School Mathematics Project

BOOK 1 [METRIC]

CAMBRIDGE UNIVERSITY PRESS

THE
SCHOOL MATHEMATICS PROJECT

When the S.M.P. was founded in 1961, its objective was to devise radically new mathematics courses, with accompanying G.C.E. syllabuses and examinations, which would reflect, more adequately than did the traditional syllabuses, the up-to-date nature and usages of mathematics.

The first stage of this objective is now more or less complete. *Books 1–5* form the main series of pupil's texts, starting at the age of 11+ and leading to the O-level examination in 'S.M.P. Mathematics', while, *Books 3T, 4* and *5* give a three-year course to the same O-level examination. (*Books T* and *T4*, together with their Supplement, represent the first attempt at this three-year course, but they may be regarded as obsolete.) *Advanced Mathematics Books 1–4* cover the syllabus for the A-level examination in 'S.M.P. Mathematics' and in preparation are five (or more) shorter texts covering the material of various sections of the A-level examination in 'S.M.P. Further Mathematics'. There are two books for 'S.M.P. Additional Mathematics' at O-level. Every book is accompanied by a Teacher's Guide.

For the convenience of schools, the S.M.P. has an arrangement whereby its examinations are made available by every G.C.E. Examining Board, and it is most grateful to the Secretaries of the eight Boards for their co-operation in this. At the same time, most Boards now offer their own syllabuses in 'modern mathematics' for which the S.M.P. texts are suitable.

By 1967, it had become clear from experience in comprehensive schools that the mathematical content of the S.M.P. texts was suitable for a much wider range of pupil than had been originally anticipated, but that the presentation needed adaptation. Thus it was decided to produce a new series, *Books A–H*, which would serve as a secondary-school course starting at the age of 11+. These books are specially suitable for pupils aiming at a C.S.E. examination; however, the framework of the C.S.E. examinations is such that it is inappropriate for the S.M.P. to offer its own examination as it does for the G.C.E.

The completion of all these books does not mean that the S.M.P. has no more to offer to the cause of curriculum research. The team of S.M.P. writers, now numbering some thirty school and university mathematicians, is continually testing and revising old work and preparing for new. At the same time, the effectiveness of the S.M.P.'s work depends, as it always has done, on obtaining reactions from active teachers—and also from pupils—in the classroom. Readers of the texts can therefore send their comments to the S.M.P. in the knowledge that they will be warmly welcomed.

Finally, the year-by-year activity of the S.M.P. is recorded in the annual Director's Reports which readers are encouraged to obtain on request to the S.M.P. Office at Westfield College, University of London, London N.W. 3.

The S.M.P. texts have been developed as a result of the collaboration of the following schools:

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BATTERSEA GRAMMAR SCHOOL
CHARTERHOUSE
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MARLBOROUGH COLLEGE
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and grateful acknowledgement is made of the valuable contributions made by the mathematical staffs of these schools.

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BOOK 1

[METRIC]

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PREFACE

This is the first of the four books which together will cover the complete S.M.P. course leading to the Elementary Mathematics examination at O level; that is, it is intended for pupils at the beginning of their secondary-school career. Pupils using this Book 1 will be able to use Books 2, 3 and 4 in succession up to O level; if they then continue with their mathematics, Books 6 and 7 will take them through the S.M.P. A-level course, or Book 5 will prepare them for the S.M.P. Additional Mathematics examination.

When writing the four main-school texts, the authors had especially in mind the needs of pupils who will not carry their mathematics beyond O level; and the aim is to give such pupils a knowledge of the nature of mathematics and its uses in the modern world. The texts are, however, designed also for those who intend to study the subject at more advanced levels; indeed the whole O-level course is an essential preliminary to the A-level S.M.P. course.

In comparison with traditional texts, these texts pay more attention to an understanding of fundamental concepts, but change in content is not our whole objective. We hope that our texts are more attuned than is usual to the child's natural learning processes, and that they will encourage teachers to try out fresh teaching methods. Thus, for example, each new section in this book is introduced by a series of preparatory examples in which the pupil is led to discover the relevant mathematics for himself; again, many of the book's abundant exercises are designed to help the pupil to learn by exploration. Of course, there is also a need for consolidation and so revision exercises have been spaced regularly throughout the book.

The difference between this and traditional texts is most marked, however, in content, for there is much in the book which until recently would not have been considered suitable for O level, let alone for the first forms. Naturally many basic topics which have always been a part of a secondary-school syllabus are retained, though treated sometimes in an unfamiliar way. Thus the subject matter of the very first chapter of the book is arithmetic and its purpose is to revise the fundamental arithmetic operations; but this is done not by repeating work already dealt with at primary school but by introducing arithmetic in various number scales. It is not considered that ability to calculate in bases other than the decimal is important in itself, but work in these scales can do much to promote a proper understanding of the ordinary processes of arithmetic. There are three more chapters in the book which are mainly concerned with arithmetic. Chapter 4 is devoted to a conventional look at fractions; the bulk of the work is contained in the exercises and the chapter is intended to be a do-it-yourself revision course. Decimal fractions are considered in Chapter 9; the introduction to these is more detailed than is the custom and some teachers may consider it to be somewhat overdone. These teachers can, of course, omit the careful build-up, but this, in our experience, is certainly necessary for most children in an average class. Chapter 14 does not introduce any new techniques of arithmetic but stresses thought processes; simple flow diagrams, of a type which will later be used in problems concerned with computers, are introduced to illustrate these processes and the examples have been chosen so as to make the work relevant to children's own experiences.

Arithmetic is, of course, the supreme mathematical tool, but in Chapters 2 and 3 the pupil encounters two other aids which he will be called upon to use time and time again throughout the course. The first of these is the notation and language of sets. At this stage (and for several years to come) the emphasis is on set-language and *not* on set-theory as that term should be understood. Chapter 2 introduces the basic idea of a set, considers ways of describing sets and discusses possible relations between two sets. New notation has been kept to a minimum; for example, the introduction of the set-union symbol has been left until a later book and the use of the 'set-builder notation' is deferred until Chapter 7, but some elementary work is included on Venn diagrams. Co-ordinates are the second aid just referred to and they are introduced in Chapter 3. The problem of specifying a position in the classroom or on a map leads to the idea of a co-ordinate system and the teacher is then given an opportunity to develop this idea to cover co-ordinate systems in three-dimensional space. In Chapter 3, equations such as $x = 3$ and $y = 4$ are related to straight lines, more complicated equations and relations being deferred until Chapters 7 and 11.

Pure geometry is met for the first time in Chapter 5, and this chapter is not unlike many to be found in traditional courses. Angle is introduced as a turn-measure and plenty of opportunities for discussion and problem-solving are provided before degree-measure is defined. The definition of angle allows the extension of the work of Chapter 3 on co-ordinates, and another feature of this chapter is the introduction of accurate scale drawing which is further developed as part of the work on surveying in Chapter 15.

Geometry is also studied in Chapters 8, 10 and 13. The aim of Chapter 10 is to emphasize understanding of area rather than to teach a number of formulae. One way in which this is achieved is by focusing attention on the region bounded by a curve rather than on the curve itself and by considering primarily regions bounded by non-rectilinear curves. Symmetry, which is studied in Chapter 13, is familiar as a concept but some teachers may not have had much experience in considering how it can be discussed usefully in the classroom or how its mathematical features can best be developed. It is interesting to remark that the Mathematical Association recommended as long ago as 1923 that symmetry should have a larger place in the teaching of geometry; perhaps its introduction has been delayed because of the havoc it can play with 'formal proofs' but, since it is not the policy of the School Mathematics Project to encourage geometrical proofs at the O-level stage, such an objection need no longer deter us. The symmetries studied in this chapter are symmetry in planes and lines and rotational symmetry about axes and points; we wish from the start to help pupils to visualize three-dimensional space. Co-ordinate systems in three dimensions have already been mentioned and more examples designed to develop further appreciation of spatial relationships can be found in Chapter 8. This is very much 'scissors-and-paste' work and there is much fun to be had with the construction of models. We want to integrate such work into the main course rather than to treat it as an end-of-term diversion; if the entertainment-value of this chapter is high, its mathematical value is certainly not low—the two are not inversely proportional!

Finally, some chapters concerned with number. In Chapter 6 number is divorced from computation. Different types of numbers are considered, such as prime, square and triangle numbers, and seeds for later work on sequences and series are sown. Although series are not encountered again in this book, the work on sequences is developed in the following chapter where simple relations between ordered pairs are used to introduce algebraic relations. In addition to representing linear relations graphically, simple ideas of ordering are established. This work on graphs is extended in Chapter 11 to include the algebraic formulation of day-to-day problems which give rise to linear relationships and this in turn is used as a stepping stone to Chapter 12, where negative numbers are introduced. There is nothing novel in this lead-in to negative numbers but we have adopted notation which distinguishes between negative numbers and the operation of subtraction applied to positive numbers; this is not a distinction which is traditionally made, and at first sight it might be thought more to confuse the pupil than help him. Our experience, however, is that this is not the case.

Answers to exercises are not printed at the end of the book but are contained in the companion Teacher's Guide which gives a chapter-by-chapter commentary on the pupil's text.

A NOTE ON THE METRIC EDITION

With the 1969 reprint of this book, some changes have been made in the notation and units used.

(i) All quantities of money have been expressed in pounds (£) and new pence (p).

(ii) All measures have been expressed in metric units. The fundamental units of the Système International (that is the metric system to be used in Great Britain) are the metre, the kilogram and the second. These units have been used in the book except where practical classroom considerations or an estimation of everyday practice in the years to come have suggested otherwise.

(iii) The notation used for the abbreviations of units and on some other occasions conforms to that suggested in the British Standard publications PD 5686: 1967 and BS 1991: Part 1: 1967.

Where units and numbers have been changed in the texts, the corresponding changes in the Teacher's Guides have been listed in a small leaflet which will be available with the present Guides. Changes involving notation only will not be so listed. The contents of the leaflet will be incorporated into the Teacher's Guides when they are next reprinted.

