the simple modules are the quotients R/I with I a maximal left ideal.

Examples.
(i) The simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.
(ii) If D is a division ring, that is every non-zero element is invertible, then ${}_D D$ is a simple D-module.
(iii) K^n considered as column vectors becomes a simple $M_n(K)$-module.

Schur's Lemma. Any homomorphism between simple modules must either be zero or an isomorphism, so if S is simple, $\mathrm{End}_R(S)$ is a division ring. Moreover if R is a K-algebra, with K an algebraically closed field, and S is finite-dimensional over K, then $\mathrm{End}_R(S) = K$.

Proof. The last part holds because any f.d. division algebra D over an algebraically closed field is equal to K. Namely, if $d \in D$ then left multiplication by D gives a linear map $D \to D$, and it must have an eigenvalue λ. Then $d - \lambda 1$ is not invertible, so must be zero, so $d \in K1$.

Theorem/Definition. A module M is said to be *semisimple* (or completely reducible) if it satisfies the following equivalent conditions.
(a) M is isomorphic to a direct sum of simple modules.
(b) M is a sum of simple modules.
(c) Any submodule of M is a direct summand.

Sketch. For fuller details see P.M.Cohn, Algebra 2, §4.2.

(a) implies (b) is trivial. Assuming (b), say $M = \sum_{i \in I} S_i$ and that N is a submodule of M, one shows by Zorn's lemma that $M = N \oplus \bigoplus_{i \in J} S_i$ for some subset J or I. This gives (a) and (c).

The property (c) is inherited by submodules $N \subseteq M$, for if $L \subseteq N$ and

$M = L \oplus C$ then $N = L \oplus (N \cap C)$. Let N be the sum of all simple submodules. It has complement C, and if non-zero, then C has a non-zero finitely generated submodule F. Then F has a maximal proper submodule P. Then P has a complement D in F, and $D \cong F/P$, so it is simple, so $D \subseteq N$. But $D \subseteq C$, so its intersection with N is zero.

Corollary 1. Any submodule or quotient of a semisimple module is semisimple.

Proof. We showed above that condition (c) passes to submodules. Now if M is semisimple and M/N is a quotient, then N has a complement C in M, and $M/N \cong C$, so it is semisimple.

Corollary 2. If K is a field, or more generally a division ring, every K-module is free and semisimple (hence the theory of vector spaces).

Proof. K is a simple K-module, and it is the only simple module up to isomorphism, since if S is a simple module and $0 \neq s \in S$ then the map $K \to S$, $r \mapsto rs$ must be an isomorphism. Thus free = semisimple. The result follows.

1.8 Jacobson radical

Theorem/Definition. The *(Jacobson) radical* $J(R)$ of R is the ideal in R consisting of all elements x satisfying the following equivalent conditions.
(i) $xS = 0$ for any simple left module S.
(ii) $x \in I$ for every maximal left ideal I
(iii) $1 - ax$ has a left inverse for all $a \in R$.
(iv) $1 - ax$ is invertible for all $a \in R$.
(i')-(iv') The right-hand analogues of (i)-(iv).

Proof (i) implies (ii). If I is a maximal left ideal in R, then R/I is a simple left module, so $x(R/I) = 0$, so $x(I + 1) = I + 0$, so $x \in I$.

(ii) implies (iii). If there is no left inverse, then $R(1 - ax)$ is a proper left ideal in R, so contained in a maximal left ideal I by Zorn's Lemma. Now $x \in I$, and $1 - ax \in I$, so $1 \in I$, so $I = R$, a contradiction.

(iii) implies (iv) $1 - ax$ has a left inverse u, and $1 + uax$ has a left inverse v. Then $u(1 - ax) = 1$, so $u = 1 + uax$, so $vu = 1$. Thus u has a left and right inverse, so it is invertible and these inverses are equal, and are themselves invertible. Thus $1 - ax$ is invertible.

(iv) implies (i). If $s \in S$ and $xs \neq 0$, then $Rxs = S$ since S is simple, so $s = axs$ for some $a \in R$. Then $(1 - ax)s = 0$, but then $s = 0$ by (iv).

(iv) implies (iv'). If b is an inverse for $1 - ax$, then $1 + xba$ is an inverse for $1 - xa$. Namely $(1 - ax)b = b(1 - ax) = 1$, so $axb = b - 1 = bax$, and then $(1 + xba)(1 - xa) = 1 + xba - xa - xbaxa = 1$, and $(1 - xa)(1 + xba) = 1 - xaxba - xa + xba = 1$.

Example. The maximal (left) ideals in $\mathbb{Z}$ are $p\mathbb{Z}$, p prime, so $J(\mathbb{Z}) = \bigcap_p \mathbb{Z}p = 0$.

Lemma (added later). If I is an ideal in R with $I \subseteq J(R)$ then $J(R/I) = J(R)/I$. In particular, $J(R/J(R)) = 0$.

Proof. The maximal left ideals of R contain I, so correspond to maximal left ideals of R/I.

Notation. If M is an R-module and I an ideal in R, we write IM fot the set of sums of products im. The powers of an ideal are defined inductively by $I^1 = I$ and $I^{n+1} = II^n$. An ideal is *nilpotent* if $I^n = 0$ for some n, or equivalently $i_1 \dots i_n = 0$ for all $i_1, \dots, i_n \in I$. An ideal I is *nil* if every element is nilpotent.

Lemma 1. For an ideal I, we have I nilpotent $\Rightarrow I$ nil $\Rightarrow I \subseteq J(R)$.

Proof. The first implication is clear. If $x \in I$ and $a \in R$ then $ax \in I$, so $(ax)^n = 0$ for some n. Then $1 - ax$ is invertible with inverse $1 + ax + (ax)^2 + \dots + (ax)^{n-1}$. Thus $x \in J(R)$.

Lemma 2. If I is a nil ideal in a ring R, then any idempotent in R/I lifts to one in R.

Proof. There is a formal power series $p(x) = a_1 x + a_2 x^2 + \dots$ with integer coefficients satisfying

$$(1 + 4x)p(x)^2 - (1 + 4x)p(x) + x = 0.$$

Namely, either solve recursively for the a_i, or use the formula for a quadratic,

$$p(x) = \frac{1 - \sqrt{1 - 4\frac{x}{1+4x}}}{2},$$

expand as a power series, and observe that the coefficients are integers.

Now an idempotent in R/I lifts to an element $a \in R$ with $b = a^2 - a \in I$. Since b is nilpotent and commutes with a, the element $e = a(1-2p(b))+p(b)$ makes sense and

$$e^2 - e = a^2(1-2p(b))^2 + 2ap(b)(1-2p(b)) + p(b)^2 - a(1-2p(b)) - p(b).$$

Writing $a^2 = a + b$ and collecting terms, this becomes

$$a[(1-2p(b))^2 + 2p(b)(1-2p(b)) - (1-2p(b)] + b(1-2p(b))^2 + p(b)^2 - p(b)]$$

$$= (1+4b)p(b)^2 - (1+4b)p(b) + b = 0.$$

Nakayama's Lemma. Suppose M is a finitely generated R-module.
(i) If $J(R)M = M$, then $M = 0$.
(ii) If $N \subseteq M$ is a submodule with $N + J(R)M = M$, then $N = M$.

Proof. (i) Suppose $M \neq 0$. Let $m_1, \dots, m_n$ be generators with n minimal. Since $J(R)M = M$ we can write $m_n = \sum_{i=1}^n r_i m_i$ with $r_i \in J(R)$. This writes $(1-r_n)m_n$ in terms of the others. But $1 - r_n$ is invertible, so it writes m_n in terms of the others. Contradiction.

(ii) Apply (i) to M/N.

Lemma/Definition. R is a *local ring* if it satisfies the following equivalent conditions.
(i) $R/J(R)$ is a division ring.
(ii) The non-invertible elements of R form an ideal (which is $J(R)$).
(iii) There is a unique maximal left ideal in R (which is $J(R)$).

Proof. (i) implies (ii). The elements of $J(R)$ are not invertible, so it suffices to show that any $x \notin J(R)$ is invertible. Now $J(R) + x$ is an invertible element in $R/J(R)$, say with inverse $J(R) + a$. Then $1 - ax, 1 - xa \in J(R)$. But this implies ax and xa are invertible, hence so is x.

(ii) implies (iii). Clear.

(iii) implies (i). Assuming (iii), $J(R)$ is the unique maximal left ideal, so $\overline{R} = R/J(R)$ is a simple R-module, and so a simple $\overline{R}$-module. Then $\overline{R} \cong \mathrm{End}_{\overline{R}}(\overline{R})^{op}$, which is a division ring by Schur's Lemma.

Examples. (a) The set $R = \{q \in \mathbb{Q} : q = a/b, b \text{ odd}\}$ is a subring of $\mathbb{Q}$. The ideal $(2) = \{q \in \mathbb{Q} : q = a/b, a \text{ even}, b \text{ odd}\}$ is the set of all non-invertible elements. Thus R is local and $J(R) = (2)$.

(b) The set of upper triangular matrices with equal diagonal entries is a subalgebra of $M_n(K)$, e.g.

$$\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K\}$$

The set of matrices with $a = 0$ form a nil ideal I, so $I \subseteq J(R)$. The map sending such a matrix to a defines an isomorphism $R/I \cong K$. Thus $I = J(R)$ and $R/J(R) \cong K$ so R is local.
(c) The ring $M_n(K)$ has no 2-sided ideals other than 0 and $M_n(K)$, but it is not local.

1.9 Finite-dimensional algebras

In this section K is a field, and we consider f.d. algebras and modules.

Wedderburn's Theorem/Definition. A f.d. algebra R is *semisimple* if the following equivalent conditions hold
(i) $J(R) = 0$.
(ii) R is semisimple as an R-module.
(iii) Every R-module is semisimple.
(iv) Every short exact sequence of R-modules is split.
(v) $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ with the D_i division algebras.

Proof. If (i) holds, then since $J(R) = 0$ the intersection of the maximal left ideals is zero. Since R is f.d., a finite intersection of them is zero, say $I_1 \cap \cdots \cap I_n = 0$. Then the map $R \to (R/I_1) \oplus \ldots (R/I_n)$ is injective. Thus (ii).

If (ii) then $R = \bigoplus_{i \in I} S_i$. Now for $j \in I$ the sum $M_j = \bigoplus_{i \neq j} S_i$ is a maximal left ideal in R, and $\bigcap_{j \in I} M_j = 0$, giving (i).

Now (ii) implies that every free module is semisimple, and since any module is a quotient of a free module, (iii) follows.

The equivalence of (iii) and (iv) is easy, using that a short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is split if and only if $\operatorname{Im} f$ is a direct summand of M, and that a module is semisimple if and only if every submodule is a direct summand.

If (iii) holds then we can write ${}_R R$ as a finite direct sum of simples, and collecting terms we can write

$$R \cong S_1 \oplus \cdots \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_2 \oplus \cdots \oplus S_r \oplus \cdots \oplus S_r$$

where $S_1, \ldots, S_n$ are non-isomorphic simples, and there are n_i copies of each S_i. The Peirce decomposition of the endomorphism ring of this direct sum gives $\mathrm{End}_R(R) \cong \prod_{i=1}^n M_{n_i}(\mathrm{End}_R(S_i))$. Now use Schur's lemma and take the opposite ring to get (v).

If (v) holds, say $R \cong \prod_{i=1}^n M_{n_i}(D_i)$ then $R = \bigoplus_{i=1}^n \bigoplus_{j=1}^{n_i} I_{ij}$ where I_{ij} is the left ideal in $M_{n_i}(D_i)$ consisting of matrices which are zero outside the jth column. This is isomorphic to the module consisting of column vectors $D_i^{n_i}$, and for D_i a division algebra, this is a simple module, giving (ii).

Remarks. (i) The modules $S_i = D_i^{n_i}$ in the Wedderburn decomposition are a complete set of non-isomorphic simple R-modules.

(ii) If K is algebraically closed, we get $R \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K)$ since there are no non-trivial f.d. division algebras over K.

(iii) This generalizes to artinian rings with the Artin-Wedderburn Theorem.

Proposition 1. If R is a f.d. algebra, then $R/J(R)$ is semisimple and $J(R)$ is nilpotent, in fact it is the unique largest nilpotent ideal in R.

Proof. The intersection of the maximal left ideals in $R/J(R)$ is zero, so it is semisimple. Since R is f.d., we have

$$J(R) \supseteq J(R)^2 \supseteq \cdots \supseteq J(R)^n = J(R)^{n+1} = \ldots$$

for some n. Then $J(R)J(R)^n = J(R)^n$. Now $J(R)^n$ is a f.d. vector space, so clearly f.g. as an R-module. Thus $J(R)^n = 0$ by Nakayama's lemma.

Example 1. If R is the subalgebra of $M_3(K)$ consisting of matrices of shape

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

then $R = S \oplus I$ where S and I consist of matrices of shape

$$S = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that I is a nilpotent ideal in R, so $I \subseteq J(R)$. Also S is a subalgebra in R, and it is clearly isomorphic to $M_2(K) \times K$, so semisimple. Then $J(R)/I$ is a nilpotent ideal in $R/I \cong S$, so it is zero. Thus $J(R) \subseteq I$. Thus $J(R) = I$ and $R/J(R) \cong S$.

Definition. If R is any K-algebra and M is a f.d. R-module, the *character* χ_M of M is the composition

$$R \xrightarrow{x \mapsto \ell_x^M} \mathrm{End}_K(M) \xrightarrow{\mathrm{tr}} K$$

where $\ell_x^M(m) = xm$ and $\mathrm{tr}(\theta)$ is the trace of an endomorphism θ.

Proposition 2.
(i) $\chi_M(xy) = \chi_M(yx)$ for $x, y \in R$.
(ii) If $0 \to L \to M \to N \to 0$ is an exact sequence, then $\chi_M = \chi_L + \chi_N$. In particular, $\chi_{L \oplus N} = \chi_L + \chi_N$.
(iii) If K has characteristic zero, the characters of the simple modules are linearly independent in the vector space $\mathrm{Hom}_K(R, K)$, so semisimple modules with the same character are isomorphic.

Proof. (i) $\chi_M(xy) = \mathrm{tr}(\ell_{xy}^M) = \mathrm{tr}(\ell_x^M \ell_y^M) = \mathrm{tr}(\ell_y^M \ell_x^M)$.

(ii) Take a basis of L and extend it to a basis of M. It induces a basis of N. With respect to this basis, the matrix of ℓ_x^M has block form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is the matrix of ℓ_x^L and C is the matrix of ℓ_x^N with respect to the induced basis of N.

(iii) The statement actually holds for R infinite-dimensional. Namely, any linear relation involves finitely many finite-dimensional modules, say with direct sum M. Then, replacing R by $R/\{x \in R : xM = 0\}$, we reduce to the case R is f.d.

Replacing R by $R/J(R)$ if necessary, we may suppose that R is semisimple.

The simple modules are $S_i = D_i^{n_i}$ corresponding to the factors in the Wedderburn decomposition of R. Say $\sum_i a_i \chi_{S_i} = 0$. Let e_j be the idempotent in R which corresponds to the identity matrix in the jth factor. Then $0 = \sum a_i \chi_{S_i}(e_j) = a_j \dim S_j$, so $a_j = 0$. Here we use characteristic zero.

Now let R be f.d. Consider the symmetric bilinear form $R \times R \to K$ defined by $\langle x, y \rangle = \chi_R(xy)$. It is *non-degenerate* if $\langle x, y \rangle = 0$ for all y implies $x = 0$.

If $b_1, \ldots, b_n$ is a K-basis of R, then the form is non-degenerate if and only if the matrix $(\langle b_i, b_j\rangle)_{ij}$ is non-singular.

Lemma 1. If $\langle -, - \rangle$ is non-degenerate, then R is semisimple.

Proof. If $x \in J(R)$ then for $y \in R$ we have $\langle x, y\rangle = \mathrm{tr}(\ell^R_{xy}) = 0$ since xy is nilpotent. Thus $x = 0$ by non-degeneracy. Thus $J(R) = 0$.

Maschke's Theorem. If G is a finite group and K is a field of characteristic 0 (or not dividing $|G|$), then KG is semisimple.

Proof. $\chi_{KG}(1) = |G|$ and $\chi_{KG}(g) = 0$ for $g \neq 1$. Now the matrix $(\langle g, h\rangle)_{g,h}$ has entry $|G|$ where $g = h^{-1}$ and other entries zero, so it is invertible.

To specify a character χ_M for a group algebra $\mathbb{C}G$, it suffices to give the values $\chi_M(g)$ for $g \in G$, and this only depends on the conjugacy class of g. The *character table* of a finite group G has columns given by the conjugacy classes in G, rows given by the simple $\mathbb{C}G$-modules, and entries given by the value of the character.

Example 2. Suppose G is cyclic of order n, say with generator σ and $K = \mathbb{C}$. Let $\epsilon = e^{2\pi\sqrt{-1}/n}$. Since G is abelian, we must have $\mathbb{C}G \cong \mathbb{C} \times \cdots \times \mathbb{C}$. There are 1-dimensional simple modules $S_0, \ldots, S_{n-1}$ with σ acting on S_i as multiplication by ϵ^i. Since there are n of them, they must be all of the simple modules. One easily checks that

$$\sum_{j=0}^{n-1} \epsilon^{ij} = \begin{cases} n & \text{(if } n \text{ divides } i\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

as in the second case its product with $\epsilon^i - 1$ is $\epsilon^{in} - 1 = 0$. It follows that the elements

$$e_i = \frac{1}{n} \sum_{j=0}^{n-1} \epsilon^{ij} \sigma^j \in \mathbb{C}G \quad (0 \leq i < n)$$

are a complete family of orthogonal idempotents. They must be linearly independent, so a basis for $\mathbb{C}G$.

For $n = 3$ the character table is

1	σ	σ^2
1	1	1
1	ϵ	ϵ^2
1	ϵ^2	ϵ

Example 3. The symmetric group S_3 of order 6 is non-abelian, so by dimensions we must have $\mathbb{C}S_3 \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$. The character table is

1	(..)	(...)
1	1	1
1	−1	1
2	0	−1

There are two easy 1-dimensional simple modules, the trivial representation, on which all group elements act as 1, and the sign representation, on which each permutation acts as multiplication by the sign of the permutation. The remaining simple module is 2-dimensional. Its character can be deduced from that of the regular module $\mathbb{C}S_3$. Alternatively, identifying S_3 with the dihedral group D_3, it is the natural 2-dimensional representation with D_3 preserving an equilateral triangle.

The Atlas of Finite Groups, gives character tables for some finite simple groups. For example the Fischer group Fi_{23}, a sporadic simple group of order $2^{18}.3^{13}.5^2.7.11.13.17.23$ has 98 simple modules over $\mathbb{C}$ of dimensions 1, 782, 3588, ..., 559458900. It has a simple module over $\mathbb{F}_3$ of dimension 253.

Theorem 3. For a f.d. algebra R, the following are equivalent.
(i) R is local.
(ii) R has no idempotents apart from 0 and 1.
(iii) Every element of R is nilpotent or invertible.

Proof. (i)⇒(ii). If R is local, the non-invertible elements form an ideal I. If e is an invertible idempotent then $e = e1 = e(ee^{-1}) = ee^{-1} = 1$. Thus if e is an idempotent $\neq 0, 1$ then $e, 1 - e \in I$, so $1 \in I$. Contradiction.

(ii)⇒(i). If not local, then $R/J(R)$ is not a division ring, so its Wederburn decomposiotn has more than one factor, or matrices. Thus it contains a non-trivial idempotent. This lifts to an idempotent in R since $J(R)$ is nil.

(i)⇒(iii). Since $J(R)$ is nil.

(iii)⇒(i). If is x not invertible, nor is ax, so it is nilpotent, so $1 - ax$ is invertible, so $x \in J(R)$.

Example 4. The *augmentation ideal* $\Delta(G)$ of a group algebra KG is the kernel of the algebra homomorphism

$$KG \to K, \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g.$$

If G is a finite p-group and K is a field of characteristic p, we show that $\Delta(G)$ is nilpotent. Thus it is equal to $J(KG)$ and KG is local.

For a cyclic group $G = \langle x \rangle$ of order p, $\Delta(G)$ is spanned by $x^i - 1 = (x - 1)(1 + x + \cdots + x^{i-1})$, so generated by $x - 1$. Then since KG is commutative $\Delta(G)^p$ is generated by $(x-1)^p = x^p - 1^p = 0$ (by since all other binomial coefficients are zero in K.)

In general, by induction. Choose a central subgroup H which is cyclic of order p. Then there is a homomorphism $\theta : KG \to K(G/H)$. Now $\theta(\Delta(G)) = \Delta(G/H)$, and $\Delta(G/H)^N = 0$. Then $\Delta(G)^N \subseteq \operatorname{Ker}\theta = KG.\Delta(H)$. Since H is central, the pth power of this vanishes.

1.10 Noetherian rings

Lemma/Definition. A module M is *noetherian* if it satisfies the following equivalent conditions
(i) Any ascending chain of submodules $M_1 \subseteq M_2 \subseteq \dots$ becomes stationary, that is, for some n one has $M_n = M_{n+1} = \dots$.
(ii) Any non-empty set of submodules of M has a maximal element.
(iii) Any submodule of M is finitely generated.

Proof. (i) $\implies$ (ii) because otherwise we choose M_1 to be any of the submodules, and iteratively, since M_i isn't maximal, we can choose $M_i < M_{i+1}$. This gives an ascending chain which doesn't become stationary.
(ii) $\implies$ (iii). Let N be a submodule, let L be a maximal element of the set of finitely generated submodules of N, and $n \in N$. Then $L + Rn$ is also a finitely generated submodule of N, so equal to L by maximality. Thus $n \in L$, so $N = L$, so it is finitely generated.
(iii) $\implies$ (i) Choose a finite set of generators for $N = \bigcup_i M_i$. Some M_i must contain each of these generators, so be equal to N. Thus $M_i = M_{i+1} = \dots$.

Lemma. If L is a submodule of M then M is noetherian if and only if L and M/L are noetherian. If $M = L + N$ and L, N are noetherian, then so is M.

Proof. If M is noetherian then clearly L and M/L are noetherian. Now suppose $M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of submodules of M. If L and M/L are noetherian, then $L \cap M_i = L \cap M_{i+1} = \dots$ and $(L + M_i)/L = (L + M_{i+1})/L = \dots$ for some i, so $L + M_i = L + M_{i+1} = \dots$. Now if $m \in M_{i+1}$, then $m = \ell + m'$ with $\ell \in L$ and $m' \in M_i$. Then $\ell = m - m' \in$

$L \cap M_{i+1} = L \cap M_i$, so $m \in M_i$. Thus $M_i = M_{i+1} = \dots$. For the last part, use that $(L+N)/L \cong N/(L \cap N)$.

Definition. A ring R is *left noetherian* if it satisfies the following equivalent conditions
(a) ${}_RR$ is noetherian (so R is has the ascending chain condition on left ideals, or any left ideal in R is finitely generated).
(b) Any finitely generated left R-module is noetherian (equivalently any submodule of a finitely generated left module is finitely generated).

Proof of equivalence. For (a) $\implies$ (b), any finitely generated module is a quotient of a finite direct sum of copies of R.

Definition. A ring is *noetherian* if it is left noetherian and right noetherian (i.e. noetherian for right modules, or equivalently R^{op} is left noetherian).

Remarks. (1) Division rings and principal ideal domains such as $\mathbb{Z}$ are noetherian. Hilbert's Basis Theorem says that if K is noetherian, then so is $K[x]$. The free algebra $R = K\langle x, y\rangle$ is not left noetherian, since the ideal (x) is not finitely generated as a left ideal.

(2) If $R \to S$ is a ring homomorphism and M is an S-module such that ${}_RM$ is noetherian, then M is noetherian. Thus if ${}_RS$ is a finitely generated R-module, and R is left noetherian, then so is S. Thus, for example, if R is noetherian, so is $M_n(R)$.

(3) If K is noetherian and R is a finitely generated commutative K-algebra, then R is noetherian, as it is a quotient of a polynomial ring $K[x_1, \dots, x_n]$. This is not true for R non-commutative. But we have the following.

Artin-Tate Lemma. Let A be a finitely generated K-algebra with K noetherian, and let Z be a K-subalgebra of $Z(A)$. If A is finitely generated as a Z-module, then Z is finitely generated as a K-algebra, hence Z and A are noetherian rings.

Proof. Let $a_1, \dots, a_n$ be algebra generators of A. Now $A = Zb_1 + \dots + Zb_m$, so we can write $a_i = \sum_j z_{ij} b_j$ and $b_i b_j = \sum_k z_{ijk} b_k$ with $z_{ij}, z_{ijk} \in Z$. Let Z' be the K-subalgebra of Z generated by the z_{ij} and z_{ijk}. It is a finitely generated commutative K-algebra, so noetherian. Now A is generated as a K-module by products of the a_i, so $A = Z'b_1 + \dots + Z'b_m$, so it is a finitely generated Z'-module. Then $Z \subseteq A$ is a finitely generated Z'-module. In particular it is finitely generated as a Z'-algebra, and hence also as a K-algebra.

1.11 Tensor products

If X is a right R-module and Y is a left R-module, the tensor product $X \otimes_R Y$ is defined to be the additive group generated by symbols $x \otimes y$ ($x \in X, y \in Y$) subject to the relations:
- $(x + x') \otimes y = x \otimes y + x' \otimes y$,
- $x \otimes (y + y') = x \otimes y + x \otimes y'$,
- $(xr) \otimes y = x \otimes (ry)$ for $r \in R$.

Properties. (1) By definition, an arbitrary element of $X \otimes_R Y$ can be written as a finite sum of tensors $\sum_{i=1}^n x_i \otimes y_i$, but this expression is not unique. You may need more than one term in the expression.

(2) If R is a K-algebra, then $X \otimes_R Y$ is a K-module via $\lambda(x \otimes y) = x(\lambda 1) \otimes y = x \otimes (\lambda 1)y$.