The drawings in this book are by Cecil Keeling

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CONTENTS

1

A NEW LOOK AT ARITHMETIC

One for sorrow,
Two for joy,
Three for a letter,
Four for a boy.

Five for silver,
Six for gold,
Seven for a secret
That's never been told.

1. COUNTING

1.1 Early numerals

Let us suppose that we live in a strange land. We have schools, pencils, exercise books, but no number words (one, two, three) or numerals (number symbols, 1, 2, 3, ...).

Discuss how we would:

(*a*) find out how many exercise books are needed for a class;

(*b*) find out whether more or fewer books are needed for the class next door;

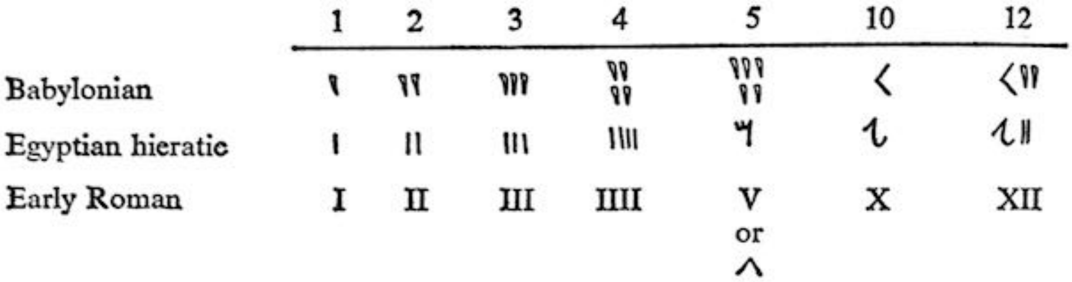
(*c*) record the number of books a class needs, so that we shall know how many to order next year;

(*d*) record the number of children in the whole school of many classes.

Primitive man was in much the same state as the people in the strange land mentioned above. He counted a number of objects using his fingers, by raising one finger for each object. He would record the total by scratching, in the sand, the number of raised fingers or by cutting notches in a piece of wood. He also counted by matching, one by one, each object with a stick. He could record the total by keeping the bundle of sticks.

Number words and numerals provide us with a much simpler way of counting and recording than do raised fingers and sticks.

Many numerals show that they started as pictures of fingers. The table below shows the symbols for the numbers one to five, ten and twelve in some early scripts.



Exercise A

1. Write out an explanation of the meaning of the first four Roman numbers.

2. Look at the fingers of one hand. What do you think the Roman symbol for five may have meant?

3. Suggest a connection between the Roman symbols for five and for ten.

4. Why did special symbols appear for 5, 10, ...?

5. State one way in which you think the Egyptian hieratic notation is better than the Babylonian.

6. What is the later Roman symbol for four? In what way is it better? In what way is it worse?

7. Write the numbers 6 and 14 in all three scripts.

8. Does the order of the symbols matter in (*a*) Babylonian, (*b*) Early Roman?

9. Write out the number

. . . . .
. . . . .
. . . . .
. . . .

in Babylonian, Early Roman and later Roman.

10. Record, in all three scripts, the number of people in your house.

1.2 The Hindu–Arabic system

Our own counting system is called the Hindu–Arabic system.

Our system has the very important property that the position in which a numeral occurs can affect its meaning. This means that we can write numbers as large as we like without having to introduce new digits. Compare our number 1966 with the Roman figures for the same number. Our system also makes arithmetic very much simpler, as we shall see later.

Exercise B

1. Find out how many different words we use in writing down the numbers from one to ninety-nine. (Ninety-nine consists of two words.)

2. Our own numerals are, of course, 1, 2, 3, 4.... Other people use different ones. The Arabs write ١, ٢, ٣, ٤. What other system of numerals do we sometimes use? Give examples of where we use this system.

3. Find out how many different numerals we use in writing down the numbers from 1 to 99. (99 consists of the same numeral, 9, written twice.)

4. Telephone numbers are usually said in a special way. Find out in what way. Explain how we could write down the number words from one to ninety-nine using fewer words than in Question 1. Which words could we do without? Is there any reason why we should not all use this system?

5. What is a teen-ager? What does teen mean? Find out the French and German words for 18. Break the words into parts, what exactly do they mean? Compare them with the English word for 18. Is there any difference?

6. Compare the words for 12, in English and French, with the words for 18. What do you notice? What extra word have we for 12?

7. What is peculiar to the French words for the numbers from 60 to 79?

8. What is special about the way we write ten compared with all the previous numbers? What is the next number that is special in the same kind of way?

9. Write this number of dots

. . . . .
. . . . .
. . . .

in words and symbols and explain their meaning.

10. What does 8 mean in (*a*) 28, (*b*) 84, (*c*) 853?

11. Does it matter in what order the digits of our numbers are written?

1.3 The spike abacus

Figure 1 shows a spike abacus. Discs with holes through them are placed on the spikes. You could make one very easily, or you may have a child's toy like this at home. It is a practical (and very ancient) way of representing numbers and doing sums. Other versions of the abacus are still in use in parts of the East.

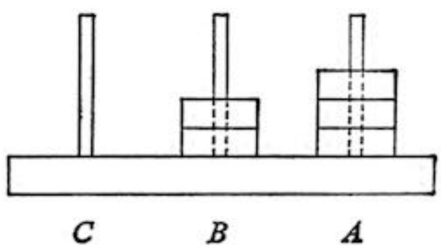


Fig. 1

Only a certain number of discs will go on each spike. In the abacus shown in Figure 1, the number is five. We will agree that we will count 1, 2, 3, 4, 5 by placing discs on spike *A*. We will represent 6's by discs on spike *B*. For example, one disc on *B* and two discs on *A* will represent one six and two, that is, the number 8, and so on.

(*a*) What number is represented in Figure 1?

(*b*) What is the largest number that can be represented, using spikes *A* and *B*?

(*c*) What do you think a single disc on *C* would mean?

(*d*) What is the largest number that could be represented by discs on all three spikes?

(*e*) What would we have to do, if even larger numbers were required?

(*f*) Sketch a spike abacus with three discs on *C*, four on *B* and none on *A*. What number is it representing?

Let us find out how to represent a number, say 29, on the abacus. We must find out how many 6's it contains so that we know how many discs to put on *B*.

$$6\overline{)29}$$
$$4 \text{ remainder } 5.$$

Figure 2 shows how the abacus would look.

Sketch an abacus showing 18 and one showing 34. What do you find when you try to represent 43?

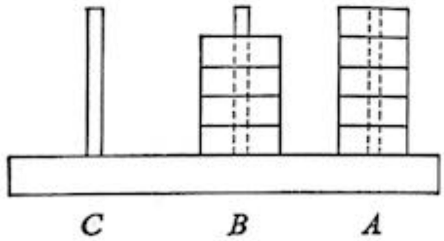


Fig. 2

1.4 Place value

Drawing a picture is not a convenient way of recording the number shown on an abacus. Let us try to find an easier method.

What does 18 *mean*? Does it mean $1+8$, 1×8, $1\div8$? None of these is right is it? You have written many times:

tens	units
1	8

In fact you know that 18 means 'one ten and eight' or $(1\times10)+8$. Similarly, 36 means 'three tens and six' or $(3\times10)+6$, and so on. Write in words and symbols the meaning of 27 and 90.

We are using the *principle of place value*. If we write 3 in the second column it means three tens, or thirty, not 3 as it would in the first column. What does 3 mean in the third column?

We can use this same principle to illustrate numbers on our abacus, provided we are careful to say what we are doing. Look again at Figure 1. It represents the number 15, but what it shows is $(2\times6)+3$. Suppose we write this as 23. We shall mean 'two sixes and three' and not 'two tens and three' as in ordinary arithmetic. We can do this by writing the *number base* after the number and a little lower, like this:

$$23_6 \text{ means } (2\times6)+3,$$

$$23_{10} \text{ means } (2\times10)+3.$$

Note. For clarity we shall write the number base in base 10.

A number written after another number and a little lower down is called a *suffix*. We use the suffix 6 to show that the number is to be read to base 6 (some people say in the scale of 6). Our ordinary numbers are to base 10. We call this the *decimal* system. Some people call the numbers *denary* numbers but we shall use the word decimal.

Example 1. Convert into decimal 45_6.

$$45_6 \text{ means } 4\ 6\text{'s and } 5,$$

that is,

$$\begin{aligned} 45_6 &= (4\times 6)+5 \\ &= 24+5 \\ &= 29_{10}. \end{aligned}$$

Example 2. Convert 19_{10} into base 6.

$$\begin{array}{r} 6\underline{)19} \\ 3 \end{array} \quad \text{remainder } 1,$$

$$\begin{aligned} 19_{10} &= (3\times 6)+1 \\ &= 31_6. \end{aligned}$$

Exercise C

1. Convert into ordinary decimal (denary) numbers: 15_6, 30_6, 23_6, 2_6.
2. Convert into base 6: 9_{10}, 19_{10}, 32_{10}, 5_{10}.
3. What does the 2 mean in 12_6, 24_6?
4. Write down in words the meaning of 33_6 and convert it into decimal.
5. Write the number in decimal and in base 6.

   . . . . . .
   . . . . . .
   . . . . . .
   . .

6. Write down in words the meaning of 40_6. Why is the 0 there?
7. Convert into decimal: 114_6, 231_6, 104_6, 540_6.
8. Explain the difference between 14_6 and 41_6.
9. If the spike abacus held only 4 discs, what number would be represented by 2 discs on *B* and 3 on *A*?
10. Write in words the meaning of 15_7, 123_4, 20_8.
11. Convert into decimal: 33_5, 14_5, 10_5.
12. Convert into base 5: 18_{10}, 11_{10}, 20_{10}, 26_{10}.
13. Convert into decimal: 13_4, 25_7, 53_8, 108_9.