If Z is a K-module, a map $\phi : X \times Y \to Z$ is K-*bilinear* if it is K-linear in each argument, and R-*balanced* if $\phi(xr, y) = \phi(x, ry)$ for all x, y, r. The map $X \times Y \to X \otimes_R Y$, $(x, y) \mapsto x \otimes y$ is K-bilinear and R-balanced. Moreover there is a bijection

$$\mathrm{Hom}_K(X \otimes_R Y, Z) \cong \{\text{set of } K\text{-bilinear } R\text{-balanced maps } X \times Y \to Z\}.$$

It sends $\theta \in \mathrm{Hom}_K(X \otimes_R Y, Z)$ to the map bmap ϕ with $\phi(x, y) = \theta(x \otimes y)$, and sends ϕ to the map θ with $\theta(\sum_i x_i \otimes y_i) = \sum \phi(x_i, y_i)$

(3) If X is an S-R-bimodule, then $X \otimes_R Y$ becomes an S-module via $s(x \otimes y) = (sx) \otimes y$, and for a left S-module Z, there is a natural isomorphism

$$\mathrm{Hom}_S(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z)).$$

Both sides correspond to the K-bilinear R-balanced maps $X \times Y \to Z$ which are also S-linear in the first argument.

Dually, if Y is an R-T-bimodule, then $X \otimes_R Y$ is a right T-module and

$$\mathrm{Hom}_T(X \otimes Y, Z) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_T(Y, Z)).$$

(4) There are natural isomorphisms $X \otimes_R R \cong X$, $x \otimes r \mapsto xr$ and $R \otimes_R Y \cong Y$, $r \otimes y \mapsto ry$. There are natural isomorphisms

$$\left(\bigoplus_{i \in I} X_i\right) \otimes_R Y \cong \bigoplus_{i \in I} (X_i \otimes_R Y), \;\; X \otimes_R \left(\bigoplus_{i \in I} Y_i\right) \cong \bigoplus_{i \in I} (X \otimes_R Y_i).$$

Thus $X \otimes_R {}_R R^{(J)} \cong X^{(J)}$ and $R_R^{(I)} \otimes_R Y \cong Y^{(I)}$, so $R_R^{(I)} \otimes_R {}_R R^{(J)} \cong R^{(I \times J)}$.

In particular if K is a field and V and W are K-vector spaces with bases $(v_i)_{i \in J}$ and $(w_j)_{j \in J}$ then $V \otimes_K W$ is a K-vector space with basis $(v_i \otimes w_j)_{(i,j) \in I \times J}$. Thus $\dim(V \otimes_K W) = (\dim V)(\dim W)$.

(5) If $\theta : X \to X'$ is a map of right R-modules and $\phi : Y \to Y'$ is a map is left R-modules, then there is a map

$$\theta \otimes \phi : X \otimes_R Y \to X' \otimes_R Y', \ x \otimes y \mapsto \theta(x) \otimes \phi(y).$$

If θ is a map of S-R-bimodules, then this is a map of S-modules, etc.

(6) If $X' \subseteq X$ is an R-submodule of X then $(X/X') \otimes_R Y$ is isomorphic to the quotient of $X \otimes_R Y$ by the subgroup generated by all elements of the form $x' \otimes y$ with $x' \in X'$, $y \in Y$ (so the cokernel of the map $X' \otimes_R Y \to X \otimes_R Y$). Similarly for $X \otimes_R (Y/Y')$ if Y' is a submodule of Y.

Thus if I is a right ideal in R,

$$(R/I) \otimes_R Y \cong (R \otimes_R Y)/\operatorname{Im}(I \otimes_R Y \to R \otimes_R Y) \cong Y/IY.$$

Similarly if J is a left ideal in R then $X \otimes_R (R/J) \cong X/XJ$.

Thus $(R/I) \otimes_R (R/J) \cong R/(I + J)$. eg. $(\mathbb{Z}/2\mathbb{Z}) \otimes_Z (\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/\mathbb{Z} = 0$.

(7) If X is a right S-module, Y a S-R-bimodule and Z a left R-module, then there is a natural isomorphism

$$X \otimes_S (Y \otimes_R Z) \cong (X \otimes_S Y) \otimes_R Z.$$

(8) Tensor product of algebras. If R and S are K-algebras, then $R \otimes_K S$ becomes a K-algebra via $(r \otimes s)(r' \otimes s') = (rr') \otimes (ss')$. For example $M_n(K) \otimes_K S \cong M_n(S)$. An R-S-bimodule (for which the two actions of K agree) is the same thing as a left $R \otimes_K S^{op}$-module.

(9) Base change. If S is a commutative K-algebra then $R \otimes_K S$ is naturally an S-algebra.

1.12 Catalgebras

Sometimes it is useful to consider non-unital associative algebras, but usually one wants some weaker form of unital condition, and there are many possibilities, for example “rings with local units”. The version below, I call “catalgebras”, since we will see later that they correspond exactly to small K-categories. (In categorical language, this is the theory of “rings with several objects”).

Definition. By a *catalgebra* we mean a K-module R with a multiplication $R \times R \to R$ which is associative and K-linear in each variable, such that there exists a family $(e_i)_{i\in I}$ of orthogonal idempotents which is *complete* in the sense that for all $r \in R$ only finitely many of the elements re_i are nonzero and only finitely many of the elements $e_i r$ are nonzero and $r = \sum_{i\in I} re_i = \sum_{i\in I} e_i r$.

By a *left module* for a catalgebra we mean a K-module M with a map $R \times M \to M$ which is an action, K-linear in each variable, and unital in the sense that $RM = M$.

Lemma 1. For a catalgebra R and a left module M we have $R = \bigoplus_{i,j\in I} e_i R e_j$ and $M = \bigoplus_{i\in I} e_i M$.

Proof. Straightforward. For example if $m \in M$ then $RM = M$ implies $m = \sum_{s=1}^{t} r_s m_s$. Now each $r_s = \sum_{i\in I} e_i r_{si}$. Thus $m = \sum_i e_i (\sum_s r_{si} m_s) \in \sum_{i\in I} e_i M$.

Examples.
(i) Any algebra is a catagebra with family (1) or a finite complete set of idempotents. Modules are the same as modules for an algebra.

(ii) A catalgebra with a finite family of idempotents is unital, so an ordinary algebra.

(iii) An arbitrary direct sum of algebras (or catalgebras) $\bigoplus_{i\in I} R_i$ is a catalgebra, with the idempotents given by the unit elements in the R_i or by combining the families for the R_i.

(iv) If I is a set and R an algebra or catalgebra, write $R^{(I\times I)}$ for the set of matrices with entries in R, with rows and columns indexed by I, and only finitely many non-zero entries. It is a catalgebra.

(v) If $(I, \leq)$ is a partially ordered set, there is an associated catalgebra KI which has as basis the pairs $(x, y) \in I^2$ with $y \leq x$ and product given by $(x, y)(x', y') = (x, y')$ if $y = x'$ and otherwise 0. The elements (x, x) are a complete set of orthogonal idempotents.

It is an algebra if and only if I is finite. In this case it is the opposite of the *incidence algebra* of I, introduced in combinatorics by G.-C. Rota. We call it the *poset algebra*.

In case $I = \mathbb{R}$ with the usual ordering, or $\mathbb{R}^n$, modules for KI are known as *persistence modules*, and appear in topological data analysis.

Lemma 2. If R is a catalgebra, then $R_1 = R \oplus K$ becomes an algebra with unit element $(0,1)$ under the multiplication $(r,\lambda)(r',\lambda') = (rr' + \lambda r' + \lambda' r, \lambda\lambda')$, and we can identify R as an ideal in R_1. Moreover there is a 1:1 correspondence

$$\{R\text{-modules } M\} \leftrightarrow \{R_1\text{-modules } M \text{ with } RM = M\}\,.$$

If L is a submodule of an R_1-module M, then then $RM = M$ if and only if $RL = L$ and $R(M/L) = M/L$.

Proof. Straightforward.

Thus modules for a catalgebra are nothing new. Henceforth, everything I do for algebras, you might think about possible generalizations to catalgebras.

2 Examples and constructions of algebras

We consider K-algebras, where K is a commutative ring. Maybe $K = \mathbb{Z}$, so we consider rings.

2.1 Path algebras

A *quiver* is a quadruple $Q = (Q_0, Q_1, h, t)$ where Q_0 is a set of vertices, Q_1 a set of arrows, and $h, t : Q_1 \to Q_0$ are mappings, specifying the head and tail vertices of each arrow,

$$\overset{t(a)}{\bullet} \xrightarrow{\ a\ } \overset{h(a)}{\bullet}.$$

A *path* in Q of length $n > 0$ is a sequence $p = a_1 a_2 \dots a_n$ of arrows satisfying $t(a_i) = h(a_{i+1})$ for all $1 \le i < n$,

$$\bullet \xleftarrow{a_1} \bullet \xleftarrow{a_2} \bullet \cdots \bullet \xleftarrow{a_n} \bullet.$$

The head and tail of p are $h(a_1)$ and $t(a_n)$. For each vertex $i \in Q_0$ there is also a *trivial path* e_i of length zero with head and tail i.

We write KQ for the free K-module with basis the paths in Q. It has a multiplication, in which the product of two paths given by $p \cdot q = 0$ if the tail of p is not equal to the head of q, and otherwise $p \cdot q = pq$, the concatenation of p and q.

This makes KQ into a catalgebra in which the trivial paths are a complete family of orthogonal idempotents. Normally we assume Q_0 is finite, so KQ is unital, $1 = \sum_{i \in Q_0} e_i$, so an algebra.

Examples 1. (i) The path algebra of the quiver $1 \xrightarrow{a} 2$ with loop b at 2 has basis $e_1, e_2, a, b, ba, b^2, b^2a, b^3, b^3a, \dots$.
(ii) The algebra of lower triangular matrices in $M_n(K)$ is isomorphic to the path algebra of the quiver

$$1 \to 2 \to \cdots \to n$$

with the matrix unit e^{ij} corresponding to the path from j to i.
(iii) The free algebra $K\langle X\rangle \cong KQ$ where Q has one vertex and $Q_1 = X$.

KQ-modules are essentially the same thing as representations of Q.

A *representation* of Q is a tuple $V = (V_i, V_a)$ consisting of a K-module V_i for each vertex i and a K-module map $V_a : V_i \to V_j$ for each arrow $a : i \to j$ in Q. If there is no risk of confusion, we write $a : V_i \to V_j$ instead of V_a.

If V is a KQ-module, then $V = \bigoplus e_i V$. We get a representation, also denoted V, with $V_i = e_i V$, and, for any arrow $a : i \to j$, the map $V_a : V_i \to V_j$ is the map given by left multiplication by $a \in e_j KQ e_i$.

Conversely any representation V determines a KQ-module via $V = \bigoplus_{i\in Q_0} V_i$, with the action of KQ given as follows:
- For $v = (v_i)_{i\in Q_0} \in V$ we have $e_i v = v_i \in V_i \subseteq V$. That is, the trivial path e_i acts on V as the projection onto V_i, and
- $a_1 a_2 \dots a_n v = V_{a_1}(V_{a_2}(\dots(V_{a_n}(v_{t(a_n)}))\dots)) \in V_{h(a_1)} \subseteq V$.

Under this correspondence:

(1) KQ-module homomorphisms $\theta : V \to W$ correspond to tuples (θ_i) consisting of a K-module map $\theta_i : V_i \to W_i$ for each vertex i satisfying $W_a \theta_i = \theta_j V_a$ for all arrows $a : i \to j$.

(2) KQ-submodules W of V correspond to tuples (W_i) where each W_i is a K-submodule of V_i, such that $V_a(W_i) \subseteq W_j$ for all arrows $a : i \to j$. Then W corresponds to the representation $(W_i, V_a|_{W_i} : W_i \to W_j)$ and V/W to the representation $(V_i/W_i, \overline{V_a} : V_i/W_i \to V_j/W_j)$.

(3) Direct sums of modules $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$ correspond to direct sums of representations $(\bigoplus_\lambda V_i^\lambda, \bigoplus_\lambda V_a^\lambda)$.

Notation.
(a) $(KQ)_+$ is the K-span of the non-trivial paths. It is an ideal, and $(KQ)/(KQ)_+ \cong K^{(Q_0)}$.
(b) We write $P[i]$ for the KQ-module KQe_i, so $KQ = \bigoplus_{i\in Q_0} P[i]$. Considered as a representation of Q, the vector space at vertex j has basis the paths from i to j.
(c) We write $S[i]$ for the representation with $S[i]_i = K$, $S[i]_j = 0$ for $j \neq i$ and all $S[i]_a = 0$. It corresponds to the module $KQe_i/(KQ)_+e_i$.

For the rest of this section, suppose K is a field. If V is a finite-dimensional representation, its *dimension vector* is $\underline{\dim}\, V = (\dim V_i) \in \mathbb{N}^{Q_0}$.

Example 2. Let Q be the quiver $1 \xrightarrow{a} 2$.

(i) $S[1]$ is the representation $K \to 0$ and $S[2]$ is the representation $0 \to K$.

(ii) $P[1]$ is the representation $K \xrightarrow{1} K$ and $P[2] \cong S[2]$.

(iii) $\mathrm{Hom}(S[1], P[1]) = 0$ and $\mathrm{Hom}(S[2], P[1]) \cong K$. The subspaces $(K \subseteq V_1, 0 \subseteq V_2)$ do not give a subrepresentation of $V = P[1]$, but the subspaces $(0 \subseteq V_1, K \subseteq V_2)$ do, and this subrepresentation is isomorphic to $S[2]$.

(iv) There is an exact sequence $0 \to S[2] \to P[1] \to S[1] \to 0$.

(v) $S[1] \oplus S[2] \cong K \xrightarrow{0} K$ and for $0 \neq \lambda \in K$ we have $K \xrightarrow{\lambda} K \cong P[1]$.

(vi) Every representation of Q is isomorphic to a direct sum of copies of $S[1]$, $S[2]$ and $P[1]$. For a f.d. representation $V_1 \xrightarrow{a} V_2$ one can see it as follows. Taking bases of V_1 and V_2, the representation is isomorphic to $K^n \xrightarrow{A} K^n$ for some $m \times n$ matrix A. Now there are invertible matrices P, Q with PAQ^{-1} of the form

$$C = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

with I an identity matrix. Then (Q, P) gives an isomorphism from $K^n \xrightarrow{A} K^n$ to $K^n \xrightarrow{C} K^n$, and this last representation is a direct sum as claimed.

Lemma.
(i) The $P[i]$ are non-isomorphic indecomposable modules.
(ii) The $S[i]$ are non-isomorphic simple modules.
(iii) KQ is f.d. if and only if Q is finite and has no oriented cycles. If so, then $(KQ)_+$ is nilpotent, so it is the Jacobson radical of KQ and the $S[i]$ are all of the simple KQ-modules.

Proof. (i) Clearly the spaces $P[i] = KQe_i$, e_jKQe_i and e_jKQ have as K-bases the paths with tail at i and/or head at j.

If $0 \neq f \in KQe_i$ and $0 \neq g \in e_iKQ$ then $fg \neq 0$. Explicitly if p and q are paths of maximal length involved in f and g, then the coefficient of pq in fg must be non-zero.

Now $\mathrm{End}(P[i])^{op} \cong e_iKQe_i$, and by the observation above this is a domain (products of non-zero elements are non-zero). Thus it has no non-trivial idempotents, so $P[i]$ is indecomposable.

If $P[i] \cong P[j]$, then there are inverse isomorphisms, so elements $f \in e_jKQe_i$ and $g \in e_iKQe_j$ with $fg = e_j$ and $gf = e_i$. But by the argument above, f and g can only involve trivial paths, so $i = j$.

(ii) Clear.

(iii) First part clear. Now there is a bound on the length of any path, so $(KQ)_+$ is nilpotent. Since $KQ/(KQ)_+ \cong K^{(Q_0)}$, it is semisimple, $J(KQ) \subseteq (KQ)_+$, so we have equality. The simples are indexed by Q_0, so there are no simples other than the $S[i]$.

Definition. Suppose Q is finite. An ideal $I \subseteq KQ$ is *admissible* if
(1) $I \subseteq (KQ)_+^2$, and
(2) $(KQ)_+^n \subseteq I$ for some n.

Examples 3.
(i) If Q has no oriented cycles, $I = 0$ is admissible.

(ii) Let Q be the quiver with one vertex and one loop x, so $KQ = K[x]$. The admissible ideals in KQ are (x^n) for $n \geq 2$.
(iii) The poset algebra of the poset

$$\begin{array}{ccc} & 4 & \\ 2 & & 3 \\ & 1 & \end{array}$$

has basis $(1,1),(2,2),(3,3),(4,4),(2,1),(3,1),(4,2),(4,3),(4,1)$. It is isomorphic to $KQ/(ca-db)$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "b"'] & 2 \arrow[d, "c"'] \\ 3 \arrow[r, "d"] & 4 \end{tikzcd}$$

This is trivially an admissible ideal. Modules correspond to representations of the quiver satisfying the commutativity relation $ca = db$.

Theorem (Gabriel). If I is an admissible ideal in KQ then $R = KQ/I$ is f.d., $J(R) = (KQ)_+/I$, and $R/J(R) \cong K \times \cdots \times K$. Conversely, if R is a f.d. algebra and $R/J(R) \cong K \times \cdots \times K$, then $R \cong KQ/I$ for some finite quiver Q and admissible ideal I.

Proof. First part is easy. Sketch for the second part.

Trivial fact (add to section 1.5). If $e \in R$ is idempotent and N is a submodule of an R-module M, then $e(M/N) \cong eM/eN$.

Let $J = J(R)$. Observe that if $S = K^n = K \times \cdots \times K$ (n factors), then it has a basis consisiting of orthogonal idempotents $f_i = (0, \dots, 0, 1, 0, \dots, 0)$, and if $f = (1, 1, \dots, 1, 0) = f_1 + \dots + f_{n-1}$, then $fSf = \{(a_1, \dots, a_{n-1}, 0)\} \cong K^{n-1}$.

We show by induction on n that if $R/J \cong K^n$ then there are orthogonal idempotents e_i in R lifting the idempotents f_i in R/J. Let e be a lift of f. Then eJe is a nilpotent ideal in eRe, so $eJe \subseteq J(eRe)$. Then $J(eRe/eJe) = J(eRe)/eJe \cong J(eRe/eJ(R)e) = e(R/J)e = f(R/J)f \cong K^{n-1}$. Thus $J(eRe) = eJe$, and by induction there are idempotents $e_1, \dots, e_{n-1}$ in eRe inducing the idempotents $f_1, \dots, f_{n-1}$ in R/J. Then take $e_n = 1 - e$.

Let $Q_0 = \{1, \dots, n\}$. We have $J = \bigoplus_{i,j \in Q_0} e_j J e_i$. Then J/J^2 is an R-R-bimodule, so we can decompose it as

$$J/J^2 = \bigoplus_{i,j \in Q_0} e_j (J/J^2) e_i = \bigoplus_{i,j \in Q_0} (e_j J e_i)/(e_j J^2 e_i)$$

Let the arrows in Q from i to j correspond to elements of $e_j J e_i$ inducing a K-basis of $(e_j J e_i)/(e_j J^2 e_i)$. We get an induced homomorphism $\theta : KQ \to R$.

Let $U = \theta(KQ_+) \subseteq J$. Now $U + J^2 = J$, so by Nakayama's Lemma, $U = J$. It follows that θ is surjective. Let $I = \operatorname{Ker} \theta$. If m is sufficiently large that $J^m = 0$, then $\theta(KQ_+^m) = U^m = 0$, so $KQ_+^m \subseteq I$. Suppose $x \in I$. Write it as $x = u + v + w$ where u is a linear combination of e_i's, v is a linear combination of arrows, and w is in KQ_+^2. Since $\theta(e_i) = e_i$ and $\theta(v), \theta(w) \in J$, we must have $u = 0$. Now $\theta(v) = -\theta(w) \in J^2$, so that $\theta(v)$ induces the zero element of J/J^2. Thus $v = 0$. Thus $x = w \in KQ_+^2$.

2.2 Diamond lemma

The diamond lemma is due to Max Newman. There is an exposition in P.M.Cohn, Algebra, volume 3. We explain a (trivial) quiver version of G.M.Bergman, The diamond lemma for ring theory, Advances in Mathematics 1978. It helps to find a K-basis for an algebra R given by generators and relations. See also, for example, Farkas, Feustel and Green, Synergy in the theories of Gröbner bases and path algebras, Canad. J. Math. 1993.

Setup.
(1) Write $R = KQ/(S)$ for some quiver Q. For $f \in KQ$ we have $f = \sum_{i,j} e_j f e_i$ so $(f) = (e_i f e_j : i, j \in Q_0)$. Thus we may assume that each element of S is a linear combination of parallel paths (same start and end).

(2) We fix a well-ordering on the set of paths, such that if w, w' are parallel and $w < w'$, then $uwv < uw'v$ for all compatible products of paths. (A well-ordering is a total ordering with the descending chain condition, so every non-empty subset has a minimal element.)

If Q is finite, this can be done by choosing a total ordering on the vertices $1 < 2 < \cdots < n$ and on the arrows $a < b < \dots$ and using the *length-lexicographic* ordering on paths, so $w < w'$ if
- length $w <$ length w', or
- $w = e_i$ and $w' = e_j$ with $i < j$, or
- length $w =$ length $w' > 0$ and w comes before w' in the dictionary ordering.

(3) We suppose that the relations in S can be written in the form

$$w_j = s_j \quad (j \in J)$$

where each w_j is a path and s_j is a linear combination of paths w parallel to w_j with $w < w_j$. (This is always possible if K is a field.)

Example 1. Consider the algebra $R = K\langle x, y\rangle/(S)$ where S is given by

$$x^2 = x, \quad y^2 = 1, \quad yx = 1 - xy$$

and the alphabet ordering $x < y$.

Definition. Given a relation $w_j = s_j$ and paths u, v such that uw_jv is a path, the associated *reduction* is the linear map $KQ \to KQ$ sending uw_jv to us_jv and any other path to itself. We write $f \rightsquigarrow g$ to indicate that g is obtained from f by applying reduction with respect to some $w_j = s_j$ and u, v.

Example 1 (continued). $f = x^2 + xy^2 \rightsquigarrow x^2 + x \rightsquigarrow x + x = 2x$, or $f \rightsquigarrow x + xy^2 \rightsquigarrow 2x$, and $g = yx^2 \rightsquigarrow yx \rightsquigarrow 1 - xy$, or $g \rightsquigarrow (1 - xy)x = x - xyx \rightsquigarrow x - x(1 - xy) = x^2y \rightsquigarrow xy$.

Lemma 1. If $f \rightsquigarrow g$ and u', v' are paths, then $u'fv' \rightsquigarrow u'gv'$ or $u'fv' = u'gv'$.

Proof. Suppose g is the reduction of f with respect to u, v and the relations $w_j = s_j$. If $u'u$ or vv' are not paths, then $u'fv' = u'gv'$. Else $u'gv'$ is the reduction of $u'fv'$ with respect to $u'u$, vv' and the relation $w_j = s_j$.

Definition. We say that f is *irreducible* if $f \rightsquigarrow g$ implies $g = f$. It is equivalent that no path involved in f can be written as a product uw_jv.

Lemma 2. Any $f \in KQ$ can be reduced by a finite sequence of reductions to an irreducible element.

Proof. Any $f \in KQ$ which is not irreducible involves paths of the form uw_jv. Amongst all paths of this form involved in f, let $\mathrm{tip}(f)$ be the maximal one. Consider the set of tips of elements which cannot be reduced to an irreducible element. For a contradiction assume this set is non-empty. Then by well-ordering it contains a minimal element. Say it is $\mathrm{tip}(f) = w = uw_jv$. Writing $f = \lambda uw_jv + f'$ where $\lambda \in K$ and f' only involving paths different from uw_jv, we have $f \rightsquigarrow g$ where $g = \lambda us_jv + f'$. By the properties of the ordering, us_jv only involves paths which are less than $uw_jv = w$, so $\mathrm{tip}(g) < w$. Thus by minimality, g can be reduced to an irreducible element, hence so can f. Contradiction.

Definition. We say that f is *reduction-unique* if there is a unique irreducible element which can be obtained from f by a sequence of reductions. If so, the irreducible element is denoted $r(f)$.

Lemma 3. The set of reduction-unique elements is a K-submodule of KQ, and the assignment $f \mapsto r(f)$ is a K-module endomorphism of it.

Proof. Consider a linear combination $\lambda f + \mu g$ where f, g are reduction-unique and $\lambda, \mu \in K$. Suppose there is a sequence of reductions (labelled (1))

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h$$

with h irreducible. Let a be the element obtained by applying the same reductions to f. By Lemma 2, a can be reduced by some sequence of reductions (labelled (2)) to an irreducible element. Since f is reduction-unique, this irreducible element must be $r(f)$.

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f).$$

Applying all these reductions to g we obtain elements b and c, and after applying more reductions (labelled (3)) we obtain an irreducible element, which must be $r(g)$.

$$g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} b \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} c \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(g).$$

But $h, r(f)$ are irreducible, so these extra reductions don't change them:

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} h,$$

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f) \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(f).$$

Now the reductions are linear maps, hence so is a composition of reductions, so $h = \lambda r(f) + \mu r(g)$. This shows that $\lambda f + \mu g$ is reduction-unique and that $r(\lambda f + \mu g) = \lambda r(f) + \mu r(g)$.

Definition. We say that two reductions of f, say $f \rightsquigarrow g$ and $f \rightsquigarrow h$, satisfy the *diamond condition* if there exist sequences of reductions starting with g and h, which lead to the same element, $g \rightsquigarrow \cdots \rightsquigarrow k$, $h \rightsquigarrow \cdots \rightsquigarrow k$. (You can draw this as a diamond.)

In particular we are interested in this in the following two cases:

An *overlap ambiguity* is a path w which can be written as $w_i v$ and also as $u w_j$ for some i, j and some paths $u, v \neq 1$, so that w_i and w_j overlap. There are reductions $w \rightsquigarrow s_i v$ and $w \rightsquigarrow u s_j$.

An *inclusion ambiguity* is a path w which can be written as w_i and as $u w_j v$ for some $i \neq j$ and some u, v. There are reductions $w \rightsquigarrow s_i$ and $w \rightsquigarrow u s_j w$.

Example 1 (continued). For the relations $x^2 = x$, $y^2 = 1$, $yx = 1 - xy$ the ambiguities are:

$$\underline{x\overline{x}}\overline{x} \quad \underline{y\overline{y}}\overline{y} \quad \underline{y\overline{y}}\overline{x} \quad \underline{y\overline{x}}\overline{x}.$$

The diamond condition fails for the last ambiguity.

Example 2. For the relations $x^2 = x$, $y^2 = 1$, $yx = y - xy$ the ambiguities are the same.