14. Convert 25_{10} into base 5, base 6, base 11.

15. Convert 53_7 into base 5.

16. Convert 112_3 into base 9.

17. If we break the usual law and say that the number pq shall mean the number whose first digit is p, and whose second digit is q, write out the meaning of pq_6.

18. Using the same special convention as in Question 17, write down the meaning of abc_8. What can you say about the magnitude of a, b, c

2. NON-DECIMAL CALCULATION

The metric system to which Britain is changing is a system which works in base 10. But in the past we have used a mixture of bases such as:

12 pence = 1 shilling (money);
20 shillings = 1 pound (money);
16 ounces = 1 pound (mass);
14 pounds = 1 stone (mass);
8 pints = 1 gallon (capacity).

Can you think of measures which we use today which are still not in base 10? We shall make use of some of these old measures (called 'imperial') in this chapter.

2.1 'Carrying'

(*a*) Copy down the following additions:

(i) $\begin{array}{r} 47+ \\ \underline{39} \\ \underline{86} \end{array}$ (ii) $\begin{array}{r} 47+ \\ \underline{39} \\ \underline{84} \end{array}$ (iii) $\begin{array}{r} 47+ \\ \underline{39} \\ \underline{80} \end{array}$

Tick the one that looks correct. Rewrite (ii) with shillings heading the first column, pence the second. The numbers have not changed, but is the addition now correct? Write down why. Rewrite (iii) with headings to the columns so that it, too, is correct. Why will shillings and pence not do?

(*b*) The answer to the sum $\begin{array}{r} 25+ \\ \underline{29} \end{array}$

depends on what we mean by 25 and 29. 25 could mean 2*s*. 5*d*., 2 stone 5 lb., 2 ft. 5 in., or it could be a decimal number. Copy the sum down twice and, by giving difference meanings to the columns, obtain two different answers.

(*c*) Changing the headings does not always change the answer. Copy the following sum down twice $\begin{array}{r} 23+ \\ \underline{42} \\ \underline{\quad} \end{array}$

Think of the columns are meaning first, years and months, secondly, gallons and pints. Write down the reason why your two answers contain the same figures.

You will have realized by now that we are discovering the facts about 'carrying'. In ordinary arithmetic (decimal), we carry if the total of a column is ten or more. This is why it is called arithmetic to base 10, or in the scale of 10. If the total of a column is 13, we have a 'put down digit', which is 3 (units), and a 'carry digit', which is 1 (ten). The 'carry digit' has to be added to the next column to the left.

In shillings and pence arithmetic, we carry if the total of a column is 12 or more. If the total of a pence column is 13 we have a 'put down digit' of 1 (penny) and a 'carry digit' of 1 (shilling) to be added to the next (shillings) column on the left. This is arithmetic to base 12. It is sometimes called *duodecimal* arithmetic. What units apart from shillings and pence are worked in duodecimal? We shall pay particular attention to this important scale later.

Exercise D

1. Copy and answer:

(*a*)

ft.	in.
2	7+
5	8

(*b*)

s.	*d.*
5	8+
2	7

(*c*)

yr.	mth.
2	7+
5	8
	9

2. Add 5 weeks 3 days, 1 week 2 days and 1 week 6 days.

3. Obtain two different answers to the sum $24+35$ by giving different meanings to the columns.

4. Make up a problem with units to illustrate that $35_8+17_8 = 54_8$.

5. Work out the following:

£	*s.*
5	8+
2	7

What do you notice if you write this down without column headings?

6. What is wrong with this: 5 weeks 4 days plus 3 weeks 6 days come to 9 weeks 3 days; so $54_7+36_7 = 93_7$?

7. What is wrong with writing 11*s.* 4*d.* as 114 in base 12?

8. Find column headings to make the following multiplications correct:

(*a*) $\begin{array}{r} 13\times \\ 3 \\ \hline 42 \end{array}$ (*b*) $\begin{array}{r} 13\times \\ 3 \\ \hline 41 \end{array}$

9. Here are two ways of doing a subtraction in decimal:

(*a*) $\begin{array}{rr} \not{3}^{2} & 5- \\ 1 & 6 \\ \hline 1 & 9 \end{array}$ (*b*) $\begin{array}{rr} 3 & 5- \\ \not{1}^{2} & 6 \\ \hline 1 & 9 \end{array}$

Write down the reason for the crossing out in each method. Why do both give the same answer?

10. Subtract 5 gallons 6 pints from 7 gallons 3 pints, explaining what you have to 'borrow'.

11. Find column headings to make the following subtractions correct:

$$(a)\quad \begin{array}{r} 64- \\ 58 \\ \hline 8 \\ \hline \end{array} \qquad\qquad (b)\quad \begin{array}{r} 71- \\ 42 \\ \hline 22 \\ \hline \end{array}$$

2.2 Martian arithmetic

The Martians have learned our numerals, 1, 2, 3, and so on, but every time they send sums over the inter-planetary radio link they seem to get the answers wrong! Many of the sums use extra symbols that we do not recognize. This one is composed of symbols we do know $18+9 = 22.$

How many fingers (or tentacles) do you suppose a Martian has? What makes you think that they do not have 10? Suppose that they have 14. Translate the above sum into numbers base 14 and see if this makes it right.

Do you agree that it does not? Presumably, however, by 18 they mean 'one something and eight'. Translate the above sum into words again using base 'something'. Try 11, 15, 18 for 'something'. How many fingers do they have?

Exercise E

1. Turn the Martian numbers 46, 81 into Earth numbers.

2. Turn the Earth numbers 34, 65 into Martian numbers.

3. Make up another Martian sum, with its answer.

4. On Venus 24 means $(2\times 25)+4$. What number base are they using? Turn the Venusian number 51 into an Earth number. Make up a correct Venusian subtraction.

5. What does 27 mean on Mercury if $27+8 = 32$ there?

6. Try to turn the Earth number 28 into a Martian number. What do you find? How does this help to explain the symbols we do not recognize.

7. Try to turn the Earth number 28 into a Venusian number. Do we now require extra symbols?

8. Why would you prefer to learn your multiplication tables on Earth rather than on other planets?

9. Martian ray guns cost 39 copliks each. How much do two cost? (39 is in Martian numbers.)

10. A flying saucer costs 490 copliks (490 is in Martian numbers, too). A Martian has saved 489 copliks. How many more copliks has he to save?

2.3 Counting in twelves

A number system of particular interest is the *duodecimal*, or base 12. What language does the word come from? What does it mean? Many people think that it would be more convenient if we always counted in twelves. In Britain, there is a Duodecimal Society whose members try to convince the general public of this.

We have already noticed that the word 'twelve', like the French and German words for ::::, is rather peculiar. After all it might have been called 'twoteen'! You have also noticed that there were no less than three sets of British units which were in the scale of 12. Which are they? Count the number of knuckles you have on the fingers (not thumbs) of one hand. Could this have anything to do with it? What?

We are going to study duodecimal numbers rather more deeply now, and we start with some worked examples.

Example 3. Convert 245_{12} into decimal.

$$\begin{aligned} 245_{12} &= (2\times 144)+(4\times 12)+5 \\ &= \quad 288 \quad + \quad 48 \quad +5 \\ &= 341_{10}. \end{aligned}$$

Example 4. Convert 245_{10} into duodecimal.

As usual, we find the number of 12's by division,

$$\begin{array}{rrr} 12)\underline{245} & & \\ 12)\underline{\ \ 20} & \text{remainder} & 5 \\ 12)\underline{\ \ \ 1} & & 8 \\ 0 & & 1 \end{array}$$

$$245_{10} = (1\times 144)+(8\times 12)+5 = 185_{12}.$$

We have done one more division than is strictly necessary (1 divided by 12), but this makes it easier to see that we can read the duodecimal number from the bottom, up.

To understand why, compare the same work done in decimal

$$\begin{array}{rrr} 10)\underline{245} & & \\ 10)\underline{\ \ 24} & \text{remainder} & 5 \\ 10)\underline{\ \ \ 2} & & 4 \\ 0 & & 2 \end{array}$$

$$245_{10} = (2\times 100)+(4\times 10)+5.$$

Exercise F

1. Convert the following duodecimal numbers into decimal: 34, 77, 116.
2. Convert the following decimal numbers into duodecimal: 25, 39, 113.

3. Try to turn 47_{10} into duodecimal. What do you find?
4. Try to turn 130 into duodecimal. What do you find?
5. What numbers have the same meaning both in decimal and duodecimal?
6. How many discs would a duodecimal abacus hold on a spike?

2.4 New numerals

(*a*) In Questions 3 and 4 of Exercise F, we found it difficult to express the numbers 10_{10} and 11_{10} in duodecimal. Explain why we cannot write them as 10_{12} and 11_{12}.

(*b*) We noted earlier that one way to build up duodecimal arithmetic, would be to count on the knuckles of the fingers of one hand.

Make a rough copy of Figure 3.

Starting at the lowest knuckle of the little finger, number the knuckles 1, 2, 3, 4, 5, 6, 7, 8, 9. Explain why the upper knuckle of the first finger must be numbered 10 in duodecimal.

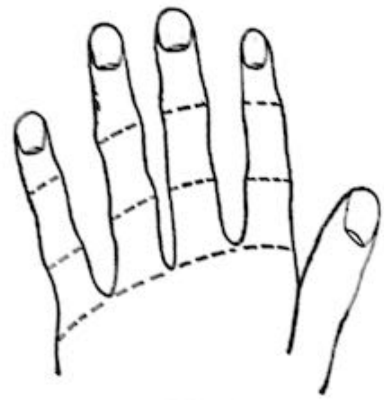


Fig. 3

(*c*) What shall we have to do about the remaining knuckles? Do some inventing!

(*d*) Use a different coloured ink, or re-copy the diagram, to number the knuckles from 11 to 20 in duodecimal. What shall we have to number the two remaining knuckles this time?

It is obvious that we need two new symbols (or numerals) and names for the numbers 10_{10} and 11_{10} in duodecimal.

Suppose we call them @ (pronounced 'at') and * (pronounced 'star'), then we can write $10_{10} = @_{12}$ and $11_{10} = *_{12}$.

Note. These new numbers may be given *any* two symbols you like. *Do not* memorize these particular ones.

We count our knuckles 1, 2, 3, 4, 5, 6, 7, 8, 9, @, *, 10. We have to be careful, of course, not to read 10 as 'ten'. It means $(1 \times 12)+0$, and could be read as 'one nought' or as 'dozen' or even 'duz', if you like new words!

Exercise G

1. Convert into decimal: 2@, 3*, @@, 5*, *00.
2. Convert into duodecimal: 23, 34, 99.
3. Write out the duodecimal numbers from 11_{12} to 20_{12}, and from 41_{12} to 50_{12} using @ and *.
4. We could read 11_{12}, 12_{12}, 13_{12}, ... as 'one one', 'one two', 'one three', and so on, or we could invent some new words such as 'duzone', 'duztwo', 'duzthree'. What might we call 20_{12}? Invent some words for 50_{12}, 74_{12}. (*The Language of Mathematics*, by Frank Land, which has new words for base 8 numbers, might help you.)
5. What word do we already know for 100_{12}?
6. Invent a new word for 1000_{12} and use it to write, in words, 1234_{12}.

2.5 Addition and subtraction

In Sections 2.5 and 2.6 all numbers are duodecimal unless otherwise stated

Example 5. Calculate 16*5 + 359.

We set down the 9 under the 5 (not the 3 under the 1—why?), as in (*a*). Remember that 5 + 9 = (1 dozen and 2) = 12. Write down 2, as in (*b*), and carry 1 dozen into the dozens column.

```
(a) 16*5+          (b)   1
     359               16*5+
    ----                359
                        ----
    ----                   2
                        ----
```

Now 1 + * + 5 = 15. Write down 5, as in (*c*), and carry a further 1 into the dozen dozens column. Finally, 1 + 6 + 3 = @, and the sum is done.

```
      1
(c) 16*5+          (d) 16*5+
     359                359
    ----               ----
      52               1@52
    ----               ----
```

Example 6. Calculate 41@2 − 345.

Again we set the 5 under the 2, since the units figures must come in the same column,

```
41@2−
 345
----
   ?
----
```

Since 2 − 5 is impossible here, we must 'borrow' 1 dozen and say (1 dozen and 2) − 5 = 9. We have to 'pay back' this dozen. You may be used either to taking 1 from the dozens figure at the top, or to paying back 1 to the dozens figure at the bottom. We show both methods. Why does it not matter which method you use?

```
(a) 41 ⁹@ 2−       (b) 41@ 2−
    3 4  5              34⁵5   (4 struck out)
    ------             -----
    . .  9              ... 9
    ------             -----
```

The remainder of the work is easily done, it goes

```
41@2−
 345
----
3@59
----
```

Example 7. We can use duodecimal to help work in shillings and pence, feet and inches and so on. For example, 5*s*. 11*d*. + 4*s*. 5*d*. + 6*s*. 10*d*. can be written, in pence, as

```
 5*+
 45
 6@
----
152
----
```

Remember $15_{12} = 17_{10}$. The answer is 17*s*. 2*d*.

Exercise H

Calculate:

1. 15+19.
2. 1@+19+142.
3. 5618+*21@.
4. *4+1@40+@.
5. 97−34.
6. 197−39.
7. 16*−88.
8. 2×*.

Use duodecimal notation to work out the following old-unit problems (given in decimal):

9. 3*s.* 8*d.*+5*s.* 9*d.*

@. 8*s.* 11*d.*+4*s.* 3*d.*+12*s.* 9*d.* (be careful!).

*. 15*s.*−9*s.* 4*d.*

10. 7 ft. 8 in.+3 ft. 4 in.+1 ft. 3 in.

11. 14 ft. 3 in.−6 ft. 11 in.

12. Make up and answer another easy example in shillings and pence.

13. Correct, if necessary,

```
16@2+
 374
 119
----
1*83
```

14. In self-service stores and elsewhere, you will often see the bill being added up on a small £ *s. d.* adding machine. Could you use this for duodecimal addition?
Why would you have to be careful?

15. Make up two sums in base 12, one of which could be added on an £ *s. d.* adding machine and the other of which could not.

16. Make up a question, whose answer is 4, using each of the numbers 8, 9, @ and * once only.

2.6 Multiplication

It is easy to work out additions in the duodecimal system in one's head. If we wished to carry out multiplications, though, it would mean learning a new set of tables. Would this set of tables be more complicated than those for multiplication in base 10?

For simplicity, we shall only use the two times and three times tables. Before reading further, you should write out these two tables. You can use them for checking the worked example below, and for doing the problems in Exercise I.

Example 8. Calculate 2154×2↓3.

Taking care to write the units figure of the multiplier (marked with an arrow) under the last figure of the number to be multiplied, we write

```
2154×
  23
----

----
```

Does it matter whether we multiply first by the 2 or by the 3? When we multiply by the 2, we insert a 0 in the units column, and start underneath the 2. Why? The calculation is

```
 2,154 ×
    23
 ------
 6,440
42,@80
------
49,300
------
```

Why did we not write

```
  23 ×
2154
----
---- ?
```

Does it surprise you that the answer ends in 0? What sort of multiplications in decimal give an answer ending in 0? What other sort of multiplications in duodecimal will have answers ending in 0?

Exercise I

1. Calculate:

(*a*) 151×12; (*b*) 151×10; (*c*) 347×13;
(*d*) $1{*}3 \times 220$; (*e*) 208×302.

2. A boy had to calculate $3189 \times 42{*}$. By mistake, he worked out $3199 \times 42{*}$. By how much was his answer too large? (Obtain the answer without working out both multiplications.)

2.7 Spiders' arithmetic

Intelligent spiders would count in 8's, of course. This is called *octal* arithmetic. Here are some spiders' problems for you to do.

Exercise J

All numbers are in octal, unless otherwise stated

1. Convert into decimal: 10, 25, 361.

2. Convert into octal: 41_{10}, 519_{10}.

Calculate:

3. $16+25$.
4. $143+2060$.
5. $45-30$.
6. $512-407$.
7. 64×2.
10. 315×22.
11. 41×100.
12. $63+27-14-7$.

13. Flies cost 3r. each, bees cost 12r. each. Find the cost of a spider's dinner consisting of 6 flies and 4 bees.

14. Correct, if necessary:

(*a*)
```
 451 −
 276
 103
```

(*b*)
```
  256 ×
   23
 1012
 5340
 6352
```

15. What multiples of numbers (octal) give answers ending in 0?

16. How many days are there in a spider's year? A spider's week is 10 days. How many full weeks are there in a year?

17. In King Spider's army there are 1000 soldiers. Every fourth soldier has lost a leg. How many legs does his army have altogether?

20. 1 gallon = 8_{10} pints. A certain school takes 3 gallons 5 pints of milk each day. Working in octal, calculate how many gallons and pints are needed for a five day week. Give your answer in decimals.

21. Electronic machines now exist which can count events that occur very rapidly (like the hits of atomic particles on a screen). They usually give the total by lighting up the correct numbers on a set of dials, which give the answer in octal. Octal counting makes for easier electrical circuits.

Figure 4 shows the dials at the end of a count. Dial *A* counts in units, *B* in 8's, *C* in 64's. What does *D* count in?

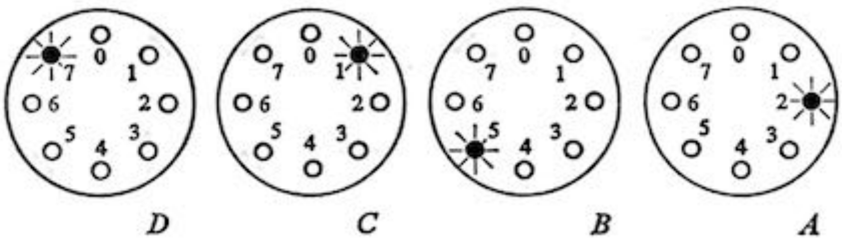


Fig. 4

Write down the reading shown in Figure 4, in octal and decimal. Give, in decimal, the largest number these dials can count.

3. THE LANGUAGE OF 0 AND 1

3.1 Computers

The most revolutionary invention of our time is the electronic computer. You may feel inclined to disagree and want to nominate the television set, the atomic bomb or the space satellite. A revolution, however, changes things. Nothing is changing our world, in as many different ways, as the computer.

A computer enables us to solve problems. The problems can be enormously complicated, but the machine does not mind. Business men can solve problems to tell them where to build factories, how to plan them, how many lorries they will need for transport. Weather forecasters are improving their ability to predict the weather. Special computers can be designed to train airline pilots on the ground with an enormous saving, in money and sometimes lives. Engineers use computers to assist them in designing aircraft, and, incidentally, television sets, atomic bombs and space satellites. Manufacturing processes can actually be controlled by computers; they guide the machine tools, correct slight errors, test the final products and reject any which are not of a sufficiently high standard. This is called automation.

The computer does what it is told to do, nothing more nor less. The thinking has still to be done by men, so has the talking. The latter is done by coding instructions as punched holes in cards or a reel of tape. All the skilled work is done on a problem before the computer receives it. This is (*a*) putting the problem into mathematical form, and deciding what sort of sums are needed to solve it; and (*b*) punching the instructions. The demand for men and women who can do the thinking work will grow enormously as more computers are built.

The speed of calculation is very high. How long would it take you to multiply 869 531 047 by 375 493 248? Quite an ordinary computer could do 1000 calculations like this in one second!

The work of a computer is to add and subtract. Multiplication is just repeated addition. Division is done by continued subtraction. Essentially, the business of preparing a problem for a computer to solve consists of splitting it into lots of small steps, consisting mainly of addition or subtraction sums.

You will learn something about hand computers in Chapter 9.

3.2 Binary numbers

Electronic computers work, of course, by electricity. A lamp is either on or off, according to whether the current is, or is not, flowing. The flow is controlled by a switch, which can break the circuit.