Does the diamond condition hold?
$\underline{xx}x \rightsquigarrow xx \rightsquigarrow x$ and $x\overline{xx} \rightsquigarrow xx \rightsquigarrow x$. Yes.
$\underline{yy}y \rightsquigarrow 1y = y$ and $y\overline{yy} \rightsquigarrow y1 = y$. Yes.
$\underline{yy}x \rightsquigarrow 1x = x$ and $y\overline{yx} \rightsquigarrow y(y - xy) = y^2 - yxy = y^2 - (yx)y \rightsquigarrow y^2 - (y - xy)y = xyy = x(yy) \rightsquigarrow x1 = x$. Yes.
$\underline{yx}x \rightsquigarrow (y - xy)x = yx - xyx \rightsquigarrow yx - x(y - xy) = yx - xy + xxy \rightsquigarrow yx - xy + xy = yx \rightsquigarrow \ldots$ and $y\overline{xx} \rightsquigarrow yx \rightsquigarrow \ldots$. Yes.

Diamond Lemma. The following conditions are equivalent:
(a) The diamond condition holds for all overlap and inclusion ambiguities.
(b) Every element of KQ is reduction-unique.
In this case the algebra $R = KQ/(S)$ has K-basis given by the irreducible paths, with multiplication given by $f.g = r(fg)$.

Example 2 (continued). Consider our example of R generated by x, y subject to $x^2 = x$, $y^2 = 1$, $yx = y - xy$. The irreducible paths $1, x, y, xy$ form a K-basis of A with multiplication table

	1	x	y	xy
1	1	x	y	xy
x	x	x	xy	xy
y	y	$y - xy$	1	$1 - x$
xy	xy	0	x	0

For example $y(xy) = (yx)y \rightsquigarrow (y - xy)y = yy - xyy \rightsquigarrow 1 - xyy \rightsquigarrow 1 - x$, and $(xy)(xy) = x(yx)y \rightsquigarrow x(y - xy)y = xyy - xxyy \rightsquigarrow x - xxyy \rightsquigarrow x - xyy \rightsquigarrow x - x = 0$.

Example 3. (P. Shaw, Appendix A, Generalisations of Preprojective algebras, Ph. D. thesis, Leeds, 2005. Available from homepage of WCB.) The algebra with generators b, c and relations $b^3 = 0$, $c^2 = 0$ and $cbcb = cb^2c - bcbc$ fails the diamond condition for the overlap $cbc(b^3) = (cbcb)b^2$. But this calculation shows that the equation $cb^2cb^2 = bcb^2cb - b^2cb^2c$ holds in the algebra, and if you add this as a relation, the diamond condition holds.

Proof of Diamond Lemma. (b)⇒(a) is trivial, so we prove (a)⇒(b). Since the reduction-unique elements form a subspace, it suffices to show that every *path* is reduction-unique. For a contradiction, suppose not. Then there is a minimal path w which is not reduction-unique. Let $f = w$. Suppose that f reduces under some sequence of reductions to g, and under another sequence of reductions to h, with g, h irreducible. We want to prove that $g = h$, giving a contradiction.

Let the elements obtained in each case by applying one reduction be f_1 and g_1. Thus

$$f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g, \qquad f \rightsquigarrow h_1 \rightsquigarrow \cdots \rightsquigarrow h.$$

By the properties of the ordering, g_1 and h_1 are linear combinations of paths which are less than w, so by minimality they are reduction-unique. Thus $g = r(g_1)$ and $h = r(h_1)$.

It suffices to prove that the reductions $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$ satisfy the diamond condition, for if there are sequences of reductions $g_1 \rightsquigarrow \cdots \rightsquigarrow k$ and $h_1 \rightsquigarrow \cdots \rightsquigarrow k$, combining them with a sequence of reductions $k \rightsquigarrow \cdots \rightsquigarrow r(k)$, we have $g = r(g_1) = r(k) = r(h_1) = h$.

Thus we need to check the diamond condition for $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$. Recall that $f = w$, so these reductions are given by subpaths of w of the form w_i and w_j. There are two cases:

(i) If these paths overlap, or one contains the other, the diamond condition follows from the corresponding overlap or inclusion ambiguity. For example w might be of the form $u'w_ivv' = u'uw_jv'$ where $w_iv = uw_j$ is an overlap ambiguity and u', v' are paths. Now condition (a) says that the reductions $w_iv \rightsquigarrow s_iv$ and $uw_j \rightsquigarrow us_j$ can be completed to a diamond, say by sequences of reductions $s_iv \rightsquigarrow \cdots \rightsquigarrow k$ and $us_j \rightsquigarrow \cdots \rightsquigarrow k$. Then Lemma 1 shows that the two reductions of w, which are $w = u'w_ivv' \rightsquigarrow u's_ivv'$ and $w = u'uw_jv' \rightsquigarrow u'vs_jv'$, can be completed to a diamond by reductions leading to $u'kv'$.

(ii) Otherwise w is of the form uw_ivw_jz for some paths u, v, z, and $g_1 = us_ivw_jz$ and $h_1 = uw_ivs_jz$ (or vice versa). Writing s_i as a linear combination of paths, $s_i = \lambda t + \lambda' t' + \dots$, we have

$$r(g_1) = r(us_ivw_jz) = \lambda r(utvw_jz) + \lambda' r(ut'vw_jz) + \dots \ .$$

Reducing each path on the right hand side using the relation $w_j = s_j$, we have $utvw_jz \rightsquigarrow utvs_jz$, and $ut'vw_jz \rightsquigarrow ut'vs_jz$, and so on, so

$$r(g_1) = \lambda r(utvs_jz) + \lambda' r(ut'vs_jz) + \dots \ .$$

Collecting terms, this gives $r(g_1) = r(us_ivs_jz)$. Similarly, writing s_j as a linear combination of paths, we have $r(h_1) = r(us_ivs_jz)$. Thus $r(h_1) = r(g_1)$, so the diamond condition holds.

For the last part we show that $r(f) = 0$ if and only if $f \in (S)$. If $f \rightsquigarrow g$ then $f - g \in (S)$, so $f - r(f) \in (S)$ giving one direction. For the other, (S) is spanned by expressions of the form $u(w_j - s_j)v$, and $uw_jv \rightsquigarrow us_jv$ so $r(uw_jv) = r(us_jv)$, so $r(u(w_j - s_j)v) = 0$.

Thus r defines a K-module isomorphism from $KQ/(S)$ to the K-span of the irreducible paths.

2.3 Tensor algebras and variations

Definitions. An algebra R is *($\mathbb{Z}$-)graded* if it is equipped with a decomposition $R = \bigoplus_{n\in\mathbb{Z}} R_n$ satisfying $R_nR_m \subseteq R_{n+m}$, and it is $\mathbb{N}$-*graded* if $R_n = 0$ for $n < 0$.

If R is graded, an R-module M is *graded* if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $R_n M_m \subseteq M_{n+m}$.

An element of R or M is *homogeneous* of *degree* n if it belongs to R_n or M_n.

A submodule N of M is *graded* or *homogeneous* if $N = \bigoplus N_n$ where $N_n = N \cap M_n$. Similarly for an ideal I in R.

A homomorphism is *graded* if it sends homogeneous elements to homogeneous elements of the same degree.

Example. The path algebra $R = KQ$ is graded with $R_n =$ the K-span of the paths of length n.

Proposition. Let R be a graded algebra.
(i) $1 \in R_0$.
(ii) A submodule or ideal is homogeneous if and only if it is generated by homogeneous elements.
(iii) A quotient of a module or algebra by a homogeneous submodule or ideal is graded.
(iv) Graded R-modules are the same thing as modules for the sub-catalgebra of $R^{(\mathbb{Z}\times\mathbb{Z})}$ consisting of matrices (a_{ij}) with $a_{ij} \in R_{i-j}$.

$$\begin{pmatrix} \ddots & & & & \\ & R_0 & R_{-1} & R_{-2} & \\ & R_1 & R_0 & R_{-1} & \\ & R_2 & R_1 & R_0 & \\ & & & & \ddots \end{pmatrix}$$

Proof. (i) if $1 = \sum r_n$ and $r \in R_i$ then $r = r1 = 1r$ gives $r = rr_0 = r_0 r$, so r_0 is a unity for R.

(ii)-(iv) Straightforward.

Definitions. Let V be a K-module. The *tensor powers* are

$$T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n},$$

where tensor products are over K and $T^0(V) = K$.
(i) The *tensor algebra* is the graded algebra

$$T(V) = \bigoplus_{n\in N} T^n(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

with the multiplication given by $T^n(V) \otimes_K T^m(V) \cong T^{n+m}(V)$.
(ii) The *symmetric algebra* is the graded algebra

$$S(V) = T(V)/(v \otimes w - w \otimes v : v, w \in V) = \bigoplus_{d \geq 0} S^d(V).$$

(iii) The *exterior algebra* is the graded algebra

$$\Lambda(V) = T(V)/(v \otimes v : v \in V) = \bigoplus_{d \geq 0} \Lambda^d(V).$$

We write $v \wedge w$ for the product in the exterior algebra. We have $v \wedge w = -w \wedge v$ since $v \wedge v = 0$, $w \wedge w = 0$ and $(v + w) \wedge (v + w) = 0$.

(iv) A mapping $q : V \to K$ is a *quadratic form* if $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in K$ and $x \in V$ and the map $V \times V \to K$, $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is a bilinear form in x and y. The associated *Clifford algebra* is

$$C(V, q) = T(V)/(v \otimes v - q(v)1 : v \in V).$$

Lemma. If V is a free K-module with basis $x_1, \dots, x_n$, then
(i) $T(V) \cong K\langle x_1, \dots, x_n\rangle$.
(ii) $S(V) \cong K[x_1, \dots, x_n]$.
(iii) $\Lambda(V) \cong K\langle x_1, \dots, x_n\rangle/(x_i^2, x_i x_j + x_j x_i)$. $\Lambda^d(V)$ is the free K-module with basis the products $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_d}$ with $i_1 < i_2 < \cdots < i_d$. In particular, $\Lambda^n(V) = K\ x_1 \wedge x_2 \wedge \cdots \wedge x_n$ and $\Lambda^m(V) = 0$ for $m > n$. If K is a field then $\Lambda(V)$ is local with $J(\Lambda(V)) = \bigoplus_{n>0} \Lambda^n(V)$.
(iv) If $q(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n a_i \lambda_i^2$, then $C(V, q) \cong K\langle x_1, \dots, x_n\rangle/(x_i^2 - a_i 1, x_i x_j + x_j x_i)$. It has basis the products $x_{i_1} x_{i_2} \dots x_{i_d}$ with $i_1 < i_2 < \cdots < i_d$. If K is a field, then $C(V, q)$ is semisimple if and only if all $a_i \neq 0$.

Proof. (i), (ii) Clear.

(iii) For the exterior algebra use the Diamond Lemma with the relations $x_i^2 = 0$ and $x_j x_i = -x_i x_j$ for $j > i$ and check the ambiguities, $x_k x_j x_i$ for $k > j > i$, $x_j x_j x_i$ and $x_j x_i x_i$ for $j > i$ and $x_i x_i x_i$.

Clearly $I = \bigoplus_{n>0} \Lambda^n(V)$ is nilpotent, and $\Lambda(V)/I \cong K$, so $I = J(\Lambda(V))$ (or use that the exterior algebra is defined by an admissible ideal).

(iv) The Diamond Lemma again. If $a_i = 0$ then x_i generates a nilpotent ideal. For the converse, use the same argument as Maschke's Theorem.

The exterior algebra is useful for studying determinants and differential forms. For example $\theta \in \mathrm{End}_K(V)$ induces a graded algebra homomorphism $\Lambda(V) \to \Lambda(V)$ and the map on top exterior powers $\Lambda^n(V) \to \Lambda^n(V)$ is multiplication by $\det \theta$.

For V an $\mathbb{R}$-vector space with basis x_1, x_2 and quadratic form $q(\lambda_1 x_1 + \lambda_2 x_2) = -\lambda_1^2 - \lambda_2^2$, we have $C(V, q) \cong \mathbb{H}$ via $x_1 \leftrightarrow i$, $x_2 \leftrightarrow j$, $x_1 x_2 \leftrightarrow k$.

Remark. More generally, if S is an algebra and V is an S-S-bimodule. One defines the tensor algebra

$$T_S(V) = S \oplus V \oplus (V \otimes_S V) \oplus (V \otimes_S V \otimes_S V) \oplus \dots.$$

One can show that the path algebra KQ is a tensor algebra, where $S = \bigoplus_{i \in Q_0} K e_i$ and $V = \bigoplus_{a \in Q_1} K a$.

2.4 Power series

Let Q be a finite quiver. The formal power series path algebra $K\langle\langle Q \rangle\rangle$ is the algebra whose elements are formal sums

$$\sum_{p \text{ path}} a_p p$$

with $a_p \in K$, but with no requirement that only finitely many are non-zero. Multiplication makes sense because any path p can be obtained as a product qq' in only finitely many ways.

In the special case of a loop one gets the formal power series algebra $K[[x]]$ and the element $1 + x$ is invertible in $K[[x]]$ since it has inverse $1 - x + x^2 - x^3 + \dots$.

Properties. (1) An element of $K\langle\langle Q \rangle\rangle$ is invertible if and only if the coefficient a_{e_i} of each trivial path e_i is invertible in K.

If the condition holds one can multiply first by a linear combination of trivial paths to get it in the form $1 + x$ with x only involving paths of length ≥ 1. Then the expression $1 - x + x^2 - x^3 + \dots$ makes sense in $K\langle\langle Q \rangle\rangle$, and is an inverse.

(2) $J(K\langle\langle Q \rangle\rangle) = \{\sum_p a_p p : a_{e_i} \in J(K) \text{ for all trivial paths } e_i\}$.

(3) If K is a field, then f.d. $K\langle\langle Q\rangle\rangle$-modules correspond exactly to f.d. modules M for KQ which are *nilpotent*, meaning that $(KQ)_+^d M = 0$ for some d.

We consider restriction via the homomorphism $KQ \to K\langle\langle Q\rangle\rangle$. Clearly any nilpotent KQ-module is the restriction of a $K\langle\langle Q\rangle\rangle$-module. Conversely if M is a nilpotent KQ-module of dimension d, then by induction of d, $J^d M = 0$, where $J = J(K\langle\langle Q\rangle\rangle)$. Namely, if M is simple then $JM = 0$. Otherwise it has a submodule N of dimension e and $J^e N = 0$, $J^{d-e}(M/N) = 0$. So $J^d M = 0$.

Remark. Let $I = \{\sum_p a_p p : a_{e_i} = 0 \text{ for all trivial paths } e_i\}$. Then $I^n = \{\sum_p a_p p : a_p = 0 \text{ for all paths of length} < n\}$. The I-adic topology on $R = K\langle\langle Q\rangle\rangle$ has base of open sets the cosets $I^n + r$ $(n \geq 1, r \in R)$. To do more, one needs to take topology into account.

For example I am told that $K\langle\langle x, y\rangle\rangle/(xy - yx)$ is non-commutative. Instead one has $K\langle\langle x, y\rangle\rangle/\overline{(xy - yx)} \cong K[[x, y]]$.

2.5 Skew polynomial rings

If R is a K-algebra and M is an R-R-bimodule, a *(K-)derivation* $d : R \to M$ is a mapping of K-modules which satisfies $d(rr') = rd(r') + d(r)r'$ for all $r, r' \in R$.

Observe that for $d(1) = d(1) + d(1)$ so $d(1) = 0$. Also, for $\lambda \in K$, $d(\lambda 1) = \lambda d(1) = 0$ by linearity.

We write $\mathrm{Der}_K(R, M)$ for the set of derivations. It is naturally a K-module.

Examples. (i) For any $m \in M$ the map $r \mapsto rm - mr$ is a derivation, called an *inner derivation*.

(ii) The map $\frac{d}{dx} : K[x] \to K[x]$,

$$\frac{d}{dx}(\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n) = \lambda_1 + 2\lambda_2 x + \cdots + n\lambda_n x^{n-1}.$$

is a derivation. More generally $\partial/\partial x_i : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$.

Definition. If R is a K-algebra and $\sigma, \delta : R \to R$, we write $R[x; \sigma, \delta]$ for a K-algebra containing R as a subalgebra, which consists of all polynomials

$$r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n$$

with $r_i \in R$, with the natural addition and a multiplication satisfying

$$xr = \sigma(r)x + \delta(r)$$

for $r \in R$. If such a ring exists, the multiplication is uniquely determined. It is called a *skew polynomial ring* or *Ore extension of* R.

Theorem. $R[x; \sigma, \delta]$ exists if and only if σ is a K-algebra endomorphism of R and $\delta \in \mathrm{Der}_K(R, {}_\sigma R)$. [One says δ is a σ-derivation of R.]

Proof. If such an algebra exists, then clearly $\sigma, \delta \in \mathrm{End}_K(R)$ and

$$\begin{aligned}
\sigma(rr')x + \delta(rr') &= x(rr') \\
&= (xr)r' \\
&= (\sigma(r)x + \delta(r))r' \\
&= \sigma(r)(xr') + \delta(r)r' \\
&= \sigma(r)(\sigma(r')x + \delta(r')) + \delta(r)r'.
\end{aligned}$$

Thus $\sigma(rr') = \sigma(r)\sigma(r')$ and $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$. For the converse, identify R with the subalgebra of $E = \mathrm{End}_K(R^{\mathbb{N}})$, with $r \in R$ corresponding to left multiplication by r. Define $X \in E$ by

$$(Xs)_i = \sigma(s_{i-1}) + \delta(s_i)$$

for $s = (s_0, s_1, \dots) \in R^{\mathbb{N}}$, where $s_{-1} = 0$. Then

$$\begin{aligned}
(X(rs))_i &= \sigma(rs_{i-1}) + \delta(rs_i) \\
&= \sigma(r)\sigma(s_{i-1}) + \sigma(r)\delta(s_i) + \delta(r)s_i \\
&= \sigma(r)X(s)_i + \delta(r)s_i.
\end{aligned}$$

Thus $X(rs) = \sigma(r)X(s) + \delta(r)s$, so $Xr = \sigma(r)X + \delta(r)$. It follows that anyt element of the subalgebra S of E generated by X and the elements $r \in R$ can be written as a polynomial $f = \sum_{i=1}^{N} r_i X^i$.

The last step is to show that S is in bijection with the polynomials. This holds since if e^i denotes ith coordinate vector in $R^{\mathbb{N}}$, the infinite sequence which is 1 in the ith place and 0 elsewhere, then $X(e^i) = e^{i+1}$ so $f(e^0) = (r_0, r_1, \dots)$, so the coefficients of the polynomial f are uniquely determined by f as an element of S.

Special cases. If $\delta = 0$ the skew polynomial ring is isomorphic to $T_R(R_\sigma)$ and we denote it $R[x; \sigma]$. If $\sigma = 1$ denote it $R[x; \delta]$.

Properties. Let $S = R[x; \sigma, \delta]$.

(1) $x^n r = \sigma^n(r)x^n +$ lower degree terms. Proof by induction on n.

(2) If R is a domain (no zero-divisors) and σ is injective then the degree of a product of two polynomials is equal to the sum of their degrees. In particular S is a domain. Proof. $(r_0 + \cdots + r_n x^n)(s_0 + \cdots + s_m x^m) = r_n \sigma^n(s_m)x^{n+m} +$ lower degree terms.

(3) If R is a division ring then σ is automatically injective and S is a principal left ideal domain. Proof. Suppose I is a non-zero left ideal. It contains a non-zero polynomial $f(x)$ of least degree d, which we may suppose to be monic. If $g(x)$ is a polynomial with leading term rx^{d+n}, then $g(x) - rx^n f(x)$ has strictly smaller degree. An induction then shows that $I = Sf(x)$.

(4) If σ is an automorphism then $rx = x\sigma^{-1}(r) - \delta(\sigma^{-1}(r))$, so $S^{op} = R^{op}[x; \sigma^{-1}, -\delta\sigma^{-1}]$.

Hilbert's Basis Theorem. Assume σ is an automorphism. If R is left (respectively right) noetherian, then so is $R[x; \sigma, \delta]$.

Proof. By the observation above, it suffices to prove this for right noetherian. Let J be a right ideal in S which is not finitely generated, and take a polynomial f_1 of least degree in J. By induction, if we have found $f_1, \ldots, f_k \in J$, then since J is not finitely generated $J \setminus \sum_{i=1}^k f_i S \neq \emptyset$, and we take f_{k+1} of least possible degree. We obtain an infinite sequence of polynomials $f_1, f_2, \cdots \in J$. Let f_i have leading term $r_i x^{n_i}$. By construction $n_1 \leq n_2 \leq \ldots$. The chain

$$r_1 R \subseteq r_1 R + r_2 R \subseteq \ldots$$

must become stationary, so some $r_{k+1} = \sum_{i=1}^k r_i r'_i$ with $r'_i \in R$. Then

$$f_{k+1} - \sum_{i=1}^k f_i \sigma^{-n_i}(r'_i) x^{n_{k+1} - n_i} \in J \setminus \sum_{i=1}^k f_i S$$

and it has degree $< n_{k+1}$, contradicting the choice of f_{k+1}.

Let K be a field.

Example 1. The *first Weyl algebra* is $A_1 = K\langle x, y : yx - xy = 1\rangle$. It is a skew polynomial ring $K[x][y; d/dx]$. It has K-basis the monomials $x^i y^j$. It is isomorphic to the subalgebra of $\mathrm{End}_K(K[x])$ consisting of all differential operators of the form $\sum_{i=0}^n p_i(x) d^i/dx^i$. It is a noetherian domain.

Lemma. If K is a field of characteristic 0, then A_1 has no non-trivial ideals, that is, it is a simple ring.

Proof. The Weyl algebra has basis the elements $w = x^i y^j$. Observe that $yw - wy = ix^{i-1}y^j$ and $xw - wx = -jxy^{j-1}$.

Suppose I is a non-zero ideal. Choose $0 \neq c \in I$. Choosing an element $x^i y^j$ involved in c with non-zero coefficient λ with $i+j$ maximal. Then $i!j!\lambda \in I$. Thus $I = A_1$.

More generally, the *nth Weyl algebra* $A_n(K)$ is the K-algebra generated by $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relations

$$y_i x_j - x_j y_i = \delta_{ij}, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i$$

It has basis the monomials $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n}$. It is an iterated skew polynomial ring $K[x_1, \dots, x_n][y_1; \partial/\partial x_1][y_2; \partial/\partial x_2] \dots [y_n; \partial/\partial x_n]$.

Various rings of functions become modules for $A = A_n(K)$, for example polynomial functions $K[x_1, \dots, x_n]$, or the smooth functions $C^\infty(U)$ on an open subset of $\mathbb{R}^n$ (if $K = \mathbb{R}$) or the holomorphic functions $\mathcal{O}(U)$ on an open subset of $\mathbb{C}^n$ (if $K = \mathbb{C}$). Let F be one of these A-modules of functions. Given $P = (P_{ij}) \in M_{m\times n}(A)$ we consider the system of differential equations

$$P \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = 0$$

with $f_i \in F$. The set of solutions is identified with $\mathrm{Hom}_A(M, F)$ where M is the cokernel of the map $A^m \to A^n$ given by right multiplication by P.

Example 2. If q is invertible in K, the *coordinate ring of the quantum plane* is $\mathcal{O}_q(K^2) = K\langle x, y : yx = qxy\rangle$. It is a skew polynomial ring $K[x][y; \sigma]$, where $\sigma : K[x] \to K[x]$ is the automorphism with $\sigma(x) = qx$. It has K-basis the monomials $x^i y^j$. It is a noetherian domain.

Example 3. If R is a K-algebra, a 2 by 2 quantum matrix with entries in R is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in R$ satisfying the relations

$$ab = qba, \; ac = qca, \; bc = cb, \; bd = qdb, \; cd = qdc, \; ad - da = (q - q^{-1})bc.$$

Equivalently it is a homomorphism $\mathcal{O}_q(M_2(K)) \to R$, where the *coordinate ring of quantum 2 by 2 matrices* is $\mathcal{O}_q(M_2(K)) = K\langle a, b, c, d\rangle/I$ where I is generated by the relations. It has basis the monomials $a^i b^j c^k d^\ell$. It is an iterated skew polynomial ring, so a noetherian domain.

The quantum determinant is $\det_q = ad - qbc = da - q^{-1}cb$. This makes sense for a quantum matrix or for an element of $\mathcal{O}_q(M_2(K))$.

$\det_q$ commutes with a, b, c, d, so is a central element of $\mathcal{O}_q(M_2(K))$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = (det_q)I = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2.6 Localization

Let R be an algebra. A subset $S \subseteq R$ is *multiplicative* if $1 \in S$ and $ss' \in S$ for all $s, s' \in S$.

Lemma. (a) If S is a multiplicative subset in R, then there exists an algebra homomorphism $\theta : R \to R_S$ with the properties (i) $\theta(s)$ is invertible for all $s \in S$, and (ii) If $\theta' : R \to R'$ is an algebra homomorphism and $\theta'(s)$ is invertible for all $s \in S$, then there is a unique homomorphism $\phi : R_S \to R'$ with $\phi\theta = \theta'$.
(b) Given any other homomorphism $\theta' : R \to R'_S$, with properties (i),(ii), there is a unique isomorphism $\phi : R_S \to R'_S$ with $\phi\theta_S = \theta'$.
(c) R_S is generated as an algebra by the elements $\theta(r)$ and $\theta(s)^{-1}$ for $r \in R$ and $s \in S$.
(d) If $sr = 0$ with $s \in S$ and $r \in R$ then $\theta(r) = 0$.
(e) If M is an R-module, and multiplication by s acts invertibly on M for all $s \in S$, then the action of R on M extends uniquely to an action of R_S.

Proof. (a) Define $R_S = K\langle \{x_r : r \in R\} \cup \{i_s : s \in S\} \rangle / I$ where I is generated by the relations $x_1 = 1, x_r + x_{r'} = x_{r+r'}$, $x_r x_{r'} = x_{rr'}, \lambda x_r = x_{\lambda r}, x_s i_s = 1, i_s x_s = 1$ and $\theta(r) = x_r$. (b) Universal property. (c) Clear. (d) Clear. (e) Consider the map $R \to \mathrm{End}_K(M)$.

Definition. A multiplicative subset S in R satisfies the *left Ore condition* if for all $s \in S$ and $r \in R$ there exist $s' \in S$ and $r' \in R$ with $s'r = r's$, and it is *left reversible* if $rs = 0$ with $r \in R$ and $s \in S$ implies that there is $s' \in S$ with $s'r = 0$. Both conditions are trivial if R is commutative or more generally if $S \subseteq Z(R)$.