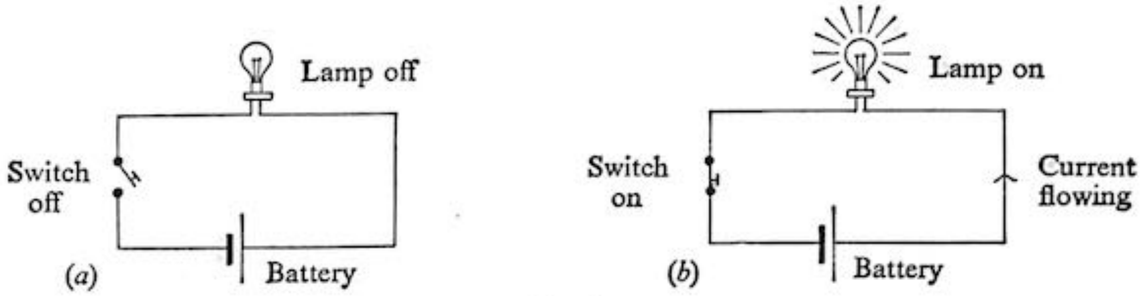


Fig. 5

Figure 5(*a*) shows the circuit with no current flowing. In a computer this can represent the number 0. Figure 5(*b*) shows current flowing; this can represent the number 1. No other numbers need be used. The arithmetic uses only the numbers 0 and 1. This is *binary arithmetic*, or arithmetic to base 2. Remember that just as in base 12 there is no symbol for 12, so in base 2 there is no symbol for 2.

In binary $10_2 =$ (one two and nought) $= (1 \times 2) + 0 = 2$,

$11_2 =$ (one two and one) $= (1 \times 2) + 1 = 3$,

$100_2 =$ one four $= 4$, and so on.

The columns will be headed as follows:

...	Eights	Fours	Twos	Units
	1	1	0	1

The number shown is $(1 \times 8) + (1 \times 4) + (0 \times 2) + 1 = 13_{10}$.
In lamps, this would look like Figure 6 (reading from left to right).

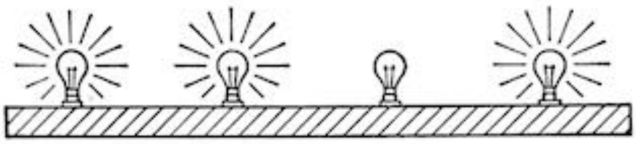


Fig. 6

In simple home-made computers, numbers are represented by lamps in this way. In large computers, the work is done with valves or transistors instead of switches. The numbers are not now represented by lamps, but the ideas are much the same.

3.3 Home-made computers

Figure 7 shows a very simple electrical device you could make for yourself. You need a battery, wire and (say) four on–off switches and lamps with holders.

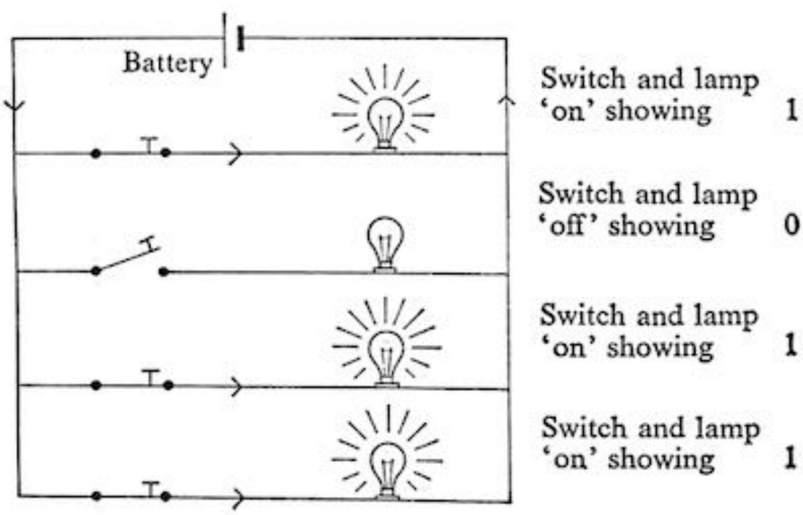


Fig. 7

You must make up your mind which way up you want to read the binary number represented. In Figure 7, the number represented could be either 1011_2 or 1101_2. Convert both of these into decimal. Sketch the device when it is showing 5_{10} and 13_{10}. What is the largest decimal number that can be represented?

A small computer that will add two single-digit binary numbers together, can also be made. The circuit is simple. Figure 8 shows $0+1=1$.

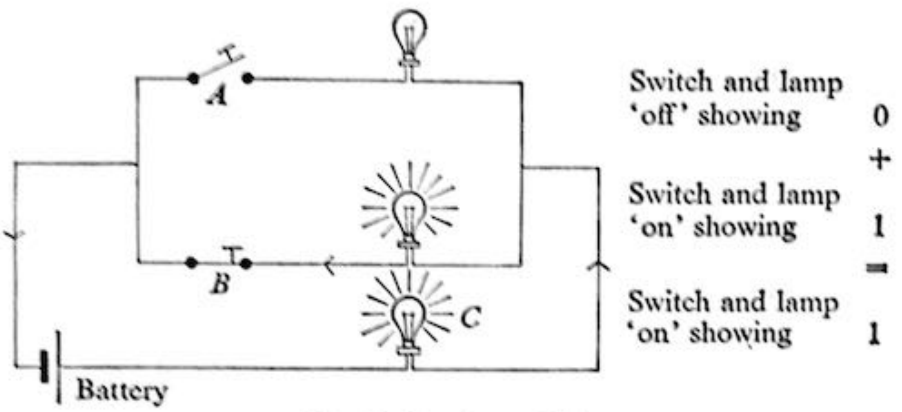


Fig. 8. Put down digit.

This circuit does not do anything special when both switches are in the 'on' position, in fact, it suggests that $1+1=1$, whereas we know that in binary $1+1=10$. To represent 10 requires two lamps, one to show the 'put down digit' and one to show the 'carry digit.'

The arithmetic is very simple. Here it is:

Number set on *A*		Number set on *B*		Solution read at *C*
1	+	0	=	1
0	+	1	=	1
0	+	0	=	0
1	+	1	=	*

* Cannot be read from this computer.

Figure 9 shows a circuit to take care of the carry digit only. It again shows $0+1$ and, of course, there is no carry digit.

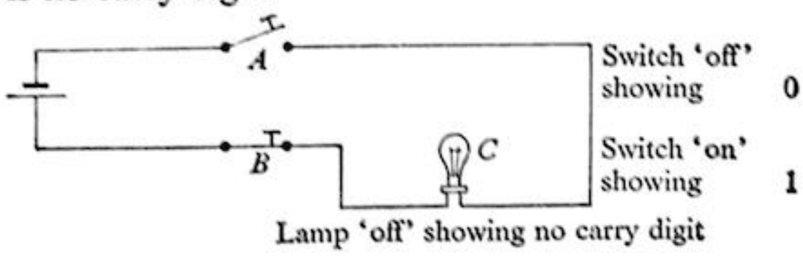


Fig. 9. Carry digit.

These circuits cannot be combined very satisfactorily because when the first circuit is set for $1+1$ it does not give 0 as the put down digit. The changes needed to take care of this are rather complicated. A circuit which adds and carries can be found in *Mathematical Models* by Cundy and Rollett.

3.4 Binary arithmetic

Example 9. Calculate 1011 + 110.

Remember the usual carrying rule, and that $1+1 = 10$

$$\begin{array}{r} 1011\,+ \\ 110 \\ \hline 10001 \\ \hline \end{array}$$

Example 10. Calculate 1011 − 110.

Remember, when we come to $0-1$, we must 'borrow'

$$\begin{array}{r} 1011\,- \\ 110 \\ \hline 101 \\ \hline \end{array}$$

Example 11. Calculate 1011 × 110.

Use the usual layout for long multiplication and remember the importance of starting each line under the number we are multiplying by.

$$\begin{array}{r} 1011\,\times \\ 110 \\ \hline 000 \\ 10110 \\ 101100 \\ \hline 1000010 \\ \hline \end{array}$$

Binary multiplication is extremely easy since the only numbers to multiply by are 1 and 0. In fact, there are virtually no multiplication tables at all: it is scarcely necessary to learn the one times!

Let us check our answer by converting the whole problem into decimal in the usual way.

$$1011 = (1\times 8)+(0\times 4)+(1\times 2)+1 = 11_{10},$$

$$110 = (1\times 4)+(1\times 2)+0 = 6_{10},$$

$$1011\times 110 = 11_{10}\times 6_{10} = 66_{10}.$$

Check that $1000010_2 = 66_{10}$ for yourself.

The figures in a binary number are called binary digits, or bits in computer language. Although the arithmetic of binary is so simple, you will have noticed a snag. It took 7 bits to write the number 66. It will take many more to represent 100_{10} and 1000_{10}. Find out how many bits are needed. This fact is of small importance in a machine, but it makes paper arithmetic rather tedious.

Exercise K

All numbers are in binary, unless otherwise stated

1. Calculate:

    (*a*) 1111 + 1001; (*b*) 1001 + 10101;
    (*c*) 1110 + 101 + 10010; (*d*) 1 + 11 + 111 + 1111.

10. Calculate:

    (*a*) 111 − 101; (*b*) 1000 − 110;
    (*c*) 10110 − 101; (*d*) 101011 − 11011.

11. Calculate:

    (*a*) 110 × 11; (*b*) 10110 × 101;
    (*c*) 10110 × 1000; (*d*) 111 × 111.

100. Calculate the following in decimal; then convert into binary and repeat the calculation, checking your answer by converting back into decimal.

    (*a*) $3_{10} + 7_{10}$; (*b*) $11_{10} + 6_{10} + 13_{10}$;
    (*c*) $17_{10} - 9_{10}$; (*d*) $15_{10} \times 9_{10}$.

101. Convert 11011 into decimal. Hence find its factors and express them in binary. Check your answer by multiplication in binary.

110. Illustrate 1011001 in lamps.

111. Arrange the following numbers in order of increasing size: 1010, 1001, 1100, 1011, 1111.

3.5 Binary division

Division, as we have already noted, is really repeated subtraction. This is obvious when the division is done in binary.

Example 12. Calculate 11011001101 ÷ 101.

We adopt the usual long division arrangement, but the work is simplified by the fact that 101 either divides once, or not at all. Remember positional notation and that we have to insert a 0 if it is necessary to bring down two figures or more. Why might this be necessary, and why the 0?

```
      101011100  remainder 1
 101)11011001101
     101
       111
       101
        1000
         101
          111
          101
           101
           101
             001
```

Exercise L

All numbers are in binary

1. Calculate $101\,101 \div 11$.

10. Calculate $11011 \div 101$.

11. Find a binary number a such that $a \times 1101 = 11010$.

100. Find a binary number b such that $110 \times b = 11001$.

101. Divide $1\,011\,010$ by (*a*) 10, (*b*) 100, (*c*) 110.

110. Write down a division to which the answer is 11.

111. Find three final digits of the number 110010... if it is to be exactly divisible by (*a*) 101, (*b*) 111.

Exercise M

Miscellaneous

1. Convert into decimal: (*a*) 26_{12}, (*b*) 109_{12}.
2. Convert into duodecimal: (*a*) 19_{10}, (*b*) 111_{10}.
3. If all the numbers are duodecimal, calculate:

(*a*) 17 + 1@; (*b*) 36×2; (*c*) 8*1 − 269; (*d*) 5*@0 ÷ 10.

4. If all the numbers are octal, calculate:

(*a*) $27 + 4 + 116$; (*b*) 373×2; (*c*) $1516 - 247$.

5. Convert: (*a*) 10110_2 into octal, (*b*) 253_8 into binary.
6. Calculate: (*a*) $101\,110_2 - 110_2$, (*b*) $110111_2 + 1011_2$.
7. Calculate $101\,011_2 \times 1101_2$.
8. Convert 2120_4 into base 16.
9. Sketch the dials of an octal counter registering 500_{10}.
10. Find the factors of 31_7, giving your answer in base 7.
11. Add 33_4 to 33_5 and give your answer in base 6.

Exercise N

Harder Miscellaneous

1. Convert 36_{15} to base 13, inventing and explaining any necessary notation.

2. If a mysterious race in the Amazonian jungle calculates that $21 + 14 = 40$, do you think it most likely that they have only one hand, or three? Give your reason.

3. Is there any sort of animal which, if it were intelligent enough, might reckon that $31 - 25 = 5$?

4. Is it true or false that $8_{10} = 10_8$, and that $12_{13} = 13_{12}$?
Find some other similar examples.

5. Calculate, in base 12, 5*1 + 37 − @ and multiply your answer by 3.

6. Can you find any number that cannot be expressed (*a*) to base 10, (*b*) to base 2, (*c*) to base 7?

7. Is it always true that, in binary, any number ending in 0 can be divided exactly by a smaller number ending in 0? Give examples.

8. A chemist has 1 g, 2 g, 4 g, and 8 g masses, but only one of each. Which ones must he place in the scale pan to balance chemicals of mass exactly (*a*) 7 g, (*b*) 11 g, (*c*) 15 g? Explain how he can find any exact number of g up to 15 g. What further mass or masses would he need, to weigh (*d*) 21 g, 22 g, 23 g; (*e*) 37 g, 38 g, 39 g?

9. Say what you mean by an even number. How can you tell an even number from an odd number (*a*) in base 6, (*b*) in base 5?

10. Explain why multiplying by 10 simply 'adds a 0' in any base.

11. Calculate 11×11 where the base is (*a*) 2, (*b*) 5, (*c*) 8. Is there an answer applicable to all bases?

12. Work out the following multiplications in Roman numerals; (*a*) X × V, (*b*) X × X, (*c*) X × L. Without converting to Hindu–Arabic, calculate LXV × XXI by 'long multiplication'.

13. Figure 10 shows a 5-hole computer tape. Each row of holes punched across the tape represents a number from 0 to 9 expressed in binary. The units figure is at the top. The small holes are for positioning only, and the bottom row of holes are used in checking. These should be ignored. Read, from left to right, the section of tape shown.

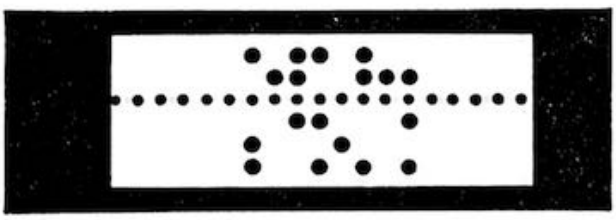


Fig. 10

Draw a portion of tape with holes expressing the number 1728.

14. Write down the single digit numbers which have the same meaning in base 3 as they have in base 5. Are there any two digit numbers with the same property?

15. Convert 189_{10} into (*a*) binary; (*b*) octal.
Set out your answers in a table like this:

	128's	64's	32's	16's	8's	4's	2's	1's
189 in base 2								
		64's			8's			1's
189 in base 8.								

What do you notice? How does this help you to convert other decimal numbers into binary?

16. Convert 5839_{10} into octal and thence into binary.

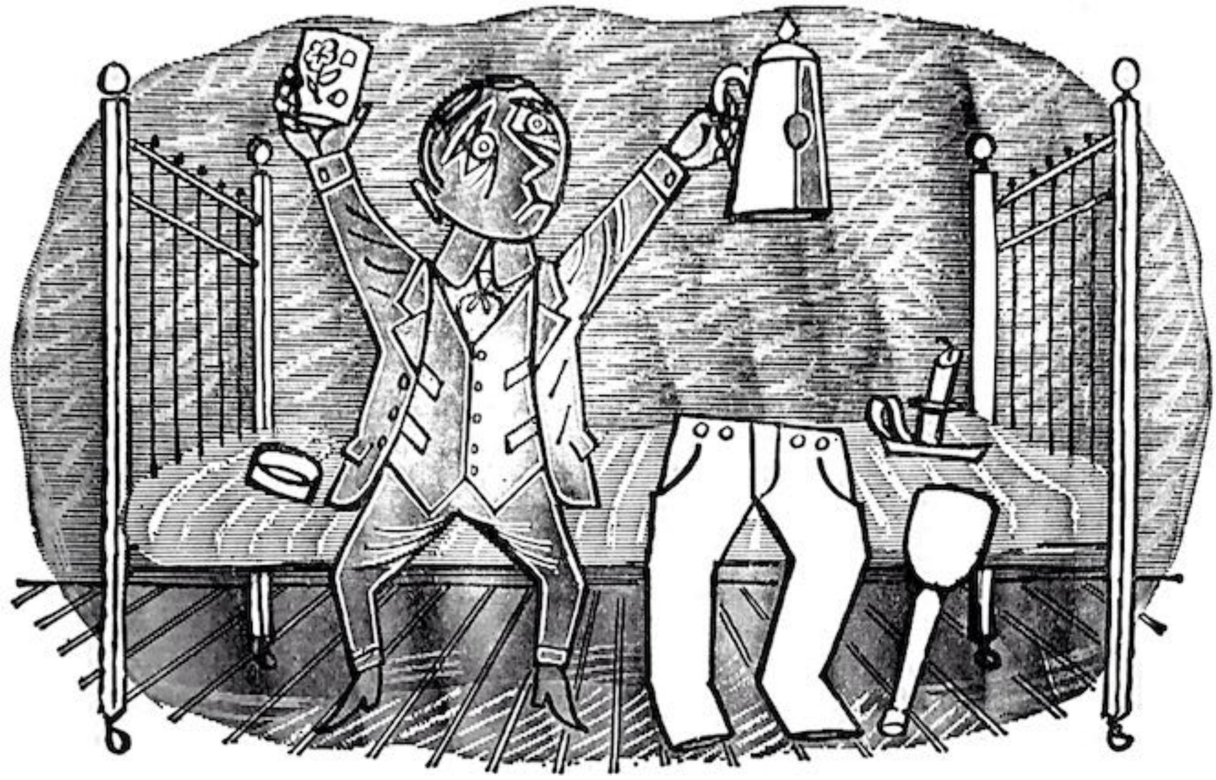
17. Convert 10111001101_2 into octal and thence into decimal.

18. Describe the relation between (*a*) base 2 and base 4, (*b*) base 2 and base 16. Convert 1011011 into (i) base 4, (ii) base 16.

19. Write 2845_9 in base 3.

2

SETS



My aunt she died a month ago
And left me all her riches:
A feather bed and a wooden leg
And a pair of calico breeches.

A coffee pot without a spout,
A mug without a handle,
A baccy box without a lid
And half a farthing candle.

When a boy leaves school to start a job, one of the first things he must do is to learn which are the best tools for his work and how to use them.

In just the same way, when you learn a new subject, whether it is French or science or mathematics, you must first learn about the appropriate tools of the subject. A tool often has many different uses, some of which only become clear with experience and it is not usually necessary to know them all at the start.

This chapter and the next will help you to discover something about two very important tools of mathematics—sets and co-ordinates.

1. SETS

1.1 Sets and members

You are all familiar with the idea of a set in everyday life as a collection of objects, for example, a *pack* of cards, a *herd* of cows, a *fleet* of ships or a *gang* of boys.

(*a*) What special names do we give to: a set of flowers; a set of stamps; a set of geese; a set of porpoises?

The objects in a set may be anything we please. It is important that we describe clearly to *which* set we are referring. For instance, the set of books in your satchel or the set of books in your desk describes exactly *which* set we mean.

The different objects which form a set are called the *members* of the set in just the same way, say, as the people who form a choir are called members of the choir.

(*b*) What are the members of the set of vowels?

(*c*) Describe a set which contains the following as some of its members:

April, August, November.