Construction. If S is a left reversible left Ore set in R and M is a left R-module, then on the set of pairs $(s, m) \in S \times M$ we consider the relation

$$(s, m) \sim (s', m') \Leftrightarrow \text{there are } u, u' \in R \text{ with } um = u'm' \text{ and } us = u's' \in S.$$

Lemma 1. This is an equivalence relation.

Proof. Exercise.

We write $s^{-1}m$ for the equivalence class of (s, m) and define $S^{-1}M$ to be the set of equivalence classes.

Lemma 2. Any finite set of elements of $S^{-1}M$ can be written with a common denominator.

Proof. It suffices to do two elements $s^{-1}m$ and $(s')^{-1}m'$. The Ore condition gives $t \in S$, $r \in R$ with $ts' = rs \in S$. Then $s^{-1}m = (rs)^{-1}rm$ and $(s')^{-1}m' = (ts')^{-1}tm'$.

Lemma 3. (a) $S^{-1}M$ becomes an R-module via

$$s^{-1}m + s^{-1}m' = s^{-1}(m + m'),$$
$$r(s^{-1}m) = (s')^{-1}(r'm) \quad \text{where } s'r = r's \text{ with } s' \in S \text{ and } r' \in R$$

(b) $s^{-1}m = 0 \Leftrightarrow$ there is $u \in R$ with $um = 0$ and $us \in S$. In particular $1^{-1}m = 0 \Leftrightarrow$ there is $u \in S$ with $um = 0$.
(c) Elements of S act invertibly on $S^{-1}M$, so $S^{-1}M$ becomes an R_S-module.
(d) An R-module map $\theta : M \to N$ induces an R_S-module map $S^{-1}M \to S^{-1}N$.
(e) If $L \xrightarrow{\theta} M \xrightarrow{\phi} N$ is exact, then so is $S^{-1}L \to S^{-1}M \to S^{-1}N$.
(f) There is a natural isomorphism $\bigoplus_{i \in I} S^{-1}M_i \cong S^{-1}(\bigoplus_{i \in I} M_i)$

Proof. (a) Straightforward.

(b) Now $s^{-1}m = 1^{-1}0 \Leftrightarrow$ there are $u, u' \in R$ with $um = u'0$ and $us = u'1 \in S$, gives the condition.

(c) If $t \in S$ then $t((st)^{-1}m) = s^{-1}m$, and if $t(s^{-1}m) = 0$, then there are s', t' with $s' \in S$, $s't = t's$ and $(s')^{-1}(t'm) = 0$. Then there is $u \in R$ with $ut'm = 0$ and $us' \in S$. Then $(ut')m = 0$ and $(ut')s = us't \in S$, so $s^{-1}m = 0$.

(d) Send $s^{-1}m$ to $s^{-1}\theta(m)$. This gives an R-module map $S^{-1}M \to S^{-1}N$, and since R_S is generated by the elements of R and the inverses of elements of S, it is an R_S-module map.

(e) If $s^{-1}m$ is sent to zero in $S^{-1}N$, then there is $u \in R$ with $u\phi(m) = 0$ and $us \in S$. Then $\phi(um) = 0$, so $um = \theta(\ell)$. Then $(us)^{-1}\ell \in S^{-1}L$ is sent to $s^{-1}m \in S^{-1}M$.

(f) holds since a finite number of fractions can be put over a common denominator.

Theorem. The following are equivalent.
(i) S is a left reversible left Ore set.
(ii) S is a left reversible left Ore set and $R_S \cong S^{-1}R$ considered as a ring with multiplication

$$(t^{-1}r)(s^{-1}u) = (s't)^{-1}r'u$$

where $s'r = r's$ with $s' \in S$ and $r' \in R$.
(iii) Every element of R_S can be written as $\theta(s)^{-1}\theta(r)$ for some $s \in S$ and $r \in R$, and $\theta(r) = 0$ if and only if $sr = 0$ for some $s \in S$.

Proof. (i) implies (ii). Define a mapping $f : S^{-1}R \to R_S$ sending $s^{-1}r$ to $\theta(s)^{-1}\theta(r)$. It is easy to see this is well-defined. By the Ore condition any composition $\theta(r)\theta(s)^{-1}$ can be rewritten as $\theta(s')^{-1}\theta(r')$. Combined with Lemma 0 (c) and Lemma 2, it follows that f is surjective. Also $S^{-1}R$ becomes an R_S-module, so there is a map $g : R_S \to S^{-1}R$ with $g\theta(r) = 1^{-1}r$. Then gf is the identity, so f is injective. Thus f is a bijection. The multiplication for R_S corresponds to that given.

(ii) implies (iii). Clear.

(iii) implies (i). Say $as = 0$. Then $\theta(a) = 0$. Thus $s'a = 0$ for some $s' \in S$, giving left reversibility. Given a, s, we have $\theta(a)\theta(s)^{-1} = \theta(s')^{-1}\theta_S(a')$ for some $s' \in S$ and $a' \in R$. Thus $\theta(s'a - a's) = 0$. Thus there is $t \in S$ with $t(s'a - a's) = 0$. Thus $(ts')a = (ta')s$, giving the Ore condition.

Remark. Similarly there is the notion of a right reversible right Ore set, for which R_S can be constructed as fractions of the form rs^{-1}.

Examples 1. (1) If σ is a K-algebra automorphism of R, then $\{1, x, x^2, \dots\}$ is a left and right reversible Ore set in $R[x;\sigma]$. The elements of $R[x;\sigma]_S$ are of the form

$$(r_0 + r_1x + \cdots + r_nx^n)x^{-m} = r_0x^{-m} + \cdots + r_nx^{n-m},$$

so Laurent polynomials.

(2) $\det_q$ is central in $\mathcal{O}_q(M_2(K))$, so we can invert $S = \{1, \det_q, \det_q^2, \dots\}$ giving $\mathcal{O}_q(\mathrm{GL}_2(K))$.

(3) From the section on local rings, $\{q \in \mathbb{Q} : q = a/b, b \text{ odd}\} = \mathbb{Z}_S$ where S is the set of odd integers.

Theorem (Special case of Goldie's Theorem). Let R be a domain which is left noetherian (or more generally has no left ideal isomorphic to $R^{(\mathbb{N})}$). Then $S = R \setminus \{0\}$ is a left reversible left Ore set, and $\theta : R \to R_S$ is an injective map to a division ring.

Proof. The left reversibility condition is trivial. If S fails the left Ore condition, then there are $a, b \neq 0$ with $Ra \cap Rb = 0$. Then $a, ab, ab^2, \dots$ are linearly independent, for if $\sum_i r_i ab^i = 0$, then by cancelling as many factors of b on the right as possible, we get

$$r_0 a + r_1 ab + \cdots + r_n ab^n = 0$$

with $r_0 \neq 0$. But then $0 \neq r_0 a \in Ra \cap Rb$. Thus $\bigoplus_i Rab^i \subseteq R$. Now R_S is a division ring for if $s^{-1}r \neq 0$ then $r \neq 0$ and $(s^{-1}r)^{-1} = r^{-1}s$.

Examples 2. (1) $\mathbb{Z}$ embeds in $\mathbb{Q}$, $K[x_1, \dots, x_n]$ embeds in $K(x_1, \dots, x_n)$, etc.

(2) $R = A_n(K)$ is a noetherian domain, so it embeds in a division ring $R_S = D_n(K)$.

(3) For $R = K\langle x, y\rangle$ the set $R\setminus\{0\}$ fails the left Ore condition since $Rx \cap Ry = 0$. There do exist embeddings of R in division rings, but they are more complicated.

2.7 Algebras from Lie theory and elsewhere

THIS SECTION IS ONLY BRIEFLY DISCUSSED IN LECTURES.

The aim is to give some examples of algebras, without details or proofs, showing the need for the Diamond Lemma, skew polynomial rings, etc.

For simplicity $K = \mathbb{C}$.

Many of these algebras come in families indexed by the simple f.d. Lie algebras, so first we need something about them.

A *Lie algebra* is a K-vector space $\mathfrak{g}$ with a K-bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$. which is skew symmetric $[x, x] = 0$ (so $[x, y] = -[y, x]$) and satisfies the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

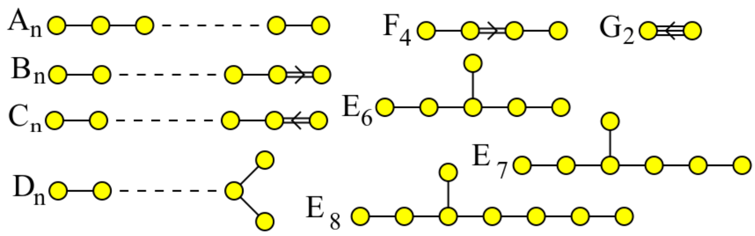
A Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{g}'$ is a K-linear map with $[\theta(x),\theta(y)] = \theta([x,y])$ for all $x,y \in \mathfrak{g}$. A Lie algebra is simple if any homomorphism from it is injective.

Examples. (i) Any associative algebra A becomes a Lie algebra with bracket $[a,b]=ab-ba$.

(ii) $\mathrm{gl}(V)=\mathrm{End}_K(V)$, $\mathrm{gl}_n = M_n(K)$ with bracket as above.

(iii) $\mathrm{sl}_n=\{a\in M_n(K) : \mathrm{tr}(a)=0\}$ and $\mathrm{so}_n=\{a\in M_n(K) : a+a^T=0\}$ are simple Lie algebras.

The simple f.d. Lie algebras are classified by Dynkin diagrams, or equivalently Cartan matrices. The Dynkin diagrams are A_m, B_m, C_m, D_m, E_6, E_7, E_8, F_4, G_2.



Source: Wikipedia

The corresponding Cartan matrices for A_m is the $m\times m$ matrix

$$C=\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

For example sl_n corresponds to A_{n-1}, so_{2n} corresponds to D_n.

Theorem (Chevalley-Serre relations). A simple Lie algebra $\mathfrak{g}$ with $m\times m$ Cartan matrix C is generated as a Lie algebra by elements h_i, e_i, f_i $(i =$

$1, \dots, m)$ with relations

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \begin{cases} h_i & (i = j) \\ 0 & (i \neq j), \end{cases} \quad [h_i, e_j] = C_{ji} e_j, \quad [h_i, f_j] = -C_{ji} f_j$$

$$\underbrace{[e_i, [\dots, [e_i}_{1-C_{ji}}, e_j] \dots]] = 0, \quad \underbrace{[f_i, [\dots, [f_i}_{1-C_{ji}}, f_j] \dots]] = 0 \quad (i \neq j)$$

Remark. The groups $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$, $\mathrm{SO}_n(K) = \{g \in \mathrm{SL}_n(K) : g^T = g^{-1}\}$ can be considered as Lie groups or algebraic groups. The tangent space at the identity becomes a Lie algebra. This gives $\mathrm{gl}_n(K)$, $\mathrm{sl}_n(K)$, $\mathrm{so}_n(K)$. A f.d. representation of a Lie group or algebraic group G is a homomorphism $G \to \mathrm{GL}(V)$ of such groups. It induces a Lie algebra map $\mathfrak{g} \to \mathrm{gl}(V)$, that is, a representation of $\mathfrak{g}$ as a Lie algebra.

Example 1. Universal enveloping algebras. Let $\mathfrak{g}$ be a Lie algebra. Its *universal enveloping algebra* is the associative algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g}).$$

If $\mathfrak{g}$ is a f.d. simple Lie algebra, $U(\mathfrak{g})$ is generated as an associative algebra by h_i, e_i, f_i subject to the Chevalley-Serre relations, now interpreted as relations in an associative algebra, where $[a, b] = ab - ba$.

For any algebra R we get a bijection $\mathrm{Hom}_{\text{algebra}}(U(\mathfrak{g}), R) \to \mathrm{Hom}_{\text{Lie algebra}}(\mathfrak{g}, R)$. Thus $U(\mathfrak{g})$-modules correspond exactly to representations of $\mathfrak{g}$.

Poincaré-Birkoff-Witt Theorem. If $\mathfrak{g}$ has basis $x_1, \dots, x_n$, then $U(\mathfrak{g})$ has basis the monomials $x_1^{i_1} \dots x_n^{i_n}$. Use the Diamond Lemma.

Example 2. Drinfeld-Jimbo quantum groups. They arose in the theory of integrable systems, in order to help find solutions of the 'quantum Yang-Baxter equation'.

Let $q \in K$ with $q \neq 0, 1, -1$. Let $\mathfrak{g}$ be a simple Lie algebra. We do the case sl_2. The quantum group $U_q(\mathrm{sl}_2)$ is the algebra generated by $\kappa, \kappa^{-1}, E, F$ subject to the relations

$$\kappa E \kappa^{-1} = q^2 E, \quad \kappa F \kappa^{-1} = q^{-2} F, \quad EF - FE = \frac{\kappa - \kappa^{-1}}{q - q^{-1}}$$

as well as $\kappa \kappa^{-1} = \kappa^{-1} \kappa = 1$.

It is an iterated skew polynomial ring $K[\kappa, \kappa^{-1}][E; \sigma][F; \sigma', \delta']$.

Example 3. Associated to a f.d. simple Lie algebra there is *Weyl group* W together with a set S of Coxeter generators. For sl_n it is the symmetric group $W = S_n$ equipped with the set $S = \{s_i = (i\ i+1) : i = 1, \dots, n-1\}$ which generates the group subject to the relations

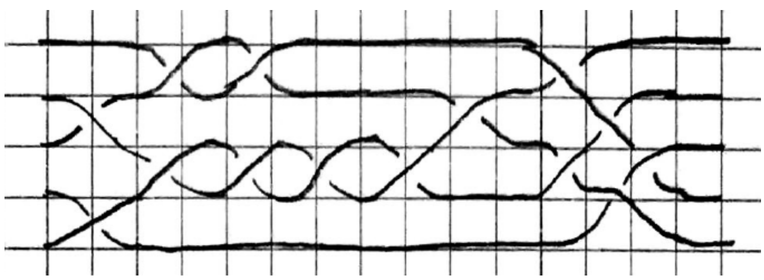
$$s_i^2 = 1, \quad s_i s_j = s_j s_i \ (|i-j| > 1), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

The group algebra KS_n is the algebra generated by $s_1, \dots, s_{n-1}$ subject to these relations.

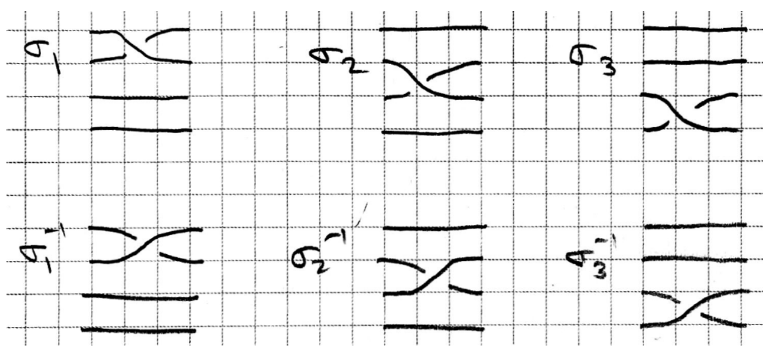
Example 4. The *(Artin) braid group* B_n is the group generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| > 1), \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

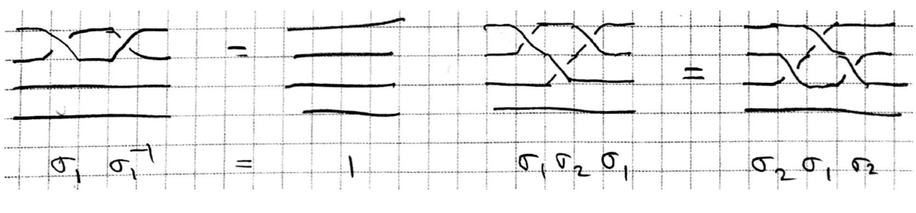
One can show that the elements of B_n can be identified with braids



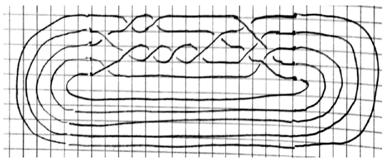
identifying two such braids if they are *isotopic*. The generators correspond to the braids



and the relations are as follows



By joining the ends of a braid, one gets a knot (or a link if it is not connected), for example



Moreover any knot arises from some braid (for some n).

The group algebra KB_n is the algebra generated by $\sigma_1, \ldots, \sigma_{n-1}$ and $\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$ subject to the relations

$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j|>1), \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

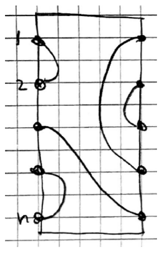
Example 5. Let $0 \neq q \in K$. The *(Iwahori-)Hecke algebra* is a deformation of the group algebra KW. For sl_n it is generated by $t_1, \ldots, t_{n-1}$ subject to the relations

$$(t_i - q)(t_i + 1) = 0, \quad t_i t_j = t_j t_i \ (|i-j|>1), \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}.$$

Now t_i is invertible, with inverse $(1/q)(1 + (1-q)t_i)$, so there is a surjective homomorphism from KB_n to the Hecke algebra sending σ_i to t_i.

One can show that the Hecke algebra has dimension $n!$, and for q not a root of 1 it is semisimple. The case $q = 1$ recovers KS_n.

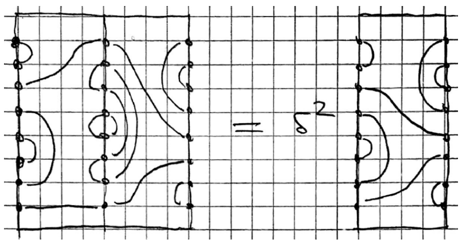
Example 6. The *Temperley-Lieb algebra* $TL_n(\delta)$ for $n \geq 1$ and $\delta \in K$ has basis the diagrams with two vertical rows of n dots, connected by n nonintersecting curves. For example



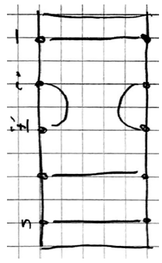
Two diagrams are considered equal if the same vertices are connected. The product is defined by

$$ab = \delta^r c$$

where c is obtained by concatenating a and b and deleting any loops, and r is the number of loops removed. For example

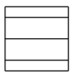
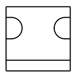
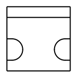
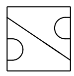
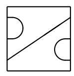


One can show that $TL_n(\delta)$ is generated by $u_1, \ldots, u_{n-1}$ where u_i is the diagram



subject to the relations $u_i^2 = \delta u_i$, $u_i u_{i\pm1} u_i = u_i$ and $u_i u_j = u_j u_i$ if $|i-j| > 1$.

The algebra $TL_n(\delta)$ is f.d., with dimension the nth Catalan number. For example for $n = 3$ the diagrams are

$1 =$ , $u_1 =$ , $u_2 =$ , $p =$ , $q =$ 

The Temperley-Lieb algebra was invented to help make computations in study Statistical Mechanics. It is now also important in Knot Theory.

The *Markov trace* is the linear map $\mathrm{tr} : TL_n(\delta) \to K$ sending a diagram to δ^{r-n} where r is the number of cycles in the diagram obtained by joining vertices at opposite ends.

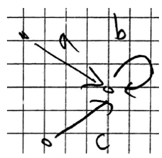
Given $0 \neq A \in K$, there is a homomorphism $\theta : KB_n \to TL_n(\delta)$ where $\delta = -A^2 - 1/A^2$, with $\theta(\sigma_i) = Au_i + (1/A)$, $\theta(\sigma_i^{-1}) = (1/A)u_i + A$.

Combined one gets a map $\mathrm{tr}\,\theta : KB_n \to K$. One can show that this only depends on the knot obtained by joining the ends of the braid, and it is a Laurent polynomial in A. It is essentially the Jones polynomial of the knot. See Lemma 2.18 in Aharonov, Jones and Landau, A polynomial quantum algorithm for approximating the Jones polynomial, Algorithmica 2009.

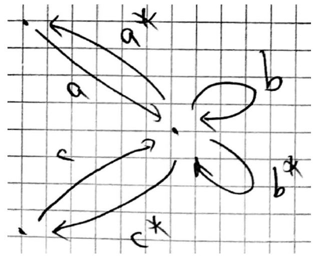
Example 7. The *preprojective algebra* for a finite quiver Q is

$$\Pi(Q) = K\overline{Q}/(\sum_{a \in Q}(aa^* - a^*a))$$

where $\overline{Q}$, the *double of* Q, is obtained by adjoining an inverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. For example if Q is the quiver



then $\overline{Q}$ is the quiver



Observe that if $c = \sum_{a \in Q}(aa^* - a^*a)$ then $e_i c e_j = 0$ if $i \neq j$, so $\Pi(Q)$ is given by the relations

$$c_i = e_i c e_i = \sum_{a \in Q, h(a)=i} aa^* - \sum_{a \in Q, t(a)=i} a^*a$$

for $i \in Q_0$.

Examples. For $Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ the relations are

$$a^*a = 0, \ aa^* = b^*b, \ bb^* = 0.$$

If Q is a loop x, then $\Pi(Q) = K\langle x, x^* \rangle/(xx^* - x^*x) \cong K[x, x^*]$.

Theorem. $\Pi(Q)$ is f.d. if and only if the underlying graph of Q is a Dynkin diagram (assuming Q is connected).

For the following, see A. Mellit, Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation, Algebra Discrete Math. 2004.

Theorem. Given $k, d_1, \dots, d_k > 0$, we have

$$K\langle x_1, \dots, x_k \rangle/(x_1 + \cdots + x_k, x_1^{d_1}, \dots, x_k^{d_k}) \cong e_0 \Pi(Q) e_0$$

where Q is star-shaped with central vertex 0 and arms $1, \ldots, k$, with vertices $(i, 1), (i, 2), \ldots, (i, d_i - 1)$ going outwards on arm i and arrows $a_{i,1}, \ldots, a_{i,d_i-1}$ pointing inwards.

Proof. Let the algebra on the left be A and the one on the right be $B = e_0\Pi(Q)e_0$. Now B is spanned by the paths in $\overline{Q}$ which start and end at vertex 0. If vertex (i, j) is the furthest out that a path reaches on arm i, then it must involve $a_{ij}a_{ij}^*$, and if $j > 1$, the relation

$$a_{ij}a_{ij}^* = a_{i,j-1}^* a_{i,j-1}$$

shows that this path is equal in B to a linear combination of paths which only reach $(i, j - 1)$. Repeating, we see that B is spanned by paths which only reach out to vertices $(i, 1)$. Thus we get a surjective map

$$K\langle x_1, \ldots x_k\rangle \to B$$

sending each x_i to $a_{i1}a_{i1}^*$. It descends to a surjective map $\theta : A \to B$ since it sends $x_1 + \cdots + x_k$ to 0 and $x_i^{d_i}$ is sent to

$$\begin{aligned}
(a_{i1}a_{i1}^*)^{d_i} &= a_{i1}(a_{i1}^* a_{i1})^{d_i-1}a_{i1} \\
&= a_{i1}(a_{i2}a_{i2}^*)^{d_i-1}a_{i1}^* \\
&= a_{i1}a_{i2}(a_{i2}^* a_{i2})^{d_i-2}a_{i2}^* a^* i1 \\
&= \cdots = \\
&= a_{i1}a_{i2} \ldots a_{i,d_i-1}(a_{i,d_i-1}^* a_{i,d_i-1})a_{i,d_i-1}^* \ldots a_{i1}^* = 0
\end{aligned}$$

since $a_{i,d_i-1}^* a_{i,d_i-1} = 0$.

To show that θ is an isomorphism it suffices to show that any A-module can be obtained by restriction from a B-module, for if $a \in \operatorname{Ker}\theta$ and $M = {}_\theta N$, then $aM = \theta(a)N = 0$. Thus if A can be obtained from a B-module by restriction, then $aA = 0$, so $a = 0$.

Thus take an A-module M. We construct a representation of $\overline{Q}$ by defining $V_0 = M$ and $V_{(i,j)} = x_i^j M$. with a_{ij} the inclusion map, and a_{ij}^* multiplication by x_i. This is easily seen to satisfy the preprojective relations, so it becomes a module for $\Pi(Q)$. Then $e_0V = M$ becomes a module for $e_0\Pi(Q)e_0 = B$. Clearly its restriction via θ is the original A-module M.

Example 8. If Q is a quiver, the *Leavitt path algebra* is $L(Q) = K\overline{Q}/I$ where I is generated by the relations

$$ab^* = \begin{cases} e_{h(a)} & (a = b) \\ 0 & (a \neq b) \end{cases} \quad \text{and} \quad \sum_{a \in t^{-1}(i)} a^* a = e_i \quad (1 < |t^{-1}(i)| < \infty).$$

(More precisely, in the literature, it is the opposite ring to this).

For example $L(1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} n) \cong M_n(K)$ sending e_i to e^{ii} a_i to $e^{i+1,i}$ and a_i^* to $e^{i,i+1}$.

Theorem (Leavitt 1962). If $R = L(Q)$ where Q has one vertex and $n+1$ loops, then $R^i \cong R^j$ as R-modules $\Leftrightarrow i \equiv j \pmod{n}$.

If the loops are $a_1, \dots, a_{n+1}$, the relations ensure that maps

$$r \mapsto (ra_1^*, \dots ra_{n+1}^*), \qquad (r_1, \dots, r_{n+1}) \mapsto \sum r_i a_i.$$

are inverse isomorphisms between R and R^{n+1}. The problem is to show that $R^i \not\cong R^j$ when $i \not\equiv j \pmod{n}$; in particular that $R \neq 0$.

3 Module categories

3.1 Categories

A *category* C consists of
(i) a collection $ob(C)$ of *objects*
(ii) For any $X, Y \in ob(C)$, a set $\mathrm{Hom}(X, Y)$ (or $C(X, Y)$, or sometimes $\mathrm{Hom}_C(X, Y)$) of *morphisms* $\theta : X \to Y$, and
(iii) For any X, Y, Z, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(\theta, \phi) \mapsto \theta\phi$.
satisfying
(a) Associativity: $(\theta\phi)\psi = \theta(\phi\psi)$ for $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z \xrightarrow{\theta} W$, and
(b) For each object X there is an identity morphism $id_X \in \mathrm{Hom}(X, X)$, with $id_Y\theta = \theta = \theta id_X$ for all $\theta : X \to Y$.

Examples.
(1) The categories of Sets, Groups, Abelian groups, Rings, Commutative rings, K-algebras, etc.
(2) The category R-Mod of R-modules for a ring R.
(3) Given a group G or a ring R, the category with one object $*$, $\mathrm{Hom}(*, *) = G$ or R and composition given by multiplication.
(4) Given a ring R, the category with objects $\mathbb{N}$, $\mathrm{Hom}(m, n) = M_{n\times m}(R)$ and composition given by matrix multiplication.
(5) Path category of a quiver Q. Objects Q_0 and $\mathrm{Hom}(i, j)$ = set of paths from i to j. The K-linear path category of Q. Objects Q_0 and $\mathrm{Hom}(i, j)$ = K-module with basis the paths from i to j.
(6) The category of correspondences. The objects are sets, $\mathrm{Hom}(X, Y)$ is the set of subsets $S \subseteq X \times Y$, the composition of morphisms $S \in \mathrm{Hom}(X, Y)$ and $T \in \mathrm{Hom}(Y, Z)$ is

$$TS = \{(x, z) \in X \times Z : (x, y) \in S \text{ and } (y, z) \in T \text{ for some } y \in Y\}.$$

The identity morphisms are the diagonal subsets $id_X = \{(x, x) : x \in X\}$.

Definition. An *isomorphism* is a morphism $\theta : X \to Y$ with an inverse, that is, there is some $\phi : Y \to X$, $\theta\phi = id_Y$, $\phi\theta = id_X$. If so, then ϕ is uniquely determined, and denoted θ^{-1}.

Definition. A *subcategory* of a category C is a category D with $ob(D) \subseteq ob(C)$ and $D(X, Y) \subseteq C(X, Y)$ for all $X, Y \in ob(D)$, such that composition in D is the same as that in C and $id_X^C \in D(X, X)$. It is a *full subcategory* if

$D(X, Y) = C(X, Y)$.

Examples.
(a) The category of finite groups in the category of all groups.
(b) The category R-mod of finitely generated R-modules inside R-Mod.
(c) The category whose objects are sets and with $\mathrm{Hom}(X, Y) =$ the injective functions $X \to Y$ is a subcategory of the category of sets.
(d) Identifying a mapping of sets $X \to Y$ with its graph, the category of sets becomes a subcategory of the category of correspondences.

Definition. If C is a category, the *opposite category* C^{op} is given by $ob(C^{op}) = ob(C)$, $C^{op}(X, Y) = C(Y, X)$, with composition of morphisms derived from that in C.

If C and D are categories, then $C \times D$ denotes the category with $ob(C \times D) = ob(C) \times ob(D)$ and $\mathrm{Hom}((X, U), (Y, V)) = C(X, Y) \times D(U, V)$.

Remark. Recall that there is no set of all sets. Thus $ob(C)$ may be a proper class. We say that C is *small* if $ob(C)$ is a set, and *skeletally small* if there is a set S of objects such that every object is isomorphic to one in S.

Example. The category of finite sets is not small, but it is skeletally small with $S = \{\emptyset, \{1\}, \{1, 2\}, \dots\}$. R-Mod is not small or skeletally small, but R-mod is skeletally small with $S = \{R^n/U : n \in \mathbb{N}, U \subseteq R^n\}$.

3.2 Monomorphisms and epimorphisms

Definition. A *monomorphism* in a category is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Z \to X$, if $\theta\alpha = \theta\beta$ then $\alpha = \beta$.

An *epimorphism* is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Y \to Z$, if $\alpha\theta = \beta\theta$ then $\alpha = \beta$.

In many concrete categories a monomorphism = injective map, epimorphism = surjective map.

Lemma. In R-Mod, monomorphism = injective map and epimorphism = surjective map.

Proof. We show epi = surjection. The other is similar.

Say $\theta : X \to Y$ is surjective and $\alpha\theta = \beta\theta$ then for all $y \in Y$ there is $x \in X$

with $\theta(x) = y$. Then $\alpha(y) = \alpha(\theta(x)) = \beta(\theta(x)) = \beta(y)$. Thus $\alpha = \beta$.

Say $\theta : X \to Y$ is an epimorphism. The natural map $Y \to Y/\operatorname{Im}\theta$ and the zero map have the same composition with θ, so they are equal. Thus $\operatorname{Im}\theta = Y$.

Example. In the category of rings, a localization map $\theta : R \to R_S$ is an epimorphism, but usually not a surjective map, for example $\mathbb{Z} \to \mathbb{Q}$.

Namely, if $\alpha, \beta : R_S \to T$ and $\alpha\theta = \beta\theta$, then $\alpha\theta$ is a map $R \to T$ which inverts the elements of S, so it can be factorized uniquely through θ. Thus $\alpha = \beta$.

Theorem. The following are equivalent for a ring homomorphism $\theta : R \to S$.
(i) θ is an epimorphism in the category of rings
(ii) $s \otimes 1 = 1 \otimes s$ in $S \otimes_R S$ for all $s \in S$.
(iii) The multiplication map $S \otimes_R S \to S$ is an isomorphism of S-S-bimodules.
(iv) Multiplication gives an isomorphism $S \otimes_R M \to M$ for any S-module M.
(v) For any S-modules M, N we have $\operatorname{Hom}_S(M, N) = \operatorname{Hom}_R(M, N)$.

Proof. (i)⇒(ii) Let M be the S-S-bimodule $S \otimes_R S$ and let $T = S \oplus M$, turned into a ring with the multiplication $(s, m)(s', m') = (ss', sm' + ms')$. The maps $\alpha, \beta : S \to T$ defined by $\alpha(s) = (s, 0)$ and $\beta(s) = (s, s \otimes 1 - 1 \otimes s)$ are ring homomorphisms with $\alpha\theta = \beta\theta$. Thus $\alpha = \beta$, so $s \otimes 1 = 1 \otimes s$.

(ii)⇒(iii) $s \mapsto s \otimes 1 = 1 \otimes s$ is an inverse. For example this map sends st to $st \otimes 1 = s(t \otimes 1) = s(1 \otimes t) = s \otimes t$.

(iii)⇒(iv) $S \otimes_R M \cong S \otimes_R S \otimes_S M \cong S \otimes_S M \cong M$.

(iv)⇒(v) $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(S, N)) \cong \operatorname{Hom}_S(S \otimes_R M, N) \cong \operatorname{Hom}_S(M, N)$.

(v)⇒(i). Say $f, g : S \to T$ have the same composition with θ. Then the identity map is an R-module map between the restrictions of ${}_fT$ and ${}_gT$. Thus it is an S-module map. Thus $f = g$.

3.3 Functors

If C, D are categories, a *(covariant) functor* $F : C \to D$ is an assignment of
(i) For each object $X \in ob(C)$, an object $F(X) \in ob(D)$, and
(ii) For each $X, Y \in ob(C)$ a map F or F_{XY} from $C(X, Y)$ to $D(F(X), F(Y))$, such that $F(\theta\phi) = F(\theta)F(\phi)$ and $F(id_X) = id_{F(X)}$.

A *contravariant functor* $F : C \to D$ is the same thing as a covariant functor $C^{op} \to D$. Thus it is an assignment of
(i) For each object $X \in ob(C)$, an object $F(X) \in ob(D)$, and
(ii) For each morphism $\theta : X \to Y$ in C a morphism $F(\theta) : F(Y) \to F(X)$ in D,
such that $F(\theta\phi) = F(\phi)F(\theta)$ and $F(id_X) = id_{F(X)}$.

Definitions. If for all $X, Y \in ob(C)$ the map $F : C(X, Y) \to D(F(X), F(Y))$ is injective, then F is *faithful*. It it is surjective then F is *full*. If every object in D is isomorphic to $F(X)$ for some object X in C, we say that F is *dense*.

The inclusion of a subcategory is a faithful functor. It is full if and only if the subcategory is full.

Definition. Let C be a category and let $\mathrm{Hom}(X, Y)$ denote the Hom sets for C. Fix an object $X \in ob(C)$. The *representable functor* $F = \mathrm{Hom}(X, -)$ is the functor $C \to Sets$ sending an object Y to $F(Y) = \mathrm{Hom}(X, Y)$, and sending a morphism $\theta \in \mathrm{Hom}(Y, Z)$ to the morphism $F(\theta) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ defined by $F(\theta)(\phi) = \theta\phi$.

Dually, fixing Y, we get a contravariant functor $\mathrm{Hom}(-, Y)$ from C to $Sets$.

Varying both X and Y, Hom defines a functor $\mathrm{Hom}(-, -) : C^{op} \times C \to Sets$.

Other examples of functors.
(1) There are many examples of "forgetful functors", which forget some structure. For example Groups to Sets, or K-Alg to K-Mod. They are faithful.
(2) Given a ring homomorphism $\theta : R \to S$, restriction defines a faithful functor S-Mod $\to R$-Mod. It is full if and only if θ is a ring-epimorphism.
(3) If M is an R-S-bimodule, then any homomorphism of S-modules $X \to X'$ gives a homomorphism $M \otimes_S X \to M \otimes_S X'$ of R-modules. Thus $M \otimes_S -$ becomes a functor from S-Mod to R-Mod.
(4) If M is an R-S-bimodule, it also gives functors $\mathrm{Hom}_R(M, -)$ from R-Mod to S-Mod and $\mathrm{Hom}_R(-, M)$ from R-Mod^{op} to S^{op}-Mod. Special case: if K is a field, then duality $V \rightsquigarrow V^* = \mathrm{Hom}_K(V, K)$ gives a contravariant functor

K-Mod to K-Mod.
(5) A functor from the path category of a quiver Q to K-Mod is exactly the same thing as a representation of Q.

3.4 Natural transformations

Definition. If F, G are functors $C \to D$, then a *natural transformation* $\Phi : F \to G$ consists of morphisms $\Phi_X \in D(F(X), G(X))$ for all $X \in ob(C)$ such that $G(\theta)\Phi_X = \Phi_Y F(\theta)$ for all $\theta \in C(X, Y)$.

We say that Φ is a *natural isomorphism* if all Φ_X are isomorphisms in D.

Examples. (1) If K is a field and V is a K-vector space, there is a natural map $V \to V^{**}$, $v \mapsto (\theta \mapsto \theta(v))$. This is a natural transformation $1_C \to (-)^{**}$ of functors from K-Mod to K-Mod. If we used K-mod, the category of finite dimensional K-vector spaces, it would be a natural isomorphism.

(2) Any morphism $\theta : X \to Y$ in a category C defines a natural transformation of representable functors $\Phi : \mathrm{Hom}(Y, -) \to \mathrm{Hom}(X, -)$, with $\Phi_Z : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Y)$, $f \mapsto f\theta$. Also it defines a natural transformation of contravariant representable functors $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$.

(3) A map of R-S-bimodules $M \to N$ gives natural transformations
(i) $\mathrm{Hom}_R(N, -) \to \mathrm{Hom}_R(M, -)$ of functors R-Mod $\to S$-Mod,
(ii) $\mathrm{Hom}_R(-, M) \to \mathrm{Hom}_R(-, N)$ of functors R-Mod$^{op} \to S^{op}$-Mod,
(iii) $M \otimes_S - \to N \otimes_S -$ of functors S-Mod $\to R$-Mod, etc.

(4) If M is an R-S-bimodule, X an R-module and Y an S-module, one gets a map

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y).$$

It is natural in X and Y, so defines natural transformations
$\mathrm{Hom}_R(X, M) \otimes_S - \to \mathrm{Hom}_R(X, M \otimes_S -)$ of functors S-Mod $\to K$-Mod, or
$\mathrm{Hom}_R(-, M) \otimes_S Y \to \mathrm{Hom}_R(-, M \otimes_S Y)$ (R-Mod$^{op} \to K$-Mod), or
$\mathrm{Hom}_R(-, M) \otimes_S - \to \mathrm{Hom}_R(-, M \otimes_S -)$ (R-Mod$^{op} \times S$-Mod $\to K$-Mod).

Yoneda's Lemma. For a functor $F : C \to Sets$ and $X \in ob(C)$ there is a 1-1 correspondence between natural transformations $\mathrm{Hom}(X, -) \to F$ and elements of $F(X)$.

Proof. A natural transformation $\Phi : \mathrm{Hom}(X, -) \to F$ gives a map $\Phi_X : \mathrm{Hom}(X, X) \to F(X)$, and hence an element $\Phi_X(id_X) \in F(X)$. Conversely, given $f \in F(X)$ and $Y \in ob(C)$ we get a map $\Phi_Y : \mathrm{Hom}(X,Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$. This defines a natural transformation Φ. These constructions are inverses.

Corollary 1. Any natural isomorphism of representable functors $\mathrm{Hom}(X,-) \cong \mathrm{Hom}(Y,-)$ is induced by an isomorphism $Y \cong X$.

Proof. If $\Phi : \mathrm{Hom}(X,-) \cong \mathrm{Hom}(Y,-)$ is a natural isomorphism, it corresponds to an element of $f \in \mathrm{Hom}(Y,X)$.

Now $\Phi_Y : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,Y)$, $\theta \mapsto \theta f$ is an isomorphism in the category of sets, so a bijection. Thus there is some $g \in \mathrm{Hom}(X,Y)$ with $gf = id_Y$.

Also $\Phi_X : \mathrm{Hom}(X,X) \to \mathrm{Hom}(Y,X)$, $\phi \mapsto \phi f$ is a bijection, and sends both fg and id_X to f, so $fg = id_X$.

Proposition/Definition. If C, D are categories, with C skeletally small, then there is a category $Fun(C,D)$ whose objects are the functors $C \to D$ and whose morphisms are the natural transformations. The isomorphisms are the natural isomorphisms.

Proof. It is straightforward to define the composition of natural transformations $F \to G$ and $G \to H$. The characterization of isomorphisms is also straightforward.

The only difficulty is to be sure that the Hom spaces are sets. Since C is a skeletally small category, every object is isomorphic to an object in a set S. Let $F, G : C \to D$ functors. A natural transformation $\Phi : F \to G$ is determined by the morphisms Φ_X for $X \in S$, for if $\theta : Y \to X$ is an isomorphism, then $\Phi_Y = G(\theta^{-1})\Phi_X F(\theta) \in D(F(Y), G(Y))$. The result follows.

Corollary 2. We get a full and faithful functor $C^{op} \to Fun(C, Sets)$, $X \mapsto \mathrm{Hom}(X,-)$.

3.5 Adjoint functors

Definition. Given functors $F : C \to D$ and $G : D \to C$, we say that (F,G) is an *adjoint pair*, or that F is *left adjoint* to G or G is *right adjoint* to F if there is a natural isomorphism $\Phi : \mathrm{Hom}(F(-),-) \cong \mathrm{Hom}(-,G(-))$ of functors $C^{op} \times D \to Sets$.

Thus one needs bijections

$$\Phi_{X,Y} : \mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, G(Y))$$

for all $X \in ob(C)$ and $Y \in ob(D)$, such that

$$\begin{array}{ccc} \mathrm{Hom}(F(X'), Y) & \xrightarrow{\Phi_{X',Y}} & \mathrm{Hom}(X', G(Y)) \\ {\scriptstyle \cdot F(\theta)}\downarrow & & {\scriptstyle \cdot\theta}\downarrow \\ \mathrm{Hom}(F(X), Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}(X, G(Y)) \end{array}$$

commutes for all $\theta : X \to X'$, and

$$\begin{array}{ccc} \mathrm{Hom}(F(X), Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}(X, G(Y)) \\ {\scriptstyle \phi\cdot}\downarrow & & {\scriptstyle G(\phi)\cdot}\downarrow \\ \mathrm{Hom}(F(X), Y') & \xrightarrow{\Phi_{X,Y'}} & \mathrm{Hom}(X, G(Y')) \end{array}$$

commutes for all $\phi : Y \to Y'$.

Examples. (1) (Hom tensor adjointness) If M is an R-S-bimodule then

$$\mathrm{Hom}_R(M \otimes_S X, Y) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(M, Y))$$

for X an S-module and Y an R-module, so $(M \otimes_S -, \mathrm{Hom}_S(M, -))$ is an adjoint pair between R-modules and S-modules.

(2) Free algebras and free modules. For K a commutative ring,

$$\mathrm{Hom}_{K\text{-alg}}(K\langle X\rangle, R) \cong \mathrm{Hom}_{Sets}(X, R),$$

for X from the category of sets and R from the category of K-algebras, so $(X \mapsto K\langle X\rangle, Forget)$ is an adjoint pair between K-algebras and sets. For R a ring

$$\mathrm{Hom}_R(R^{(X)}, M) \cong \mathrm{Hom}_{Sets}(X, M)$$

for X from the category of sets and M from the category of R-modules, so $(X \mapsto R^{(X)}, Forget)$ is an adjoint pair between R-modules and sets.

3.6 Equivalence of categories

Definition. A functor $F : C \to D$ is an *equivalence* if there is $G : D \to C$ such that $FG \cong 1_D$ and $GF \cong 1_C$.

Remark. F is an isomorphism if there is G giving equalities of functors $FG = 1_D$ and $GF = 1_C$. This is not such a useful concept.

Theorem. F is an equivalence if and only if it is full, faithful and dense.

Proof. Suppose there is a G and natural isomorphisms $\Phi : GF \to 1_C$ and $\Psi : FG \to 1_D$. For $\theta \in C(X, Y)$ we get $\theta\Phi_X = \Phi_Y G(F(\theta))$ so if $F(\theta) = F(\theta')$ then $\theta\Phi_X = \theta'\Phi_X$, so $\theta = \theta'$ since Φ_X is an isomorphism. Thus F is faithful. Similarly G is faithful. Suppose $\phi \in D(F(X), F(Y))$. Let $\theta = \Phi_Y G(\phi)\Phi_X^{-1} \in C(X, Y)$. Then $\theta\Phi_X = \Phi_Y G(F(\theta))$ gives $G(\phi) = G(F(\theta))$, so $\phi = F(\theta)$, so F is full. Also any $Y \in ob(D)$ is isomorphic to $F(G(Y))$, so F is dense.

On the other hand, if F satisfies the stated conditions, for each $Z \in ob(D)$ choose $G(Z) \in ob(C)$ and an isomorphism $\eta_Z : Z \to F(G(Z))$. We extend it to a functor $G : D \to C$ by defining $G(\theta)$ for $\theta \in D(Z, W)$ to be the unique morphism $\alpha \in C(G(Z), G(W))$ with $F(\alpha) = \eta_W \theta \eta_Z^{-1}$.

Examples. (i) If K is a field, there is an equivalence of categories from the category with objects $\mathbb{N}$ and $\mathrm{Hom}(m, n) = M_{n\times m}(K)$ to the category K-mod of finite dimensional K-vector spaces, sending n to K^n and a matrix A to the corresponding linear map.

(ii) The following three categories are equivalent for a quiver Q.
(1) KQ-Mod.
(2) The category of K-representations of Q
(3) The functor category from the path category of Q to K-Mod.

(iii) If R is a graded ring, the category of graded R-modules is equivalent to the category of modules for the associated catalgebra.

3.7 K-categories and catalgebras

Let K be a commutative ring.

Definition. A *K-category* is a category C with the additional structure that each of the sets $\mathrm{Hom}(X, Y)$ is a K-module, in such a way that composition

$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ is K-bilinear. In particular each set $\mathrm{Hom}(X,Y)$ contains a distinguished element, the zero element.

A functor $F : C \to D$ between K-categories is *K-linear*, if all of the maps $C(X,Y) \to D(F(X),F(Y))$ are K-module maps.

One uses the terminology *pre-additive category* and *additive functor* if these hold for some K (equivalently for $K = \mathbb{Z}$).

Examples.
(i) K-Mod is a K-category. If R is a K-algebra, then R-Mod is a K-category.
(ii) The K-linear path category of a quiver Q.
(iii) Warning. The category of K-algebras is NOT a K-category.

Remark/Definitions. If C is a K-category, then the representable functor $\mathrm{Hom}(X,-)$ can be considered as a K-linear functor $C \to K$-Mod. Yoneda's lemma still works for K-linear functors $C \to K$-Mod. If D is another K-category and C is skeletally small, there is a category $Fun_K(C,D)$ of K-linear functors $C \to D$.

Proposition. There is a 1-1 correspondence (an equivalence of categories) between
(i) small K-categories C, and
(ii) K-catalgebras R equipped with a complete family of orthogonal idempotents $(e_i)_{i\in I}$.
For C corresponding to R, there is an equivalence between $Fun_K(C, K\text{-Mod})$ and R-Mod.

Proof. Given a small category C, let I be the set of objects, let $R = \bigoplus_{i,j\in I} C(j,i)$. We think of the elements $r \in R$ as matrices whose entry $r_{ij} \in C(j,i)$. It becomes an algebra by matrix multiplication

$$(rr')_{ij} = \sum_k r_{ik} r'_{kj}$$

and let $e_i = 1_i \in C(i,i) \subseteq R$. Conversely, given R and $(e_i)_{i\in I}$, let C have set of objects I and $C(j,i) = e_i R e_j$, with composition induced by multiplication in R.

Given a K-linear functor $F : C \to K$-Mod we define $M = \bigoplus_{i\in I} F(i)$ and turn it into an R-module via

$$(rm)_i = \sum_{j\in I} F(r_{ij})(m_j) \in F(i)$$

Conversely given M, the unital condition guarantees that $M = \bigoplus_{i \in I} e_i M$. We define F by $F(i) = e_i M$ and for $r \in C(j,i) = e_i R e_j$, $F(r) : e_j M \to e_i M$ is multiplication by r.

Example. The K-linear path category of a quiver Q corresponds to the path algebra KQ with the trivial paths e_i.

3.8 Limits and colimits

Let C be a category.

Definition 1. Let J be a small category.

Given an object X in C, the constant functor $c_X : J \to C$ sends every object of J to X and every morphism to id_X.

There is a functor $c : J \to Fun(J,C)$ sending any object X to c_X, and any morphism $\theta : X \to Y$ to the natural transformation $\Phi : c_X \to c_Y$ with $\Phi_j = \theta$ for all objects j in J.

Given a functor $F : J \to C$, a *limit* for F is an object X in C and a natural isomorphism $\mathrm{Hom}_C(-,X) \cong \mathrm{Hom}_{Fun(J,C)}(c(-),F)$. If F has a limit, it is unique up to a unique isomorphism, so we can talk about 'the limit'.

Special cases 1. Taking taking J to the path category of a suitable quiver gives the following notions.

(a) A *product* of a family of objects X_i $(i \in I)$ is an object X equipped with morphisms $p_i : X \to X_i$ such that for any object X' and morphisms $q_i : X' \to X_i$ there is a unique morphism $\theta : X' \to X$ with $q_i = p_i\theta$, that is, the map $\mathrm{Hom}(X',X) \to \prod_i \mathrm{Hom}(X',X_i)$, $\theta \mapsto (p_i\theta)$ is a bijection.

Take the quiver with vertex set I and no arrows. A functor $F : J \to C$ is given by a collection of objects X_i $(i \in I)$. By Yoneda's Lemma, natural transformations $\mathrm{Hom}_C(-,X) \to \mathrm{Hom}_{Fun(J,C)}(c(-),F)$ correspond to collections of morphisms $X \to X_i$ for $i \in I$.

(b) An *equalizer* of a pair of morphisms $f, g : U \to W$ consists of an object X and a morphism $p : X \to U$ with $fp = gp$ and the universal property, that for all $p' : X' \to U$ with $fp' = gp'$ there is a unique $\theta : X' \to X$ with $p' = p\theta$. The quiver is $\circ \rightrightarrows \circ$.

In a K-category, the *kernel* of a morphism $f : U \to W$ is the equalizer of f and 0. Thus it is an object X and a morphism $p : X \to U$ with $fp = 0$, such that for any morphism $p' : X' \to U$ with $fp' = 0$ there is a unique morphism $\theta : X' \to X$ with $p' = p\theta$. Conversely the equalizer of f, g = kernel of $f - g$.

(c) A *pullback* of a pair of morphisms $f : U \to W$ and $g : V \to W$, consists of an object X and morphisms $p : X \to U$ and $q : X \to V$ giving a commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & W \end{array}$$

and which is univeral for such commutative squares, that is for any X', $p' : W' \to X$, $q' : W' \to Y$ with $fp' = gq'$ there is a unique $\theta : X' \to X$ with $p' = p\theta$ and $q' = q\theta$.

Examples 1. In the category Sets, K-algebras or R-modules, all limits exist.

The product is the usual one.

The kernel of $f : U \to W$ in R-Mod is $\operatorname{Ker} f \to U$.

The pullback is $\{(u, v) \in U \times V : f(u) = g(v)\}$, etc.

Lemma. In an equalizer, p is mono. In a pullback, if f is mono, so is q.

For the equalizer, suppose $\alpha, \beta : X' \to X$ and $p\alpha = p\beta = p'$. Since $fp' = gp'$, there is a unique $\theta : X' \to X$ with $p' = p\theta$. But both $\theta = \alpha$ and $\theta = \beta$ satisfy this, so $\alpha = \beta$.

For the pullback. Suppose $\alpha, \beta : X' \to X$ with $q\alpha = q\beta$. Then $gq\alpha = gq\beta$, so $fp\alpha = fp\beta$. Since f is mono, $p\alpha = p\beta$. Thus by the uniqueness part of the universal property for a pullback, $\alpha = \beta$.

Definition 2. Colimits in C are the same as limits in C^{op}.

Special cases 2.
(a) A *coproduct* of a family of objects X_i $(i \in I)$ is an object X equipped with morphisms $i_i : X_i \to X$ such that for any object X' and morphisms $j_i : X_i \to X'$ there is a unique morphism $\theta : X \to X'$ with $j_i = \theta i_i$, that is, the map $\operatorname{Hom}(X, X') \to \prod_i \operatorname{Hom}(X_i, X')$, $\theta \mapsto (\theta i_i)$ is a bijection.

(b) A *coequalizer* of a pair of morphisms $f, g : U \to W$ consists of an object

X and a morphism $p : W \to X$ with $pf = pg$ and the universal property.

In a K-category, the cokernel of a morphism $f : U \to W$ is the coequalizer of f and 0.Thus it is an object X and a morphism $p : W \to X$ with $pf = 0$, such that for any morphism $p' : W \to X'$ with $p'f = 0$ there is a unique morphism $\theta : X \to X'$ with $p' = \theta p$.

(c) A *pushout* of a pair of morphisms $f : W \to U$ and $g : W \to V$, consists of an object X and morphisms $p : U \to X$ and $q : V \to X$ giving a commutative square $pf = qg$, and which is univeral for such commutative squares, that is for any X', $p' : U \to X'$, $q' : V \to X'$ with $p'f = q'g$ there is a unique $\theta : X \to X'$ with $p' = \theta p$ and $q' = \theta q$.

Examples 2. (i) In the category Sets and R-Mod all colimits exist.