Because the words 'the set whose members are' will frequently occur, we use a special shorthand: curly brackets.

'The set whose members are the first five even numbers'

is written {the first five even numbers}.

This could equally well be written as

{2, 4, 6, 8, 10}

although, in this case, we should read it as

'the set whose members are 2, 4, 6, 8 and 10'.

(*d*) What are the members of {outdoor games played at your school}?

(*e*) How could you describe {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}?

When a set is to be referred to more than once, it is convenient to give it a label, usually a capital letter. If, for example, we let V stand for the set of vowels, then we write $V = \{\text{vowels}\}$ or $V = \{\text{a, e, i, o, u}\}$.

(*f*) What can you say about the sets A, B and C if

$A = \{\text{apple, orange, banana}\}$,

$B = \{\text{banana, orange, apple}\}$,

$C = \{\text{orange, banana, apple}\}$?

(*g*) If D = {types of fruit}, why is it wrong to say that $D = A$?

(*h*) If $E = \{2, 4, 6, 8\}$, which of the following phrases best describes the members of E:

(i) four even numbers;
(ii) some small even numbers;
(iii) the even numbers between 1 and 9?

There are two distinct ways of describing a set:

(I) by giving a list of its members;
(II) by giving a rule which the members must satisfy.

Care has to be taken when using the second method to make sure that the rule gives only the members required.

For example, if J = {January, June, July} and Y = {months of the year}, then J and Y are not equal since Y contains members (May, for instance) which are not members of J.

Exercise A

1. What are the members of the following sets:

(*a*) {types of British coins in everyday use}; (*b*) {colours of the rainbow};
(*c*) {letters in your surname}; (*d*) {houses in your school};
(*e*) {even numbers between 9 and 17}; (*f*) {subjects on your time-table};
(*g*) {children in your form who wear glasses};
(*h*) {the colours of a set of traffic lights}?

2. Give a rule which describes the members of the following sets:

(*a*) {Sunday, Monday, Tuesday}; (*b*) {2, 4, 6, 8, 10};
(*c*) {sight, hearing, smell, touch, taste};
(*d*) {January, March, May, July, August, October, December};
(*e*) {farthing, crown, guinea};
(*f*) {5, 10, 15, 20, 25};
(*g*) {Hearts, Clubs, Diamonds, Spades};
(*h*) {Pawn, Knight, Bishop, Rook, King, Queen}.

3. In (*a*)–(*f*) say whether or not the two sets are equal.

(*a*) $A = \{1, 5, 9, 13\}$, $B = \{1, 9, 13, 5\}$;
(*b*) $C = \{x, y, z\}$, $D = \{y, z, x\}$;
(*c*) $E = \{1, 3, 5, 7, 9\}$, F = {the odd numbers};
(*d*) G = {the two tallest pupils in your form},
H = {the two oldest pupils in your form};
(*e*) I = {countries in the British Isles}, J = {England, Wales};
(*f*) K = {even numbers between 9 and 17}, $L = \{12, 10, 14, 16\}$.

4. Describe three sets of which you are a member.

5. Give an example of a set which can be described in more than one way.

6. (*a*) If A and B are two sets and $A = B$, is it true that $B = A$?
(*b*) If, in addition, $B = C$ what can you say about A and C?

1.2 Venn diagrams

The pupils in your class can be divided up into sets in many different ways. Each set will have its own particular feature. For example, members of your class who wear glasses form one set, while those with birthdays in March form another.

(*a*) Suppose all the children in your class whose birthday is in March go to the front of the room and stand together. Let this set be called *M*. Now let *S*, the set of children in your class whose birthday is in September, also stand together at the front of the room. If the teacher draws a curve on the floor around each set, what can you say about the two curves? Make a sketch.

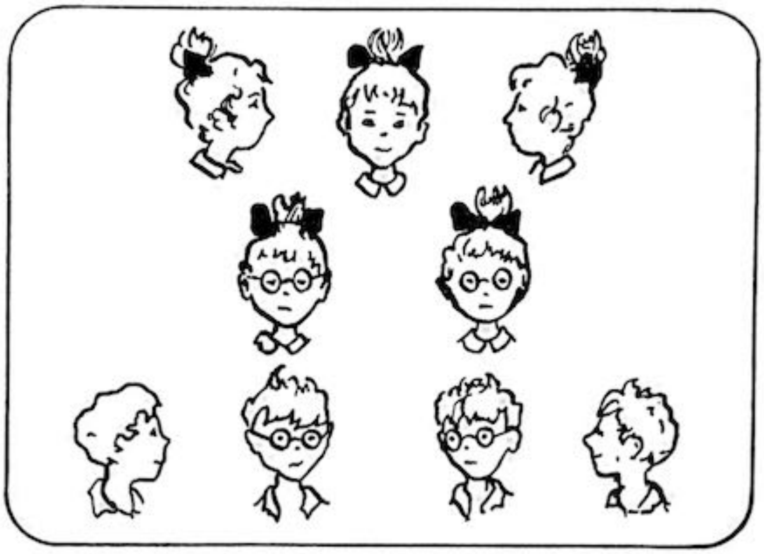


Fig. 1

(*b*) Figure 1 shows the members of a class whose parents own a car—set *C*. Let *G* be the set of children in the class who wear glasses and suppose *G* and *C* go to the front of the classroom. If we draw curves around *G* and *C* what can we say about the two curves?

(*c*) Figure 2 shows the result when the members of a class having blue eyes, *B*, and the members who played chess, *P*, went to the front of the class.

How many members of this class with blue eyes also played chess?

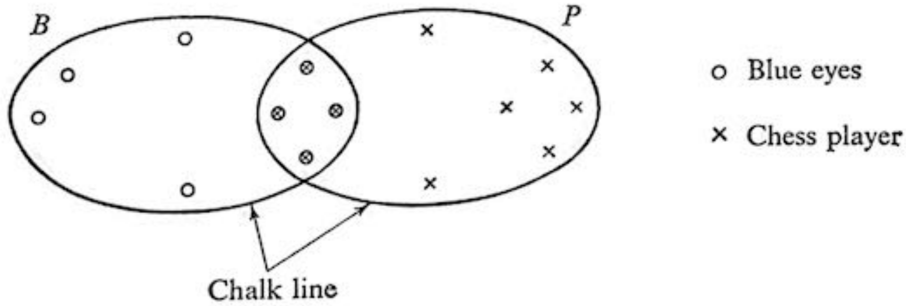


Fig. 2

(*d*) Suggest pairs of sets, whose members are pupils in your class, which would lead to the diagrams shown in Figure 3.

Diagrams like these are called *Venn diagrams* after John Venn (1834–1923), a Cambridge mathematician.

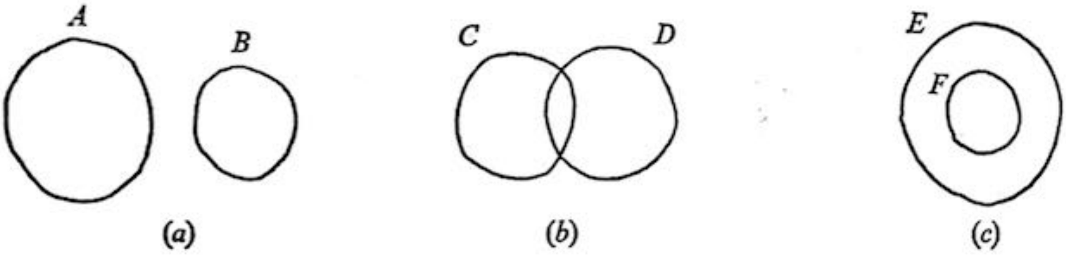


Fig. 3

(*e*) If $G = \{2, 4, 6, 8, 10\}$ and $H = \{1, 2, 3, 4, 5\}$, draw a Venn diagram which would represent the relation between these sets. Shade in the region which represents the members common to both sets.

We call the set whose members are common to the sets G and H, the *intersection* of G and H and denote it by $G \cap H$.

The symbol, which looks like a 'U' upside down, is called 'cap' so that $G \cap H$ is read as 'G cap H'.

In the above example, $G \cap H = \{2, 4\}$.

Example 1. Draw a Venn diagram to represent the sets:

$A = \{\text{Austin cars}\}$ and $B = \{\text{red cars}\}$.

Letter the region representing $A \cap B$ and describe it verbally.

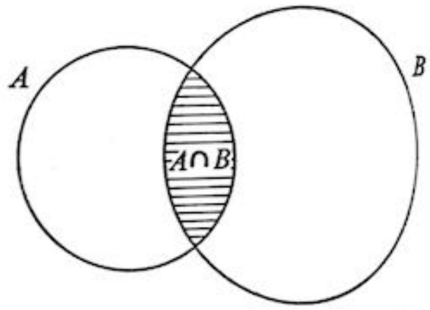


Fig. 4. $A \cap B$ is the set of red Austin cars.

It often happens that two sets have no members in common, so their intersection will have no members. When this happens, the sets are said to be *disjoint*, and their intersection is called the *empty set* and written as { }. (Some books denote the empty set by the Danish letter ø pronounced 'oe'.) For example, if $P = \{\text{pupils}\}$ and $T = \{\text{teachers}\}$, then P and T are disjoint and $P \cap T = \{\ \}$.

Exercise B

1. Illustrate the relation between the following pairs of sets by drawing Venn diagrams:

(*a*) $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$;
(*b*) $A = \{\text{vowels}\}$, $B = \{\text{the first five letters of the alphabet}\}$;
(*c*) $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 5, 3\}$;
(*d*) $A = \{\text{letters in your surname}\}$, $B = \{\text{letters in your Christian names}\}$;
(*e*) $A = \{\text{children in your class who wear glasses}\}$,
$B = \{\text{children in your class who do not wear glasses}\}$;

(*f*) A = {types of birds who nest in England}, B = {thrush, robin, wren, rook};
(*g*) $A = \{\triangle, \circ, \square, *\}$, $B = \{\square, *, \triangle, \dagger\}$;
(*h*) $A = \{3, 7, 11, 5, 9\}$, B = {odd numbers between 2 and 12}.