Coproducts: disjoint union $\bigcup X_i$, direct sum $\bigoplus X_i$.

The cokernel of a morphism $f : U \to W$ in R-Mod is $W \to W/\operatorname{Im} f$.

Pushouts: $U \cup V/\sim$ where $\sim$ is the equivalence relation generated by $f(w) \sim g(w)$ for $w \in W$, and $(U \oplus V)/\operatorname{Im}\theta$, where $\theta : W \to U \oplus V$ is $\theta(w) = (f(w), -g(w))$.

(ii) In the category of commutative K-algebras / all K-algebras we have finite coproducts. For the first, the coproduct of U and V is $U \otimes_K V$. For the second, the coproduct is $U *_K V$. For example if $U = K\langle X\rangle/I$ and $V = K\langle Y\rangle/J$, then $U *_K V = K\langle X \cup Y\rangle/(I \cup J)$.

3.9 Additive categories

Proposition 1. For objects X, $X_1, \dots, X_n$ $(n \geq 0)$ in a K-category the following are equivalent
(i) X is the product of $X_1, \dots, X_n$ for some morphisms $p_i : X \to X_i$
(ii) X is the coproduct of $X_1, \dots, X_n$ for some morphisms $i_i : X_i \to X$,
(iii) There are morphisms $p_i : X \to X_i$ and $i_i : X_i \to X$ with $p_i i_i = id_{X_i}$, $p_i i_j = 0$ for $i \neq j$ and $\sum_{i=1}^n i_i p_i = id_X$.
In this case we write $X = \bigoplus_{i=1}^n X_i$ and call it a *direct sum*.

Proof. (i)$\Rightarrow$(iii) For any object X' we have a bijection $\operatorname{Hom}(X', X) \to \prod_{i=1}^n \operatorname{Hom}(X', X_i), \phi \mapsto (p_i\phi)$. For $Z = X_j$, in the RHS we take the identity endomorphism of X_j and the zero maps in $\operatorname{Hom}(X_j, X_i)$ for $i \neq j$. This gives a map $i_j : X_j \to X$. These satisfy the conditions. For example if $\phi = \sum_{i=1}^n i_i p_i$ then $p_j \phi = \sum_{i=1}^n p_j i_i p_i = \sum \delta_{ij} 1_j p_i = p_j$, so $\phi = id_X$.

(iii)⇒(i) For any X' one has inverse bijections

$$\mathrm{Hom}(X', X) \underset{\phi \mapsto (p_i \phi)}{\overset{(\alpha_i) \mapsto \sum i_i \alpha_i}{\begin{smallmatrix}\longleftarrow \\ \longrightarrow\end{smallmatrix}}} \prod_{i=1}^n \mathrm{Hom}(X', X_i)$$

so the p_i turn X into a product.

(ii)⇔(iii) Dual.

Special case. In a K-category, X is a product or coproduct of no objects ⇔ $\mathrm{Hom}(X, X) = 0$. Such an object is called a *zero* object, denoted 0.

Definition. A category is *additive* if it is a K-category for some K (equivalently for $K = \mathbb{Z}$), if it has a zero object and every pair of objects has a direct sum (equivalently it has all finite direct sums).

Example. R-Mod, R-mod, the category of free R-modules.

Proposition 2.. If F is a K-linear functor between additive K-categories, then F preserves finite direct sums, so $F(0) = 0$ and $F(X \oplus Y) \cong F(X) \oplus F(Y)$.

Proof. Apply F to the morphisms in part (iii) of Proposition 1.

3.10 Abelian categories and exact functors

Definition. A category is *abelian* if
(i) it is additive,
(ii) every morphism has a kernel and cokernel,
(iii) every morphism factors as an epi followed by a mono, and
(iv) every mono is the kernel of its cokernel and every epi is the cokernel of its kernel.

Example. R-Mod. Also the category R-mod of finitely generated modules, for R a left noetherian ring. (The noetherian hypothesis ensures that the kernel of a morphism between f.g. modules is f.g.)

Definitions. A *subobject* of an object X in an abelian category is an equivalence class of monos to X, where $\alpha : U \to X$ is equivalent to $\alpha' : U' \to X$ ⇔ $\alpha = \alpha' \theta$ for some isomorphism $\phi : U \to U'$.

[There is possibly a set-theoretic problem here, which we ignore.]

Given a subobject $U \to X$ we write X/U for its cokernel.

Given a morphism $\theta : X \to Y$, the kernel of θ gives a subobject $\operatorname{Ker}\theta$ of X, and the kernel of the cokernel of θ gives a subobject $\operatorname{Im}\theta$ of Y.

We get analogues of the isomorphism theorems - details omitted.

Recall that a sequence of modules

$$\dots \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \dots$$

is exact at Y if $\operatorname{Im} f = \operatorname{Ker} g$. This makes sense for an abelian category too.

A short exact sequence is an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. Equivalently f the kernel of g, and g is the cokernel of f.

A short exact sequence is *split exact* if it satisfies the following equivalent conditions
(i) g has a *section*, a morphism $s : Y \to E$ with $gs = id_Y$.
(ii) f has a *retraction*, a morphism $r : E \to X$ with $rf = id_X$.
(iii) There are

$$X \underset{f}{\overset{r}{\leftrightarrows}} E \underset{s}{\overset{g}{\rightleftarrows}} Y$$

with $gs = id_Y$, $gf = 0$, $rs = 0$, $rf = id_X$ and $sg + fr = id_E$, so $E \cong X \oplus Y$.

Definition. If F is an additive functor between abelian categories, we say that F is *exact* (respectively *left exact*, respectively *right exact*) if given any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact (respectively $0 \to F(X) \to F(Y) \to F(Z)$ is exact, respectively $F(X) \to F(Y) \to F(Z) \to 0$ is exact). Similarly, if F is a contravariant functor, we want the sequence

$$0 \to F(Z) \to F(Y) \to F(X) \to 0$$

to be exact (respectively $0 \to F(Z) \to F(Y) \to F(X)$ exact, respectively $F(Z) \to F(Y) \to F(X) \to 0$ exact).

Notes. (i) Any additive functor between abelian categories sends split exact sequences to split exact sequences.

(ii) An exact functor sends any exact sequence (not just a short exact sequence) to an exact sequence. A left exact functor sends an exact sequence $0 \to X \to Y \to Z$ to an exact sequence $0 \to F(X) \to F(Y) \to F(Z)$. Similarly for right exact.

Proposition. For an abelian category, $\mathrm{Hom}(-,-)$ gives a left exact functor in each variable. That is, if M is an object and $0 \to X \to Y \to Z \to 0$ is exact, then so are

$$0 \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,Y) \to \mathrm{Hom}(M,Z)$$

and

$$0 \to \mathrm{Hom}(Z,M) \to \mathrm{Hom}(Y,M) \to \mathrm{Hom}(X,M).$$

Proof. The first sequence is exact at $\mathrm{Hom}(M,Y)$ since $X \to Y$ is a kernel for $Y \to Z$, and it is exact at $\mathrm{Hom}(M,X)$ since $X \to Y$ is a mono.

Lemma. For morphisms $f : U \to W$ and $g : V \to W$ in an abelian category, the pullback

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "q"'] & U \arrow[d, "f"] \\ V \arrow[r, "g"] & W \end{tikzcd}$$

exists. Moreover, if g is part of an exact sequence $0 \to Z \xrightarrow{\alpha} V \xrightarrow{g} W \to 0$ one gets a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "\beta"] \arrow[d, equal] & X \arrow[r, "p"] \arrow[d, "q"'] & U \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & Z \arrow[r, "\alpha"] & V \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd}$$

Dually, for morphisms $f : W \to U$ and $g : W \to V$, the pushout

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d, "q"] \\ U \arrow[r, "p"] & X \end{tikzcd}$$

exists. Moreover, if g is part of an exact sequence $0 \to W \xrightarrow{g} V \xrightarrow{\alpha} Z \to 0$ one gets a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{g} & V & \xrightarrow{\alpha} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle f}\big\downarrow & & {\scriptstyle q}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & U & \xrightarrow{p} & X & \xrightarrow{\beta} & Z & \longrightarrow & 0
\end{array}$$

Proof. We have morphisms

$$U \underset{i_U}{\overset{p_U}{\leftrightarrows}} U \oplus V \underset{i_V}{\overset{p_V}{\rightleftarrows}} V.$$

Let $\theta = fp_U + gp_V : U \oplus V \to W$. This morphism has a kernel, say $X \xrightarrow{k} U \oplus V$. Let $p = p_U k$ and $q = -p_V k$. Now $0 = \theta k = (fp_U + gp_V)k = fp - gq$, so $fp = gq$. Moreover given morphisms $p' : X' \to U$ and $q' : X' \to V$ with $fp' = gq'$, we get a morphism $\phi = i_U p' - i_V q' : X' \to U \oplus V$ with $\theta\phi = 0$. Thus ϕ factors uniquely as $k\phi'$ for some $\phi' : X' \to X$. This means that $p' = p_U\phi = p_U k\phi' = p\phi'$ and $q' = -p_V\phi = -p_V k\phi' = q\phi'$, which is the universal property for a pullback.

We have already shown that if f is mono, so is q.

Conversely, if q is mono, so is f. Namely, suppose $\alpha : Z \to U$ is a morphism with $f\alpha = 0$. Consider the zero map $Z \to V$. By the pullback property there is a morphism $\gamma : Z \to X$ with $p\gamma = \alpha$ and $q\gamma = 0$. Since q is mono, $\gamma = 0$. Thus $\alpha = 0$.

Dually there are pushouts.

Now suppose that $g : V \to W$ belongs to an exact sequence. The morphism $\alpha : Z \to V$ together with the zero morphism $Z \to U$ give a morphism $\beta : Z \to X$. We need to show $0 \to Z \to X \to U \to 0$ is exact.

Since g is an epi, so is θ. Thus the sequence

$$0 \to X \xrightarrow{k} U \oplus V \xrightarrow{\theta} W \to 0$$

is exact. Thus θ is the cokernel of k. Thus W is the pushout of p and q. Thus by properties of pushout, dual to pullbacks, since g is epi, so is p.

It remains to see that β is a kernel for p. This is straightforward.

3.11 Projective modules

Proposition/Definition. An object P in an abelian category is *projective* if it satisfies the following equivalent conditions.
(i) $\mathrm{Hom}(P, -)$ is an exact functor.
(ii) Any short exact sequence $0 \to X \to Y \to P \to 0$ is split.
(iii) Given an epimorphism $\theta : Y \twoheadrightarrow Z$, any morphism $P \to Z$ factors through θ.

Proof. (i)$\Rightarrow$(ii) $\mathrm{Hom}(P, Y) \to \mathrm{Hom}(P, P)$ is onto. A lift of id_P is a section.

(ii)$\Rightarrow$(iii) Take the pullback along the map $P \to Z$. The resulting exact sequence has P as third term, so is split. This gives a map from P to the pullback. Composing with the map to Y gives the map $P \to Y$.

(iii)$\Rightarrow$(i) Clear.

Lemma 1. Given sequences $0 \to X_i \to Y_i \to Z_i \to 0$ ($i \in I$) of R-modules, the following are equivalent.
(i) The sequences are exact for all $i \in I$.
(ii) $0 \to \prod_i X_i \to \prod_i Y_i \to \prod_i Z_i \to 0$ is exact.
(iii) $0 \to \bigoplus_i X_i \to \bigoplus_i Y_i \to \bigoplus_i Z_i \to 0$ is exact.

Proof. Straightforward.

Proposition. A direct sum of modules $\bigoplus_i M_i$ is projective $\Leftrightarrow$ all M_i are projective.

Proof. $\mathrm{Hom}(\bigoplus_i M_i, -) = \prod_i \mathrm{Hom}(M_i, -)$, so $\bigoplus_i M_i$ is projective
$\Leftrightarrow$ $0 \to \mathrm{Hom}(\bigoplus_i M_i, X) \to \mathrm{Hom}(\bigoplus_i M_i, Y) \to \mathrm{Hom}(\bigoplus_i M_i, Z) \to 0$ exact for all $0 \to X \to Y \to Z \to 0$
$\Leftrightarrow$ $0 \to \prod_i \mathrm{Hom}(M_i, X) \to \prod_i \mathrm{Hom}(M_i, Y) \to \prod_i \mathrm{Hom}(M_i, Z) \to 0$ exact
$\Leftrightarrow$ all $0 \to \mathrm{Hom}(M_i, X) \to \mathrm{Hom}(M_i, Y) \to \mathrm{Hom}(M_i, Z) \to 0$ are exact
$\Leftrightarrow$ all M_i are projective.

Theorem. Any free module is projective, and any module is a quotient of a free module. A module is projective if and only if it is a direct summand of a free module.

Proof. $Hom_R(R, X) \cong X$, so R is a projective module, hence so is any direct sum of copies of R. If $F \to P$ is onto with F free and P projective, then P is isomorphic to a summand of F.

Examples.
(i) If R is a semisimple f.d. algebra, then every submodule is a direct summand, so every short exact sequence is split, so every module is projective.
(ii) For a principal ideal domain, any finitely generated projective module is free. This follows from the usual classification of f.g. modules for a pid.
(iii) If $e \in R$ is an idempotent, then $R = Re \oplus R(1-e)$, so Re is a direct summand of ${}_R R$, so it is projective. Conversely any direct summand I of ${}_R$ is of the form Re for some idempotent e, for the projection onto I is an idempotent $e \in \mathrm{End}_R(R) \cong R^{op}$, so gives an idempotent $e \in R$ with $I = Re$.

Notation. We write $R-proj$ for the category of finitely generated projective left R-modules. Note that an R-module is finitely generated projective if and only if it is isomorphic to a direct summand of a free module R^n for some n.

Lemma 2. The functor $\mathrm{Hom}_R(-, R)$ defines an antiequivalence between $R-proj$ and $R^{op}-proj$.

Proof. There is a natural transformation

$$X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, R), R), \quad x \mapsto (\theta \mapsto \theta(x)).$$

It is an isomorphism for $X = R$, so for finite direct sums of copies of R, so for direct summands of such modules.

Lemma 3. If M is an R-S-bimodule, then there is a natural transformation

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y), \quad \theta \otimes y \mapsto (x \mapsto \theta(x) \otimes y)$$

for X an R-module and Y an S-module. It is an isomorphism if X is finitely generated projective. Moreover, if id_X is in the image of the natural map $\mathrm{Hom}_R(X, R) \otimes_R X \to \mathrm{End}_R(X)$, then X is finitely generated projective.

Proof. For the first part, reduce to the case of $X = R$. Say id_X comes from $\sum_i \theta_i \otimes x_i$, then the composition of the maps

$$X \xrightarrow{(\theta_i)} R^n \xrightarrow{(x_i)} X$$

is the identity.

3.12 Injective modules

Proposition/Definition. An object I in an abelian category is *injective* if it satisfies the following equivalent conditions.

(i) $\mathrm{Hom}(-, I)$ is an exact functor.
(ii) Any short exact sequence $0 \to I \to Y \to Z \to 0$ is split.
(iii) Given an injective map $\theta : X \hookrightarrow Y$, any map $X \to I$ factors through θ.

Proof. This is the opposite category version of the result for projectives.

Definition 1. An inclusion of R-modules $M \subseteq N$ is an *essential extension* of M if every non-zero submodule S of N has $S \cap M \neq 0$.

Theorem 1. For an R-module I, following conditions are equivalent.
(a) I is injective.
(b) (Baer's criterion) Every homomorphism $f : J \to I$ from a left ideal J of R can be extended to a homomorphism $R \to I$.
(c) I has no non-trivial essential extensions

Proof. (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c) Let $I \subseteq L$ be a non-trivial essential extension and fix $\ell \in L \setminus I$. We consider the pullback

$$\begin{tikzcd} J \arrow[r] \arrow[d] & R \arrow[d] \\ I \arrow[r] & L \end{tikzcd}$$

where $R \to L$ is the map $r \mapsto r\ell$. Then $J \to R$ is injective, so J is identified with a left ideal in R. By (b), the map $J \to I$ lifts to a map $R \to I$, say sending 1 to i. Then if $r(\ell - i) \in I$, then $r\ell \in I$, so $r \in J$, so $r\ell = ri$, so $r(\ell - i) = 0$. Thus $I \cap R(\ell - i) = 0$ and $R(\ell - i) \neq 0$, contradicting that $I \subseteq L$ is an essential extension.

(c)$\Rightarrow$(a). Given $I \subseteq Y$, we need to show that I is a summand of Y. By Zorn's Lemma, the set of submodules in Y with zero intersection with I has a maximal element C. If $I + C = Y$, then C is a complement. Otherwise, $I \cong (I + C)/C \subseteq Y/C$ is a non-trivial extension. By (c) it cannot be an essential extension, so there is a non-zero submodule U/C with zero intersection with $(I + C)/C$. Then $U \cap (I + C) = C$, so $U \cap I \subseteq C \cap I = 0$. This contradicts the maximality of C.

Proposition. A direct product of modules $\prod_i M_i$ is injective $\Leftrightarrow$ all M_i are injective

Proof. Use that $\mathrm{Hom}(-, \prod_i M_i) = \prod_i \mathrm{Hom}(-, M_i)$.

Definition 2. If K is an integral domain, then a K-module M is *divisible* if and only if for all $m \in M$ and $0 \neq a \in K$ there is $m' \in M$ with $m = am'$.

Observe that if M is divisible, so is any quotient M/N.

Theorem 2. If K is an integral domain, then any injective module is divisible. If K is a principal ideal domain, the converse holds.

Proof. Divisibility says that any map $Ka \to M$ lifts to a map $K \to M$. If K is a pid these are all ideals in K.

Now suppose that K is a field or a principal ideal domain. We define $(-)^* = \mathrm{Hom}_K(-, E)$, where

$$E = \begin{cases} K & \text{(if } K \text{ is a field)} \\ F/K & \text{(if } K \text{ is a pid with fraction field } F \neq K) \end{cases}$$

Then E is divisible, so an injective K-module, so $(-)^*$ is an exact functor. It gives a functor from R-modules on one side to R-modules on the other side.

Lemma. If M is a K-module, the map $M \to M^{**}$, $m \mapsto (\theta \mapsto \theta(m))$ is injective. (It is an isomorphism if K is a field and M is a finite-dimensional K-vector space).

Proof. Given $0 \neq m \in M$ it suffices to find a K-module map $f : Km \to E$ with $f(m) \neq 0$, for then since E is injective, f lifts to a map $\theta : M \to E$. If K is a field there is an isomorphism $Km \to E$. If K is a principal ideal domain, choose a maximal ideal Ka containing $ann(m) = \{x \in K : xm = 0\}$. Then there is a map $Km \to E$ sending xm to $K + x/a$.

If K is a field, and M is of dimension d, then so is M^*, and so also M^{**} so the map $M \to M^{**}$ must be an isomorphism.

Theorem 3. Any R-module embeds in a product of copies of R^*, and such a product is an injective R-module. A module is injective if and only if it is isomorphic to a summand of such a product.

Proof. We have R^* injective since $\mathrm{Hom}_R(-, R^*) \cong (-)^*$ is exact. Thus any product of copies is injective. Now choose a free right R-module and a surjection $R^{(X)} \to M^*$. Then M embeds in M^{**} and this embeds in $(R^{(X)})^* \cong (R^*)^X$. The last part is clear.

Corollary. Any module over any ring embeds in an injective module.

Proof. Apply the last result with $K = \mathbb{Z}$.

3.13 Flat modules

If M is an S-R-bimodule, then by a lemma in the section on tensor products, $M \otimes_R -$ defines a right exact functor from R-Mod to S-Mod which commutes with direct sums,

$$M \otimes_R \left(\bigoplus_{i \in I} X_i \right) \cong \bigoplus_{i \in I} (M \otimes_R X_i) \,.$$

Eilenberg-Watts Theorem. Any right exact functor from R-Mod to S-Mod which commutes with direct sums is naturally isomorphic to a tensor product functor for some bimodule.

Proof. Suppose that F is a right exact functor from R-Mod to S-Mod. Then $F(R)$ is an S-module, and it becomes an S-R-bimodule via the map

$$R \xrightarrow{\cong} \mathrm{End}_R(R)^{op} \xrightarrow{F} \mathrm{End}_S(F(R))^{op}.$$

Now for any R-module X there is a R-module map

$$X \xrightarrow{\cong} \mathrm{Hom}_R(R, X) \xrightarrow{F} \mathrm{Hom}_S(F(R), F(X)).$$

By hom-tensor adjointness this gives an S-module map $F(R) \otimes_R X \to F(X)$. This is natural in X, so it Φ_X for some natural transformation $\Phi : F(R) \otimes_R - \to F$. Clearly Φ_R is an isomorphism. Then for any free module $R^{(I)}$ we have $F(R^{(I)}) = F(R)^{(I)} \cong F(R) \otimes R^{(I)}$, so $\Phi_{R^{(I)}}$ is an isomorphism. Then for any module X there is a presentation $R^{(I)} \to R^{(J)} \to X \to 0$ and the first two vertical maps in the diagram

$$\begin{tikzcd}
F(R) \otimes R^{(I)} \arrow[r] \arrow[d, "\Phi_{R^{(I)}}"'] & F(R) \otimes R^{(J)} \arrow[r] \arrow[d, "\Phi_{R^{(J)}}"'] & F(R) \otimes X \arrow[r] \arrow[d, "\Phi_X"'] & 0 \\
F(R^{(I)}) \arrow[r] & F(R^{(J)}) \arrow[r] & F(X) \arrow[r] & 0
\end{tikzcd}$$

are isomorphisms. Also the rows are exact. Hence the third vertical map is an isomorphism. Thus Φ is a natural isomorphism.

Definition 1. A right R-module is *flat* if $M \otimes_R -$ is an exact functor (from R-Mod to K-Mod).

Properties.
(i) A direct sum of modules is flat if and only if each summand is flat, since

$M \otimes_R (\bigoplus_i X_i) \cong \bigoplus_i M \otimes_R X_i$.
(ii) Any projective module is flat, for $R \otimes_R X \cong X$, so R is flat. Now use the previous result.

Proposition 1. If K is a field or a pid, then an R-module M is flat if and only if M^* is injective.

Proof. We have $\mathrm{Hom}_R(X, M^*) \cong (M \otimes_R X)^*$. If M is flat, then the right hand functor is exact, so M^* is injective. Conversely, if M^* is injective then the right hand functor is exact. Suppose M is not flat. Given an exact sequence

$$0 \to X \to Y \to Z \to 0$$

we get

$$0 \to L \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0.$$

Then get

$$(M \otimes_R Y)^* \to (M \otimes_R X)^* \to L^* \to 0$$

Thus $L^* = 0$. But L embeds in L^{**}, so $L = 0$.

Proposition 2. A module M_R is flat if and only if the multiplication map $M \otimes_R I \to M$ is injective for every left ideal I in R.

Proof. If flat, the map is injective. For the converse we can work over $K = \mathbb{Z}$. If the map is injective, then the map $M^* \to (M \otimes_R I)^*$ is surjective. We can write this as $\mathrm{Hom}_R(R, M^*) \to \mathrm{Hom}_R(I, M^*)$. By Baer's criterion M^* is injective. Thus M is flat.

Example. A $\mathbb{Z}$-module is flat if and only if it is torsion-free. If $I = \mathbb{Z}n$ then $M \otimes I \to M$ is injective if and only if multiplication of M by n is injective. For example $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Proposition 3. If S is a left reversible left Ore set in R then the assignment $M \rightsquigarrow S^{-1}M$ defines an exact functor which is naturally isomorphic to the tensor product functor $M \rightsquigarrow R_S \otimes_R M$, so R_S is a flat as a right R-module.

Proof. Combine Eilenberg-Watts with results from section 2.6.

Definition 2. A module M is *finitely presented* if it is a quotient of a finitely generated free module by a finitely generated submodule. Equivalently if there is an exact sequence $R^m \to R^n \to M \to 0$.

Any f.g. projective module is finitely presented. If R is left noetherian, any finitely generated left R-module is finitely presented.

Lemma. If M is an R-S-bimodule, the natural transformation

$$\mathrm{Hom}_R(X, M) \otimes_S Y \to \mathrm{Hom}_R(X, M \otimes_S Y)$$

is an isomorphism if X is finitely presented and Y is flat.

Proof. It is clear for $X = R$. Then it follows for $X = R^n$. In general there is an exact sequence $R^m \to R^n \to X \to 0$, and in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(X, M) \otimes_S Y & \longrightarrow & \mathrm{Hom}_R(R^n, M) \otimes_S Y & \longrightarrow & \mathrm{Hom}_R(R^m, M) \otimes_S Y \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_R(X, M \otimes_S Y) & \longrightarrow & \mathrm{Hom}_R(R^n, M \otimes_S Y) & \longrightarrow & \mathrm{Hom}_R(R^m, M \otimes_S Y)
\end{array}$$

the rows are exact and the right two vertical maps are isomorphisms, hence so is the first.

Proposition 4. A finitely presented flat module is projective.

Proof. The natural map $\mathrm{Hom}_R(X, R) \otimes_R X \to \mathrm{End}_R(X)$ is an isomorphism by the last lemma. Thus id_X is the image of some element $\sum_{i=1}^n f_i \otimes x_i$. Then the f_i and x_i define maps $f : R^n \to X$ and $g : X \to R^n$ with $fg = id_X$, so X is a direct summand of R^n.

3.14 Envelopes and covers

Suppose $\mathcal{C}$ is a full subcategory of R-Mod, closed under finite direct sums and direct summands.

Definition. If M is an R-module, a $\mathcal{C}$-preenvelope is a homomorphism $\theta : M \to C$ with C in $\mathcal{C}$, such that any $\theta' : M \to C'$ with C' in $\mathcal{C}$ factors as $\theta' = \phi\theta$ for some $\phi : C \to C'$. It is a $\mathcal{C}$-envelope if in addition, for any $\phi \in \mathrm{End}_R(C)$, if $\phi\theta = \theta$, then ϕ is an automorphism.

If a $\mathcal{C}$-envelope exists, it is unique up to a (non-unique) isomorphism.

Dually, if M is an R-module, a $\mathcal{C}$-precover is a homomorphism $\theta : C \to M$ with C in $\mathcal{C}$, such that any $C' \to M$ factors through $C \to M$. It is a $\mathcal{C}$-cover if in addition, for any $\phi \in \mathrm{End}_R(C)$, if $\theta\phi = \theta$, then ϕ is an automorphism.

If a $\mathcal{C}$-cover exists, it is unique up to a (non-unique) isomorphism.

Theorem 1. Every module M has an injective envelope, $M \hookrightarrow E(M)$. Moreover $\theta : M \to I$ is an injective envelope if and only if θ is a monomorphism, I is injective and $\operatorname{Im}\theta \subseteq I$ is an essential extension.

Proof. Any module M embeds in an injective module E. Zorn's Lemma implies that the set of submodules of E which are essential extensions of M has a maximal element I.

Suppose that $I \subset J$ is a non-trivial essential extension. Then $M \subset J$ is an essential extension. Since E is injective the inclusion $I \to E$ can be extended to a map $g : J \to E$. Clearly $M \cap \operatorname{Ker} g = 0$, so since M is essential in J it follows that $\operatorname{Ker} g = 0$. Thus we can identify J with $g(J)$. But then M is essential in J, contradicting the maximality of I.

Thus I has no non-trivial essential extensions, so I is injective.

Thus the inclusion $\theta : M \to I$ satisfies the stated conditions. We show it is an injective envelope. Clearly it is a preenvelope.

Say $\phi\theta = \theta$ for some $\phi : I \to I$. Then $M \cap \operatorname{Ker}\phi = 0$, so $\operatorname{Ker}\phi = 0$. Then $\phi : I \to I$ is a monomorphism, so $I = \operatorname{Im}\phi \oplus C$ for some complement C. But then $M \cap C = 0$, so $C = 0$. Thus ϕ is an automorphism.

Corollary. Suppose R is a f.d. algebra over a field. If M is a f.d. module, so is $E(M)$. If I is an indecomposable injective module then it has a unique simple submodule S and $I \cong E(S)$. This gives a 1:1 correspondence between indecomposable injective modules and simple modules.

Proof. If M is f.d., then M^* is a f.d. R^{op}-module, so f.g., so is there is a surjection $(R^{op})^n \to M^*$, so $M \cong M^{**} \hookrightarrow (R^*)^n$, so M embeds in a f.d. injective, so $E(M)$ is f.d.. The rest is straightforward.

Theorem 2. Suppose R is a f.d. algebra over a field. Every module M has a projective cover $P(M) \twoheadrightarrow M$. If M is f.d., so is $P(M)$. If P is an indecomposable projective module, it has a unique simple quotient S, and $P \cong P(S)$. This gives a 1:1 correspondence between indecomposable projective modules and simple modules.

Sketch. Let $J = J(R)$. If M is any R-module then M/JM is an R/J-module, so semisimple.

Any endomorphism $\phi : P \to P$ induces an endomorphism $\overline{\phi} : P/JP \to P/JP$. If P is a projective module, the ring homomorphism $\operatorname{End}_R(P) \to$

$\operatorname{End}_R(P/JP)$ sending $\phi : P \to P$ to $\overline{\phi}$ is surjective. Moreover the kernel is $L = \operatorname{Hom}_R(P, JP)$. Now $L^k \subseteq \operatorname{Hom}_R(P, J^k P)$ for any k, so since J is nilpotent, so is L.

If P is indecomposable projective, then $\operatorname{End}_R(P)$ has no non-trivial idempotents. Since idempotents lift modulo nilpotent ideals, $\operatorname{End}_R(P/JP)$ has no non-trivial idempotents, so P/JP is indecomposable, and since it is semisimple, it must be simple.

Now we want to see that every simple module occurs as P/JP for some indecomposable projective. Writing ${}_R R$ as a direct sum of indecomposables $\bigoplus_{i=1}^n P_i$, we have $R/J \cong \bigoplus_{i=1}^n P_i/JP_i$. Since all simple modules occur as a summand of R/J, they all occur from some P_i.

Now any homomorphism $\theta : P \to M$ induces a homomorphism $\overline{\theta} : P/JP \to M/JM$. We show that if $\theta : P \to M$ is a homomorphism with P projective and such that $\overline{\theta}$ is an isomorphism, then θ is a projective cover. Since $\overline{\theta}$ is surjective, $M = JM + \operatorname{Im}(\theta) = J(JM + \operatorname{Im}(\theta)) + \operatorname{Im}(\theta) = J^2 M + \operatorname{Im}(\theta) = \cdots = J^k M + \operatorname{Im}(\theta)$ for all k. Since J is nilpotent, $M = \operatorname{Im}(\theta)$, so θ is surjective. It follows that it is a projective precover.

Now if $\phi \in \operatorname{End}_R(P)$ and $\theta\phi = \theta$, then $\overline{\theta}\,\overline{\phi} = \overline{\theta}$, and since $\overline{\theta}$ is an isomorphism we deduce that $\overline{\phi} = 1$. Thus $\overline{\phi - 1} = 0$. Thus $\phi - 1 \in L$, so it is nilpotent, and hence ϕ is an automorphism. Thus θ is a projective cover.

In general, given any module M, write $M/JM = \bigoplus_{i\in I} S_i$, and consider $P(M) = \bigoplus_{i\in I} P(S_i) \to M/JM$. This lifts to a map $\theta : P(M) \to M$ with $\overline{\theta}$ an isomorphism. Thus θ is a projective cover of M.

Remark. The ring for which all modules have projective covers are the 'left perfect rings'. They are also the rings for which flat = projective, so the best generalization of Theorem 2 is

Theorem of Bican, El Bashir and Enochs 2001. Every module has a flat cover.

Proof is much harder.

Example.

$R = KQ/I$

$$Q = \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\ 2 \arrow[dr, "c"'] & & 3 \arrow[dl, "d"] \\ & 4 & \end{tikzcd} \qquad I = (ca - db) \text{ admissible}$$

simples

$S[1]$ $S[2]$ $S[3]$ $S[4]$

$$\begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & K \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & 0 \arrow[dl] \\ & K & \end{tikzcd}$$

projectives $P[i] = P(S[i]) = Re_i$

$P[1]$ $P[2]$ $P[3]$ $P[4]$

$$\begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & K \arrow[dl] \\ & K & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & 0 \arrow[dl] \\ & K & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & K \arrow[dl] \\ & K & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & 0 \arrow[dl] \\ & K & \end{tikzcd}$$

injectives $I[i] = E(S[i]) = (e_i R)^*$

$I[1]$ $I[2]$ $I[3]$ $I[4]$

$$\begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & K \arrow[dl] \\ & 0 & \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & K \arrow[dl] \\ & K & \end{tikzcd}$$

So $S[1] \cong I[1]$, $P[1] \cong I[4]$, $S[4] \cong P[4]$

3.15 Morita Equivalence

I ONLY BRIEFLY DISCUSSED THIS SECTION IN LECTURES.

Definitions. An abelian category A is *cocomplete* if it has arbitrary coproducts, or equivalently arbitrary colimits. If so, then an object P is *finitely generated* if $\mathrm{Hom}(P, -)$ preserves coproducts, and P is a *generator* if for every object M there is an epimorphism $P^{(I)} \to M$.

Note that a module category R-Mod is cocomplete, finitely generated is the same as the usual definition, and R is a projective generator.

Theorem 1. If A is an abelian category and R is a ring, then A is equivalent to R-Mod if and only if A is cocomplete, and it has a finitely generated projective generator P with $R \cong \mathrm{End}(P)^{op}$.

Proof. The module category R-Mod has these properties, with $P = R$. For sufficiency, consider the functor $F = \mathrm{Hom}(P, -)$ from A to R-Mod. Given objects X and Y choose epimorphisms $p_X : P^{(I)} \to X$ and $p_Y : P^{(J)} \to Y$. Given $\theta : X \to Y$, if $F(\theta) = 0$, then the composition $P^{(I)} \to X \to Y$ is zero, so θ is zero. Thus F is faithful.

Applying F one gets $R^{(I)} \to F(X)$ and $R^{(J)} \to F(Y)$. Any R-module map $\alpha : F(X) \to F(Y)$ lifts to an R-module map $R^{(I)} \to R^{(J)}$. This corresponds to an element of $\mathrm{Hom}(P^{(I)}, P^{(J)})$. Now the composition $\mathrm{Ker}\, p_X \to P^{(I)} \to P^{(J)} \to Y$ is sent by F to zero, so since F is faithful, it is zero itself. Thus there is an induced morphism $\theta : X \to Y$ giving a commutative square. Thus $F(\theta)$ gives a commutative square with the map $R^{(I)} \to R^{(J)}$. Thus $\alpha = F(\theta)$. Thus F is full.

Now for any R-module M there is a presentation $R^{(I)} \to R^{(J)} \to M \to 0$. The first map comes from a morphism $P^{(I)} \to P^{(J)}$. Let this have cokernel X. Then since F is exact, we get $R^{(I)} \to R^{(J)} \to F(X) \to 0$. Thus $M \cong F(X)$. Thus F is dense.

Theorem 2. Let R and S be two rings. The following are equivalent.
(i) The categories R-Mod and S-Mod are equivalent
(ii) There is an S-R-bimodule M such that $M \otimes_R -$ gives an equivalence R-Mod to S-Mod
(iii) $S \cong \mathrm{End}_R(P)^{op}$ for some finitely generated projective generator P in R-Mod.

Proof. (i)$\Leftrightarrow$(iii) follows from the theorem.

(ii)$\Rightarrow$(i) is trivial. For (i)$\Rightarrow$(ii) note that an equivalence is exact, and preserves direct sums, so it must be a naturally isomorphic to a tensor product functor.

Examples. (i) R is Morita equivalent to $M_n(R)$ for $n \geq 1$. Namely the module R^n is a finitely generated projective generator in R-Mod with $\mathrm{End}_R(R^n)^{op} \cong M_n(R)$.

(ii) If $e \in R$ is idempotent, and $ReR = R$, then R is Morita equivalent to eRe. Namely, the condition ensures that the multiplication map $Re \otimes_{eRe} eR \to R$ is onto. Taking a map from a free eRe-module onto eR, say $eRe^{(I)} \to eR$, we get a map $Re^{(I)} \to R$, so Re is a generator. Then $\mathrm{End}_R(Re)^{op} \cong eRe$.

Corollary. Any f.d. algebra over a field is Morita equivalent to one with $R/J(R) \cong D_1 \times \cdots \times D_r$, a product of division algebras.

In particular if K is algebraically closed, any f.d. algebra is Morita equivalent to KQ/I for some quiver Q and admissible ideal I.

Sketch. Write ${}_RR$ as a direct sum of indecomposable projectives, and collect isomorphic summands, say

$$ {}_RR \cong P[1]^{n_1} \oplus \cdots \oplus P[r]^{n_r}. $$

Then $P = P[1] \oplus \cdots \oplus P[r]$ is a f.g. projective generator, so R is Morita equivalent to $S = \mathrm{End}_R(P)^{op}$. One can show that

$$ J(S) = \begin{pmatrix} J(\mathrm{End}(P[1])) & \mathrm{Hom}(P[2], P[1]) & \dots \\ \mathrm{Hom}(P[1], P[2]) & J(\mathrm{End}(P[2])) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, $$

so $S/J(S) \cong \prod_{i=1}^{r} \mathrm{End}(P[i])/J(\mathrm{End}(P[i]))$, a product of division algebras.

4 Homological algebra

Recommended book: C. A. Weibel, An introduction to homological algebra.

4.1 Complexes

Definition 1. Let R be a ring. A *chain complex* C (or $C.$ or C_*) consists of R-modules and homomorphisms

$$\ldots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \ldots$$

satisfying $d_n d_{n+1} = 0$ for all n. The elements of C_n are called *chains* of degree n or *n-chains*. The maps d_n are the *differential*.

If C is a chain complex, then its *homology* is defined by

$$H_n(C) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1}) = Z_n(C)/B_n(C).$$

The elements of $B_n(C)$ are *n-boundaries*. The elements of $Z_n(C)$ are *n-cycles*. If x is an n-cycle we write $[x]$ for its image in $H_n(C)$.

A chain complex C is *acyclic* if $H_n(C) = 0$ for all n, that is, if it is an exact sequence. It is *non-negative* if $C_n = 0$ for $n < 0$. It is *bounded* if there are only finitely many nonzero C_n.

Definition 2. A *cochain complex* C (or $C^{\cdot}$ or C^*) consists of R-modules and homomorphisms

$$\ldots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \ldots$$

satisfying $d^n d^{n-1} = 0$ for all n. The elements of C^n are called *cochains* of degree n or *n-cochains*.

The *cohomology* of a cochain complex is defined by

$$H^n(C) = \mathrm{Ker}(d^n)/\,\mathrm{Im}(d^{n-1}) = Z^n(C)/B^n(C).$$

The elements of $B^n(C)$ are *n-coboundaries*. The elements of $Z^n(C)$ are *n-cocycles*.

Remarks. (i) There is no difference between chain and cohain complexes, apart from numbering. Pass between them by setting $C^n = C_{-n}$.

(ii) Many complexes are zero to the right, so naturally thought of as non-negative chain complexes, or zero to the left, so naturally thought of as non-negative cochain complexes.

(iii) More generally we could replace R-modules by objects in an abelian category.

Definition 3. The *category of cochain complexes* $C(R\text{-Mod})$ has as objects the cochain complexes. A morphism $f : C \to D$ is given by homomorphisms $f^n : C^n \to D^n$ such that each square in the diagram commutes

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d} & C^{n-1} & \xrightarrow{d} & C^n & \xrightarrow{d} & C^{n+1} & \xrightarrow{d} & \dots \\
 & & {\scriptstyle f^{n-1}}\downarrow & & {\scriptstyle f^n}\downarrow & & {\scriptstyle f^{n+1}}\downarrow & & \\
\dots & \xrightarrow{d} & D^{n-1} & \xrightarrow{d} & D^n & \xrightarrow{d} & D^{n+1} & \xrightarrow{d} & \dots
\end{array}$$

There is a *shift functor* $[i] : C(R\text{-Mod}) \to C(R\text{-Mod})$ defined by $C[i]^n = C^{n+i}$ with the differential $d_{C[i]} = (-1)^i d_C$.

The category $C(R\text{-Mod})$ is abelian. (It can be identified with the category of graded $R[d]/(d^2)$-modules, where R has degree 0 and d has degree 1, so it is the category of modules for a catalgebra.)

Direct sums are computed degreewise, $(C \oplus D)^n = C^n \oplus D^n$. Also kernels and cokernels are computed degreewise. Thus a sequence $0 \to C \to D \to E \to 0$ is exact if and only if all $0 \to C^n \to D^n \to E^n \to 0$ are exact.

Lemma. A morphism of complexes $f : C \to D$ induces morphisms on cohomology $H^n(C) \to H^n(D)$, so H^n is a functor from $C(R\text{-Mod})$ to R-Mod.

Proof. An arbitrary element of $H^n(C)$ is of the form $[x]$ with $x \in Z^n(C) = \operatorname{Ker} d^n$. We send it to $[f^n(x)] \in H^n(D)$.

Definition 4. A morphism of complexes $f : C \to D$ is a *quasi-isomorphism* if the map $H^n(C) \to H^n(D)$ is an isomorphism for all n.

Example. Morphism from $\mathbb{Z} \xrightarrow{a} \mathbb{Z}$ to $0 \to \mathbb{Z}/a\mathbb{Z}$ for $a \neq 0$.

Theorem. A short exact sequence of complexes $0 \to C \to D \to E \to 0$ induces a long exact sequence on cohomology

$$\dots \to H^{n-1}(E) \to H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to \dots$$

for suitable connecting maps $c^n : H^n(E) \to H^{n+1}(C)$.

Proof. For all n we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & C^n \arrow[r] \arrow[d, "d_C^n"'] & D^n \arrow[r] \arrow[d, "d_D^n"'] & E^n \arrow[r] \arrow[d, "d_E^n"'] & 0 \\
0 \arrow[r] & C^{n+1} \arrow[r] & D^{n+1} \arrow[r] & E^{n+1} \arrow[r] & 0
\end{tikzcd}$$

and the easy part of the snake lemma gives exact sequences on kernels of the vertical maps

$$0 \to Z^n(C) \to Z^n(D) \to Z^n(E)$$

and on cokernels

$$C^{n+1}/B^{n+1}(C) \to D^{n+1}/B^{n+1}(D) \to E^{n+1}/B^{n+1}(E) \to 0$$

This holds for all n, so shows that the rows in the following diagram are exact

$$\begin{tikzcd}
 & C^n/B^n(C) \arrow[r] \arrow[d, "\overline{d}_C^n"'] & D^n/B^n(D) \arrow[r] \arrow[d, "\overline{d}_C^n"'] & E^n/B^n(E) \arrow[r] \arrow[d, "\overline{d}_E^n"'] & 0 \\
0 \arrow[r] & Z^{n+1}(C) \arrow[r] & Z^{n+1}(D) \arrow[r] & Z^{n+1}(E).
\end{tikzcd}$$

Here the vertical maps are induced by d_C^n, d_D^n and d_E^n, so the diagram commutes. Thus by the snake lemma one gets an exact sequence

$$\mathrm{Ker}(\overline{d}_C^n) \to \mathrm{Ker}(\overline{d}_D^n) \to \mathrm{Ker}(\overline{d}_E^n) \to \mathrm{Coker}(\overline{d}_C^n) \to \mathrm{Coker}(\overline{d}_D^n) \to \mathrm{Coker}(\overline{d}_E^n)$$

That is,

$$H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to H^{n+1}(E)$$

as required.

4.2 Ext

Definition 1. If M is an R-module, then a *projective resolution* of M is an exact sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

with the P_i projective modules. It is equivalent to give a non-negative chain complex P of projective modules and a quasi-isomorphism $P \to M$ (with M considered as a chain complex in degree 0),

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"'] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

Note that every module has many different projective resolutions. Choose any surjection $\epsilon : P_0 \to M$, then any surjection $d_1 : P_1 \to \operatorname{Ker} \epsilon$, then any surjection $d_2 : P_2 \to \operatorname{Ker} d_1$, etc.

If one fixes a projective resolution of M then the *syzygies* of M are the modules $\Omega^n M = \operatorname{Im}(d : P_n \to P_{n-1})$ (and $\Omega^0 M = M$). Thus there are exact sequences

$$0 \to \Omega^{n+1} M \to P_n \to \Omega^n M \to 0.$$

Dually an *injective resolution* of a module X is an exact sequence

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

with the I^n injective modules. The cosyzygies are $\Omega^{-n} X = \operatorname{Im}(I^{n-1} \to I^n)$ (and $\Omega^0 X = X$), so

$$0 \to \Omega^{-n} X \to I^n \to \Omega^{-(n+1)} X \to 0.$$

Definition 2. Given modules M and X, choose a projective resolution $P_* \to M$ of M. We define $\operatorname{Ext}^n_R(M, X) = H^n(\operatorname{Hom}_R(P_*, X))$, the nth cohomology of the cochain complex of K-modules $\operatorname{Hom}_R(P_*, X)$, which is

$$\dots \to 0 \to 0 \to \operatorname{Hom}_R(P_0, X) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, X) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_2, X) \to \dots$$

where $\operatorname{Hom}_R(P_n, X)$ is in degree n.

Properties. (i) $\operatorname{Ext}^n_R(M, X)$ is a K-module, it is zero for $n < 0$, and $\operatorname{Ext}^0_R(M, X) \cong \operatorname{Hom}(M, X)$ since the exact sequence $P_1 \to P_0 \to M \to 0$ gives an exact sequence

$$0 \to \operatorname{Hom}_R(M, X) \to \operatorname{Hom}_R(P_0, X) \to \operatorname{Hom}_R(P_1, X).$$

(ii) This definition depends on the choice of the projective resolution. But we will show that $\operatorname{Ext}^n_R(M, X)$ can also be computed using an injective resolution of X, and that will show that it does not depend on the projective resolution of M.

(iii) $\operatorname{Ext}^n_R(M, X) = 0$ for $n > 0$ if X is injective. Namely, the sequence $\dots \to P_2 \to P_1 \to P_0$ is exact, hence so is the sequence

$$\operatorname{Hom}(P_0, X) \to \operatorname{Hom}(P_1, X) \to \operatorname{Hom}(P_2, X) \to \dots$$

Lemma. A map $X \to Y$ induces a map $\operatorname{Ext}^n_R(M, X) \to \operatorname{Ext}^n_R(M, Y)$, and in this way the assignment $X \rightsquigarrow \operatorname{Ext}^n_R(M, X)$ is a K-linear functor.

Proof. It induces a map of complexes $\mathrm{Hom}_R(P_*, X) \to \mathrm{Hom}_R(P_*, Y)$, and that induces a map on cohomology.

Proposition 1. A short exact sequence $0 \to X \to Y \to Z \to 0$ induces a long exact sequence

$$0 \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(M, Y)$$

$$\to \mathrm{Ext}^1_R(M, X) \to \mathrm{Ext}^1_R(M, Y) \to \mathrm{Ext}^1_R(M, Z)$$
$$\to \mathrm{Ext}^2_R(M, X) \to \mathrm{Ext}^2_R(M, Y) \to \mathrm{Ext}^2_R(M, Z) \to \dots$$

Proof. One gets a sequence of complexes

$$0 \to \mathrm{Hom}_R(P_*, X) \to \mathrm{Hom}_R(P_*, Y) \to \mathrm{Hom}_R(P_*, Z) \to 0.$$

This is exact since each P_n is projective. Thus it induces a long exact sequence on cohomology.

Proposition 2. If $0 \to X \to I^0 \to I^1 \to I^2 \to \dots$ is an injective resolution of X, then one can compute $\mathrm{Ext}^n_R(M, X)$ as the nth cohomology of the complex $\mathrm{Hom}_R(M, I^*)$ if K-modules:

$$0 \to \mathrm{Hom}_R(M, I^0) \to \mathrm{Hom}_R(M, I^1) \to \mathrm{Hom}_R(M, I^2) \dots$$

Proof. Break the injective resolution into exact sequences

$$0 \to \Omega^{-i} X \to I^i \to \Omega^{-(i+1)} X \to 0$$

for $i \geq 0$ where $\Omega^0 X = X$. One gets long exact sequences

$$0 \to \mathrm{Hom}_R(M, \Omega^{-i} X) \to \mathrm{Hom}_R(M, I^i) \to \mathrm{Hom}_R(M, \Omega^{-(i+1)} X)$$

$$\to \mathrm{Ext}^1_R(M, \Omega^{-i} X) \to 0 \to \mathrm{Ext}^1_R(M, \Omega^{-(i+i)} X)$$
$$\to \mathrm{Ext}^2_R(M, \Omega^{-i} X) \to 0 \to \mathrm{Ext}^2_R(M, \Omega^{-(i+1)} X) \dots$$

so
$$\mathrm{Ext}^1_R(M, \Omega^{-i} X) \cong \mathrm{Coker}(\mathrm{Hom}_R(M, I^i) \to \mathrm{Hom}_R(M, \Omega^{-(i+1)}))$$
and
$$\mathrm{Ext}^j_R(M, \Omega^{-(i+1)} X) \cong \mathrm{Ext}^{j+1}_R(M, \Omega^{-i} X)$$
for $j \geq 1$. Thus (it is called *dimension shifting*)

$$\mathrm{Ext}^n_R(M, X) \cong \mathrm{Ext}^{n-1}_R(M, \Omega^{-1} X) \cong \dots \cong \mathrm{Ext}^1_R(M, \Omega^{-(n-1)} X)$$

$$\cong \operatorname{Coker}\left(\operatorname{Hom}_R(M, I^{n-1}) \to \operatorname{Hom}_R(M, \Omega^{-n}X)\right)$$

Now $0 \to \Omega^{-n}X \to I^n \to I^{n+1}$ is exact, hence so is

$$0 \to \operatorname{Hom}_R(M, \Omega^{-n}X) \to \operatorname{Hom}_R(M, I^n) \to \operatorname{Hom}_R(M, I^{n+1})$$

It follows that $\operatorname{Ext}^n_R(M, X)$ is the cohomology in degree n of the complex

$$\cdots \to \operatorname{Hom}_R(M, I^{n-1}) \to \operatorname{Hom}_R(M, I^n) \to \operatorname{Hom}_R(M, I^{n+1}) \to \dots$$

as required.

Remarks. (i) As mentioned, it follows that $\operatorname{Ext}^n_R(M, X)$ does not depend on the projective resolution of M.

(ii) Using the description in terms of an injective resolution of X it follows that the assignment $M \rightsquigarrow \operatorname{Ext}^n(M, X)$ is a contravariant K-linear functor.

Also, if $0 \to L \to M \to N \to 0$ is an exact sequence and I^* is an injective resolution of X, then one gets an exact sequence of complexes $0 \to \operatorname{Hom}_R(N, I^*) \to \operatorname{Hom}_R(M, I^*) \to \operatorname{Hom}_R(L, I^*) \to 0$, and hence a long exact sequence

$$\begin{gathered} 0 \to \operatorname{Hom}_R(N, X) \to \operatorname{Hom}_R(M, X) \to \operatorname{Hom}_R(L, X) \\ \to \operatorname{Ext}^1_R(N, X) \to \operatorname{Ext}^1_R(M, X) \to \operatorname{Ext}^1_R(L, X) \\ \to \operatorname{Ext}^2_R(N, X) \to \operatorname{Ext}^2_R(M, X) \to \operatorname{Ext}^2_R(L, X) \to \dots \end{gathered}$$

Example 0. If R is a field, or more generally a finite-dimensional semisimple algebra over a field, then all short exact sequences of R-modules are split exact, so all modules are projective and injective. Thus

$$\operatorname{Ext}^n_R(M, X) \cong \begin{cases} \operatorname{Hom}_R(M, X) & (n = 0) \\ 0 & (n > 0). \end{cases}$$

Example 1. If $0 \neq a \in \mathbb{Z}$ then $\mathbb{Z}/a\mathbb{Z}$ has projective resolution $0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to \mathbb{Z}/a\mathbb{Z} \to 0$. Thus $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X)$ is the cohomology of the complex

$$\cdots \to 0 \to \operatorname{Hom}(\mathbb{Z}, X) \xrightarrow{a} \operatorname{Hom}(\mathbb{Z}, X) \to 0 \to \dots$$

that is,

$$\cdots \to 0 \to X \xrightarrow{a} X \to 0 \to \dots$$

so $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X) = \mathrm{Hom}(\mathbb{Z}/a\mathbb{Z}, X) \cong \{x \in X : ax = 0\}$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X) \cong X/aX$ and $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, X) = 0$ for $n > 1$.

Example 2. Let $R = K[x]/(x^2)$ with K a field. Any finitely generated module is a direct sum of copies of K (with x acting as 0) and R. The module K has projective resolution

$$\to R \xrightarrow{x} R \xrightarrow{x} R \to K \to 0.$$

Now $\mathrm{Hom}_R(R, K) = K$, and we get $\mathrm{Ext}^n_R(K, K) \cong K$ for all $n \geq 0$.

Example 3. Consider the algebra $R = KQ/I$ given over a field K by a quiver Q and an admissible ideal I. For example

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

and $I = (cba, dc)$. Let $S[i]$ be the simple module at vertex i. The corresponding indecomposable projective module is $P[i] = Re_i$. It has basis the paths starting at i modulo the relations. This gives representations

$$\begin{aligned}
&S[1] : K \to 0 \to 0 \to 0 \to 0, \quad && P[1] : K \to K \to K \to 0 \to 0,\\
&S[2] : 0 \to K \to 0 \to 0 \to 0, && P[2] : 0 \to K \to K \to K \to 0,\\
&S[3] : 0 \to 0 \to K \to 0 \to 0, && P[3] : 0 \to 0 \to K \to K \to 0,\\
&S[4] : 0 \to 0 \to 0 \to K \to 0, && P[4] : 0 \to 0 \to 0 \to K \to K,\\
&S[5] : 0 \to 0 \to 0 \to 0 \to K, && P[5] : 0 \to 0 \to 0 \to 0 \to K.
\end{aligned}$$

The simples have projective resolutions:

$$\begin{aligned}
0 \to P[5] \to &S[5] \to 0,\\
0 \to P[5] \to P[4] \to &S[4] \to 0,\\
0 \to P[5] \to P[4] \to P[3] \to &S[3] \to 0,\\
0 \to P[3] \to P[2] \to &S[2] \to 0,\\
0 \to P[5] \to P[4] \to P[2] \to P[1] \to &S[1] \to 0.
\end{aligned}$$

We can compute $\mathrm{Ext}^n_R(S[i], S[j])$ as the cohomology of the complex $\mathrm{Hom}_R(P_*, S[j])$ where P_* is a projective resolution of $S[i]$. Use that

$$\mathrm{Hom}_R(P[i], S[j]) = \mathrm{Hom}_R(Re_i, S[j]) = e_i S[j] = \begin{cases} K & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

For example for $\mathrm{Ext}^n(S[1], S[4])$ we have

$$0 \to \mathrm{Hom}(P_0, S[4]) \to \mathrm{Hom}(P_1, S[4]) \to \mathrm{Hom}(P_2, S[4]) \to \mathrm{Hom}(P_3, S[4]) \to \dots$$

which is

$$0 \to \mathrm{Hom}(P[1], S[4]) \to \mathrm{Hom}(P[2], S[4]) \to \mathrm{Hom}(P[4], S[4]) \to \mathrm{Hom}(P[5], S[4]) \to 0 \to \dots$$

which is

$$0 \to 0 \to 0 \to K \to 0 \to 0 \to \dots$$

so

$$\mathrm{Ext}^n_R(S[1], S[4]) = \begin{cases} K & (n = 2) \\ 0 & (n \neq 2) \end{cases}.$$

4.3 Description of Ext^1 using short exact sequences

Definition 1. Two short exact sequences ξ, ξ' with the same end terms are *equivalent* if there is a map θ (necessarily an isomorphism) giving a commutative diagram

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\theta}\downarrow & & \| & & \\ \xi' : 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

It is easy to see that the split exact sequences form one equivalence class.

Definition 2. For any short exact sequence of modules

$$\xi : 0 \to L \to M \to N \to 0$$

we define an element $\hat{\xi} \in \mathrm{Ext}^1_R(N, L)$ as follows. The long exact sequence for $\mathrm{Hom}_R(N, -)$ gives a connecting map $\mathrm{Hom}_R(N, N) \to \mathrm{Ext}^1_R(N, L)$ and $\hat{\xi}$ is the image of id_N under this map.

Theorem 1. The assignment $\xi \mapsto \hat{\xi}$ gives a bijection between equivalence classes of short exact sequences $0 \to L \to M \to N \to 0$ and elements of $\mathrm{Ext}^1_R(N, L)$. The split exact sequences correspond to the element $0 \in \mathrm{Ext}^1_R(N, L)$.

Proof. Fix a projective resolution of N, and hence an exact sequence

$$0 \to \Omega^1 N \xrightarrow{\theta} P_0 \xrightarrow{\epsilon} N \to 0.$$

An exact sequence ξ gives a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathrm{Hom}(N,N) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(P_0,L) \arrow[r] \arrow[d, "\theta^*"'] & \mathrm{Hom}(P_0,M) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\Omega^1 N,L) \arrow[r] \arrow[d] & \mathrm{Hom}(\Omega^1 N,M) \arrow[r] & \mathrm{Hom}(\Omega^1 N,N) & \\
 & \mathrm{Ext}^1(N,L) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

and the connecting map $\mathrm{Hom}(N,N) \to \mathrm{Ext}^1(N,L)$ is given by diagram chasing, so by the choice of maps α, β giving a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0.
\end{tikzcd}$$

Then $\hat{\xi} = [\alpha]$ where $[\dots]$ denotes the map $\mathrm{Hom}(\Omega^1 N, L) \to \mathrm{Ext}^1(N,L)$.

Any element of $\mathrm{Ext}^1(N,L)$ arises from some ξ. Namely, write it as $[\alpha]$ for some $\alpha \in \mathrm{Hom}(\Omega^1 N, L)$. Then take ξ to be the pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Now if ξ, ξ' are equivalent exact sequences one gets a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1 N & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & \big\| & & \\ \xi : 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \big\| & & \big\downarrow & & \big\| & & \\ \xi' : 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

so ξ and ξ' correspond to the same map α, so $\hat{\xi} = \hat{\xi}'$. If two short exact sequences ξ, ξ' give the same element of $\mathrm{Ext}^1(N, L)$ there are diagrams with maps α, β and α', β' and with $\alpha - \alpha'$ in the image of the map $\theta^* : \mathrm{Hom}(P_0, L) \to \mathrm{Hom}(\Omega^1 N, L)$. Say $\alpha - \alpha' = \phi\theta$ with $\phi : P_0 \to L$. Then there is also a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1 N & \xrightarrow{\theta} & P_0 & \longrightarrow & N & \longrightarrow & 0 \\ & & {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle\beta - f\phi}\big\downarrow & & \big\| & & \\ \xi : 0 & \longrightarrow & L & \xrightarrow{f} & M & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

This is a pushout, so by the uniqueness of pushouts, ξ and ξ' are equivalent.

Remark. Homomorphisms $L \to L'$ and $N'' \to N$ induce maps $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N, L')$ and $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N'', L)$. One can show that these maps correspond to pushouts and pullbacks of short exact sequences. For pushouts this follows directly from the definition. For pullbacks it needs more work - omitted.

Theorem 2. The following are equivalent for a module M.
(i) M is projective
(ii) $\mathrm{Ext}^n(M, X) = 0$ for all X and all $n > 0$.
(iii) $\mathrm{Ext}^1(M, X) = 0$ for all X.

The following are equivalent for a module X.
(i) X is injective
(ii) $\mathrm{Ext}^n(M, X) = 0$ for all M and all $n > 0$.
(iii) $\mathrm{Ext}^1(M, X) = 0$ for all cyclic modules M.

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are clear. (iii)$\Rightarrow$(i) using the characterization of a projective or injective module as one for which all short exact sequences ending or starting at the module split. In the injective case we use Baer's criterion: if I is a left ideal in R, the pushout of a sequence $0 \to I \to R \to$

$R/I \to 0$ along any map $I \to X$ spits. Using the splitting one gets a lift of the map to a map $R \to X$, and then by Baer's criterion X is injective.

4.4 Global dimension

Proposition/Definition 1. Let M be a module and $n \geq 0$. The following are equivalent.
(i) There is a projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$
(ii) $\mathrm{Ext}^m(M,X) = 0$ for all $m > n$ and all X.
(iii) $\mathrm{Ext}^{n+1}(M,X) = 0$ for all X.
(iv) For any projective resolution of M, we have $\Omega^n M$ projective.
The *projective dimension*, $\mathrm{proj.\,dim}\, M$, is the smallest n with this property (or ∞ if there is none).

Let X be a module and $n \geq 0$. The following are equivalent.
(i) There is an injective resolution $0 \to X \to I^0 \to \cdots \to I^n \to 0$
(ii) $\mathrm{Ext}^m(M,X) = 0$ for all $m > n$ and all X.
(iii) $\mathrm{Ext}^{n+1}(M,X) = 0$ for all **cyclic** M.
(iv) For any injective resolution of X, we have $\Omega^{-n} X$ injective.
The *injective dimension*, $\mathrm{inj.\,dim}\, X$, is the smallest n with this property (or ∞ if there is none).

Proof (i)⇒(ii)⇒(iii) are trivial. For (iii)⇒(iv) let $P_* \to M$ be a projective resolution. For any X, dimension shifting gives

$$0 = \mathrm{Ext}^{n+1}(M,X) \cong \mathrm{Ext}^n(\Omega^1 M, X) \cong \ldots \cong \mathrm{Ext}^1(\Omega^n M, X),$$

so $\Omega^n M$ is projective. Then

$$0 \to \Omega^n M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is also a projective resolution of M, giving (i).

Lemma. If $0 \to L \to M \to N \to 0$ is exact, then

$$\mathrm{proj.\,dim}\, M \leq \max\{\mathrm{proj.\,dim}\, L, \mathrm{proj.\,dim}\, N\},$$
$$\mathrm{inj.\,dim}\, M \leq \max\{\mathrm{inj.\,dim}\, L, \mathrm{inj.\,dim}\, N\}.$$

Proof. For any X the long exact sequence for $\mathrm{Hom}(-,X)$ gives an exact sequence

$$\mathrm{Ext}^{n+1}(N,X) \to \mathrm{Ext}^{n+1}(M,X) \to \mathrm{Ext}^{n+1}(L,X)$$

and the outer terms are zero for $n = \max$.

Definition. The *(left) global dimension* of R (in $\mathbb{N} \cup \{\infty\}$) is

$$\begin{aligned}\operatorname{gl.dim} R &= \sup\{\operatorname{proj.dim} M : M \in R-Mod\} \\ &= \inf\{n \in \mathbb{N} : \operatorname{Ext}^{n+1}(M,X) = 0 \forall M, X\} \\ &= \sup\{\operatorname{inj.dim} X : X \in R-Mod\} \\ &= \inf\{n \in \mathbb{N} : \operatorname{Ext}^{n+1}(M,X) = 0 \forall M, X,\ M \text{ cyclic}\} \\ &= \sup\{\operatorname{proj.dim} M : M \text{ cyclic}\}.\end{aligned}$$

Example. $\operatorname{gl.dim} R = 0 \Leftrightarrow$ all modules are projective $\Leftrightarrow$ all short exact sequences split $\Leftrightarrow$ every submodule has a complement $\Leftrightarrow$ R is semisimple artinian.

Proposition/Definition 2. A ring R is said to be *(left) hereditary* if it satisfies the following equivalent conditions
(i) $\operatorname{gl.dim} R \leq 1$.
(ii) Every submodule of a projective module is projective.
(iii) Every left ideal in R is projective.

Proof of equivalence. (i)$\Rightarrow$(ii) If N is a submodule of P then for any X, by the long exact sequence, $\operatorname{Ext}^1(N,X) \cong \operatorname{Ext}^2(P/N,X) = 0$.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) For any X and left ideal I we have $\operatorname{Ext}^2(R/I,X) \cong \operatorname{Ext}^1(I,X) = 0$, so X has injective dimension ≤ 1.

Examples. A principal ideal domain is hereditary. If K is a field and Q is a quiver, then one can show that any KQ-module M has a *standard resolution*

$$0 \to \bigoplus_{a \in Q} KQe_{h(a)} \otimes_K e_{h(a)}M \xrightarrow{f} \bigoplus_{i \in Q_0} KQe_i \otimes e_iM \xrightarrow{g} M \to 0$$

where $g(x_i \otimes m_i) = x_i m_i$, and $f(x_a \otimes m_a) = x_a a \otimes m_a - x_a \otimes a m_a$. Thus $\operatorname{proj.dim} M \leq 1$, so KQ is hereditary.

Proposition. If R is a f.d. algebra over a field, then

$$\operatorname{gl.dim} R = \max\{\operatorname{proj.dim} S : S \text{ is a simple module}\}.$$

Proof. We show by induction on $\dim M$ that any f.d. module M has projective dimension $\leq m$, where m is the maximum of the projective dimensions of the simples. Namely if M is simple, this hold by definition. If not, it has a non-trivial proper submodule X. Now in the exact sequence $0 \to X \to M \to M/X \to 0$, the end terms have smaller dimension, so projective dimension at most m, hence $\operatorname{proj.dim} M \leq m$ by the lemma. Now every cyclic module is f.d., so has projective dimension $\leq m$, hence $\operatorname{gl.dim} R = m$.

Examples. In Example 3 of §4.2, the simple module $S[1]$ has projective resolution

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to S[1] \to 0$$

where $P_3 = P[5]$, $P_2 = P[4]$, $P_1 = P[2]$ and $P_0 = P[1]$, so $\operatorname{proj.dim} S[1] \leq 3$. In fact we have equality—for example the methods in the example show that $\operatorname{Ext}^3(S[1], S[5]) \neq 0$. The other simples have projective dimension ≤ 2. Thus $\operatorname{gl.dim} R = 3$.

For the commutative square algebra in §3.14, the simple module $S[1]$ has projective resolution

$$0 \to P[4] \to P[2] \oplus P[3] \to P[1] \to S[1] \to 0$$

so $\operatorname{proj.dim} S[1] = 2$, and the other simples have projective dimension ≤ 1, so $\operatorname{gl.dim} R = 2$.

Theorem. Consider a skew polynomial ring $S = R[x; \sigma, \delta]$ with σ an automorphism of R and δ a σ-derivation.
(i) For any S-module M there is a an exact sequence

$$0 \to S \otimes_R ({}_{\sigma^{-1}}M) \xrightarrow{f} S \otimes_R M \xrightarrow{g} M \to 0$$

where g is multiplication and $f(s \otimes m) = sx \otimes m - s \otimes xm$.
(ii) $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$.
(iii) $\operatorname{gl.dim} S = 1 + \operatorname{gl.dim} R$ if $\delta = 0$.

Proof. (i) Since σ is an automorphism, S is a free right R-module with basis $\{1, x, x^2, \dots\}$, so for any R-module N, the elements of $S \otimes_R N$ can be written uniquely as expressions $\sum x^i \otimes n_i$.

The map f is well-defined: define $f' : S \otimes_K M \to S \otimes_R M$ by $f'(s \otimes m) = sx \otimes m - s \otimes xm$. Then since $xr = \sigma(r)x + \delta(r)$ we get

$$\begin{aligned} f'(sr \otimes m) - f'(s \otimes \sigma^{-1}(r)m) &= srx \otimes m - sr \otimes xm - sx \otimes \sigma^{-1}(r)m + s \otimes x\sigma^{-1}(r)m \\ &= s(rx - x\sigma^{-1}(r)) \otimes m + s \otimes (x\sigma^{-1}(r) - rx)m \\ &= -s\delta(\sigma^{-1}(r)) \otimes m + s \otimes \delta(\sigma^{-1}(r))m = 0. \end{aligned}$$

Thus f' descends to a map f.

Exact in middle: clearly $gf = 0$. Choose an element of $\operatorname{Ker} g$ of the form $x^i \otimes m +$ lower powers of x, with $m \neq 0$ and i minimal. Then $i = 0$, for otherwise one can cancel the leading term by subtracting $f(x^{i-1} \otimes m)$. Thus the element is $1 \otimes m$. But then since the element is in $\operatorname{Ker} g$, it is zero.

Exact on left: an element of the form $x^i \otimes m +$ lower powers of x with $m \neq 0$ is sent by f to $x^{i+1} \otimes m +$ lower powers of x, which cannot be zero.

(ii) S_R is free, so flat, so a projective resolution $P_* \to N$ of an R-module N gives an S-module projective resolution $S \otimes_R P_* \to S \otimes_R N$. Using that $\operatorname{Hom}_S(S \otimes_R -, X) \cong \operatorname{Hom}_R(-, X)$ for an S-module X, it follows that

$$\operatorname{Ext}^n_S(S \otimes_R N, X) \cong \operatorname{Ext}^n_R(N, X). \qquad (*)$$

By the long exact sequence for $\operatorname{Hom}_S(-, X)$ we get

$$\operatorname{Ext}^n_S(S \otimes M, X) \xrightarrow{h} \operatorname{Ext}^n_S(S \otimes_{\sigma^{-1}} M, X) \to \operatorname{Ext}^{n+1}_S(M, X) \to \operatorname{Ext}^{n+1}_S(S \otimes M, X).$$

For $n > \operatorname{gl.dim} R$, the second and fourth terms are zero, so also the third term is zero, so $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$.

(iii) Let X be an R-module and $X \to I^*$ an injective resolution. We get cosyzygies $0 \to \Omega^{-(i-1)} X \to I^i \to \Omega^{-i} X \to 0$. Since $\delta = 0$, we can consider all of these as S-modules with x acting as 0, so for any S-module U, we get a long exact sequence

$$0 \to \operatorname{Hom}_S(U, \Omega^{-(i-1)} X) \to \operatorname{Hom}_S(U, I^i) \to \operatorname{Hom}_S(U, \Omega^{-i} X) \to \operatorname{Ext}^1_S(U, \Omega^{-(i-1)} X) \to \dots$$

Now suppose $U = S \otimes_R N$. If $j > 0$ we have $\operatorname{Ext}^j_S(U, I^i) \cong \operatorname{Ext}^j_R(N, I^i) = 0$, so as in dimension shifting, we get

$$\operatorname{Hom}_S(U, \Omega^{-n} X) \twoheadrightarrow \operatorname{Ext}^1_S(U, \Omega^{-(n-1)} X) \cong \dots \cong \operatorname{Ext}^n_S(U, X).$$

Applying this to the map f we get a commutative square

$$\begin{array}{ccc} \operatorname{Hom}_S(S \otimes M, \Omega^{-n} X) & \longrightarrow & \operatorname{Ext}^n_S(S \otimes M, X) \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ \operatorname{Hom}_S(S \otimes_{\sigma^{-1}} M, \Omega^{-n} X) & \longrightarrow & \operatorname{Ext}^n_S(S \otimes_{\sigma^{-1}} M, X) \end{array}$$

where f' is composition with f.

Since x acts as zero on M and $\Omega^{-n} X$ it follows that f' is zero. Namely $f'(\phi)(s \otimes m) = \phi f(s \otimes m) = \phi(sx \otimes m - s \otimes xm) = \phi(sx \otimes m) = sx\phi(1 \otimes m) = 0$.

Since the horizontal maps are onto, h is zero. Thus for $n = \mathrm{gl.dim}\, R$ we get $\mathrm{Ext}^{n+1}_S(M,X) \cong \mathrm{Ext}^n_R({}_{\sigma^{-1}}M,X)$, and for suitable M,X this is non-zero. Thus $\mathrm{gl.dim}\, S = 1 + \mathrm{gl.dim}\, R$.

Corollary. If K is a field, then $\mathrm{gl.dim}\, K[x_1,\dots,x_n] = n$.

4.5 Tor

Given a right R-module M and a left R-module X, choose a projective resolution $P_* \to M$ (or more generally a *flat resolution*, where we only require the P_n to be flat). We define $\mathrm{Tor}^R_n(M,X)$ to be the nth homology of the complex

$$P_* \otimes_R X : \cdots \to P_2 \otimes_R X \to P_1 \otimes_R X \to P_0 \otimes_R X \to 0$$

Since the tensor product is a right exact functor, it follows that $\mathrm{Tor}^R_0(M,X) \cong M \otimes_R X$. Moreover a short exact sequence $0 \to X \to Y \to Z \to 0$ gives a long exact sequence

$$\cdots \to \mathrm{Tor}^R_2(M,Z) \to \mathrm{Tor}^R_1(M,X) \to \mathrm{Tor}^R_1(M,Y) \to \mathrm{Tor}^R_1(M,Z) \to$$

$$\to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0.$$

Using this one can show that Tor can be computed using a projective or flat resolution of X. Thus the two modules M,X play a symmetrical role; Tor_n is a covariant functor in both arguments. This shows independence of the resolution.

Theorem. The following are equivalent for a module M.
(i) M is flat
(ii) $\mathrm{Tor}^R_n(M,X) = 0$ for all X and all $n > 0$.
(iii) $\mathrm{Tor}^R_1(M,X) = 0$ for all X.

Proposition/Definition. Let M be a module and $n \geq 0$. The following are equivalent.
(i) There is a flat resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$
(ii) $\mathrm{Tor}^R_m(M,X) = 0$ for all X and $m > n$
(iii) $\mathrm{Tor}^R_{n+1}(M,X) = 0$ for all X.
(iv) For any flat resolution of M, we have $\Omega^n M$ flat.
The *flat dimension* $\mathrm{flatdim}\, M$ is the smallest n with this property (or ∞ if there is none).

Definition. The *weak dimension* of R is

$$\text{w.}\dim R = \sup\{\operatorname{flatdim} M : \forall M\} = \inf\{n \in \mathbb{N} : \operatorname{Tor}^R_{n+1}(M, X) = 0 \forall M, X\}.$$

It is left/right symmetric.

4.6 Global dimension for noetherian rings

Proposition. (i) For M a left R-module, $\operatorname{flatdim} M \leq \operatorname{proj.dim} M$, with equality if M is finitely generated and R is left noetherian.
(ii) $\text{w.}\dim R \leq \operatorname{gl.dim} R$, with equality if R is left noetherian.
(iii) If R is (left and right) noetherian, the left and right global dimensions or R are equal.

Proof. (i) The inequality holds since any projective resolution is also a flat resolution. If R is left noetherian and M is f.g., we have a projective resolution with all P_n finitely generated. Then $\operatorname{flatdim} M \leq n$ implies $\Omega^n M$ is flat. Since it is also finitely presented, it is projective. Thus $\operatorname{proj.dim} M \leq n$.

(ii) Use that $\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim} M : M \text{ cyclic}\}$.

(iii) Clear.

THE REMAINING MATERIAL WAS ONLY BRIEFLY DISCUSSED IN THE LAST LECTURE.

Let K be a field. Recall that the first Weyl algebra is

$$R = A_1(K) = K[x][y; d/dx] = K\langle x, y\rangle/(yx - xy - 1).$$

We know $\operatorname{gl.dim} R \leq 2$. In fact more is true.

Theorem. Let K be a field of characteristic zero, and for simplicity suppose it is algebraically closed. In this case the first Weyl algebra is hereditary.

Lemma 1. $S = k[x] \setminus \{0\}$ is a left and right Ore set in R and $R_S \cong K(x)[y; d/dx]$. Thus $\operatorname{gl.dim} R_S \leq 1$.

Proof. To show S is a left Ore set, given $a \in R$ and $s \in S$ we need to find a', s' with $a's = s'a$. We do this by induction on the order of a as a differential operator. Now $[a, s]$ has smaller order, so there is a'', s'' with $a''s = s''[a, s]$. Then $(s''a - a'')s = s''sa$, so we can take $a' = s''a - a''$ and $s' = s''s$. The rest is straightforward.

Lemma 2. If M is a finitely generated R-module which is torsion-free as a $k[x]$-module, then $\operatorname{proj.dim} M \leq 1$.

Proof. Since M is torsion-free over $k[x]$, the natural map $M \to S^{-1}M$ is injective. Now $S^{-1}M$ is a module for $K(x)[y; d/dx]$ so it has a projective resolution $0 \to Q_1 \to Q_0 \to S^{-1}M \to 0$. As R_S is flat as left R-module, Q_0 and Q_1 are flat R-modules, so $\operatorname{flatdim}_R S^{-1}M \leq 1$.

Now M embeds in $S^{-1}M$ and $\operatorname{w.dim} R = \operatorname{gl.dim} R \leq 2$, so for any L the long exact sequence gives an exact sequence

$$\to \operatorname{Tor}_3^R(L, (S^{-1}M)/M) \to \operatorname{Tor}_2^R(L, M) \to \operatorname{Tor}_2^R(L, S^{-1}M) \to$$

The outside terms are zero, so $\operatorname{flatdim} M \leq 1$. Now use that M is finitely generated.

Lemma 3. If $\lambda \in K$, then the R-module $S_\lambda = R/R(x-\lambda)$ is simple and $\operatorname{proj.dim} S_\lambda \leq 1$.

Proof. Any element of R can be written uniquely as a sum $\sum_n y^n p_n(x)$, so as a K-linear combination of elements $y^n(x-\lambda)^m$. Thus S_λ can be identified with $K[y]$, with y acting by multiplication and the action of x given by $xq(y) = \lambda q(y) - q'(y)$.

To show simplicity, note that the action of $(\lambda - x)$ on $K[y]$ is as differentiation by y, so the submodule generated by any non-zero element of $K[y]$ contains 1, and hence this submodule is all of $K[y]$.

Now we have projective resolution $0 \to R \xrightarrow{\cdot(x-\lambda)} R \to S_\lambda \to 0$.

Proof of the theorem. It suffices to show that $\operatorname{proj.dim} M \leq 1$ for M cyclic.

If M is not torsion-free over $K[x]$, then some non-zero element of M is killed by a non-zero polynomial $p(x)$. Since K is algebraically closed, we can factorize this polynomial, and hence find $0 \neq m \in M$ and $\lambda \in K$ with $(x-\lambda)m = 0$. Then m generates a submodule of M isomorphic to S_λ. Repeating with the quotient module, we get an ascending chain of submodules of M, and since M is noetherian this terminates. Thus we get submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq M$$

such that each $M_i/M_{i-1} \cong S_{\lambda_i}$ and M/M_k is torsion-free as a $K[x]$-module.

The quotients M_i/M_{i-1} and M/M_k all have projective dimension ≤ 1, and hence $\operatorname{proj.dim} M \leq 1$.

Some other facts about noetherian rings.

(i) R is left noetherian $\Leftrightarrow$ any direct sum of injective modules is injective $\Leftrightarrow$ any injective module is a direct sum of indecomposable modules. See for example Lam, Lectures on modules and rings.

(ii) (Chase) Any product of flat right modules is flat if and only if R is *left coherent*, which means that any finitely generated left ideal is finitely presented. In particular this holds if R is left noetherian or left hereditary.

(iii) If R is left noetherian ring and $\operatorname{gl.dim} R < \infty$ then

$$\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim} S : S \text{ simple}\}.$$

For a proof see McConnell and Robson, Noncommutative noetherian rings, Corollary 7.1.14.