2. List the members of the sets $A \cap B$ for the pairs of sets given in Question 1.

3. Write down three pairs of sets for which Figure 5 would be a suitable Venn diagram.

4. When the members of a set B are all members of another set A, then B is called a *subset* of A.
(*a*) Draw a Venn diagram to illustrate the relation 'B is a subset of A'.
(*b*) If B is a subset of A, what is $A \cap B$?
(*c*) Give three examples of subsets.

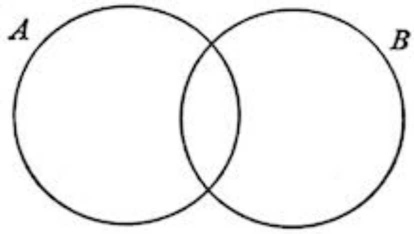


Fig. 5

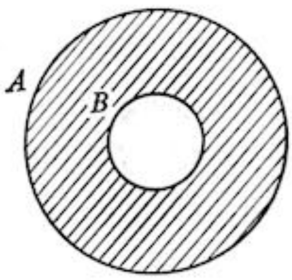


Fig. 6

5. Copy and complete the following:
(*a*) $\{a, \quad\} \cap \{b, c\} = \{c\}$;
(*b*) $\{7, 9, \quad\} \cap \{5, \quad, 2, 3\} = \{9, 3\}$;
(*c*) $\{d, \quad, \quad, b, t\} \cap \{p, \quad, d, \quad\} = \{a, \quad, e\}$.

6. If A is a complete set of playing cards and B is the subset of A consisting of the picture cards, then A and B can be represented by the Venn diagram shown in Figure 6.
(*a*) Describe, in words, the set of cards represented by the shaded region.
(*b*) Name two sets of cards C and D for which

$C \cap D$ = {the four of Hearts}.

7. If A is any set, what can you say about:

(*a*) $A \cap A$; (*b*) $A \cap \{\ \}$?

8. Let P = {even numbers greater than 7} and Q = {numbers less than 25}.
Give two members of the set $P \cap Q$.

9. In a class of 30 pupils, there are 19 who play tennis and 16 who play hockey.
(*a*) Draw a Venn diagram to illustrate the sets T and H, where T = {tennis players in the form} and H = {hockey players in the form}.
(*b*) If everyone in the class plays at least one of these games, how many members has the set $T \cap H$?

10. A paper boy delivers 27 copies of the *Daily Telegraph* and 22 copies of the *Daily Express* in a street of 40 houses.
What is (*a*) the smallest number of houses, (*b*) the largest number of houses, which could have had two papers delivered? (Assume no house receives more than 2 papers.)

Summary

1. *A set* is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object belongs to a set or not. This may be done by making a list of the objects or by giving a statement which describes them.

2. *Members.* The members of a set are the individual objects of that set.

3. {...} is read as 'the set whose members are...'. For instance, we write, A is {2, 4, 6, 8, 10}, if the members of A are the numbers 2, 4, 6, 8 and 10.

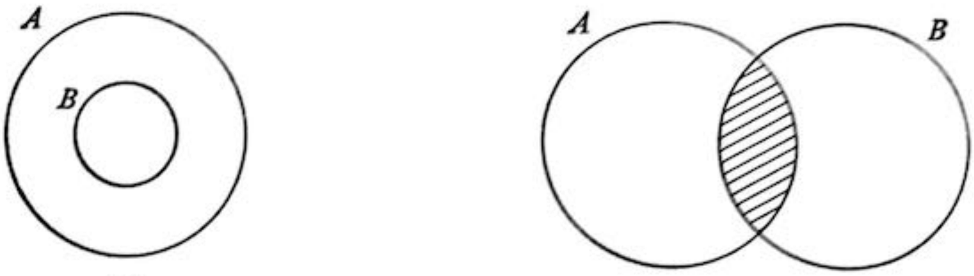
4. *Equal sets.* Two sets are equal if they have the same members. This is written $A = B$. For example, $\{3, 2, 1\} = \{1, 3, 2\}$.

5. *The empty set* is the set which contains no members. This is written { } or ø.

6. *Subsets.* B is a subset of A if every member of B is also a member of A.

7. The *intersection* of two sets A and B is the set of all members which belong to both A and B.
The intersection of A and B is written $A \cap B$ and read as 'A cap B'.

8. *Venn diagrams.*



(*a*) B is a subset of A

(*b*) Shaded area represents $A \cap B$

Fig. 7

2. THE COUNTING NUMBERS

The numbers 1, 2, 3, 4, 5, ... are called *counting numbers*.

These are the very first numbers you met, and they are so familiar that you may be wondering what there is new to learn about them. Until the last chapter, you probably thought of numbers only as things which you use in arithmetic when you add, subtract, multiply or divide. In this section, we shall not be concerned with calculating with numbers but with discovering something about the patterns which they form.

Look at the counting numbers from 1 to 99 as they appear in Figure 8. Make a neat copy of this figure and answer the following questions.

	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Fig. 8

(*a*) Describe the position of the set of numbers which contain the digit 5. What do the set of numbers in the third row have in common? What can you say about the set of counting numbers in the seventh column?

(*b*) Put a ring around every number which is exactly divisible by 3 (that is, a number which leaves no remainder when divided by 3). Did you have to divide all the numbers by 3, or was it possible to complete the question by seeing the pattern which was being formed?

(*c*) Write out the counting numbers from 1 to 99, using only 6 columns. The first three rows are:

	1	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17

Can you see any pattern in the numbers now?

Where are the numbers which are divisible by 3?

Put a ring around the numbers which have 8 as their units digit. What do you notice? Compare this with the same set of numbers in Figure 8.

You will have seen that there are still patterns in the counting numbers when you use 6 columns although, at first, they are not so obvious as with 10 columns.

(*d*) Why do you think the pattern is so clear in the first table?

Instead of writing down all the counting numbers from 1 to 99 every time we want a table like Figure 8, we make use of the following fact. The set of numbers in a row all have the same tens digit and the set of numbers in a column all have the same units digit.

In Figure 9, each square represents a counting number. The number at the top of each column indicates the units digit for all the squares beneath it. The number at the left of each row indicates the tens digit for all the squares to the right of it.

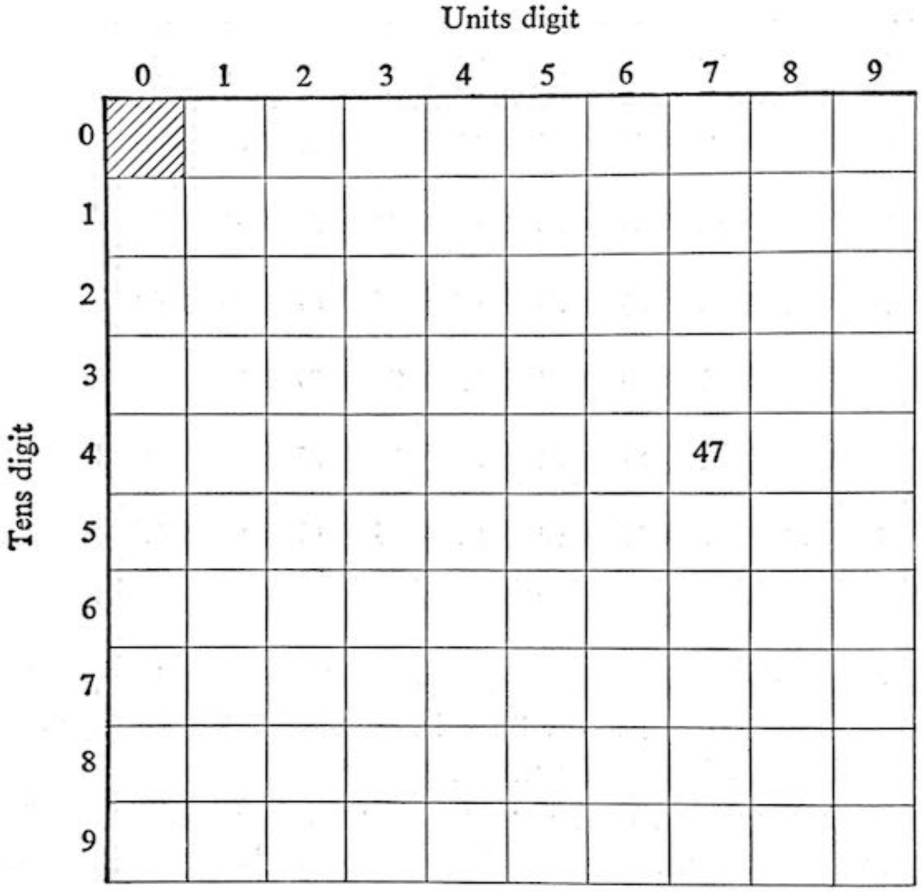


Fig. 9

For example, the square which is in the column labelled '7' and the row labelled '4' represents the number 47.

(*e*) Make a copy of Figure 9. (It is much easier to do this on graph paper.) Put an asterisk in the squares representing numbers which are divisible by 3. The regular pattern of asterisks should be identical to the pattern of ringed numbers you found in (*b*).

We call the subset of the counting numbers which are divisible by 3, the *multiples* of 3. If we label this set M_3 then

$$M_3 = \{3, 6, 9, 12, 15, \ldots\}.$$

Use a small circle or an asterisk of a different colour to indicate the set

$$M_7 = \{\text{multiples of } 7\}.$$

(*f*) Which counting numbers from 1 to 99 are members of the set $M_3 \cap M_7$?

These numbers, which are multiples of 3 and of 7, are called the *common multiples* of 3 and 7.

The smallest number of a set of common multiples is called the *lowest common multiple*, often abbreviated to L.C.M. You will find these numbers very useful in calculations with fractions.

What is the L.C.M. of the set $M_3 \cap M_7$?

Summary

1. The set of counting numbers is $\{1, 2, 3, 4, 5, ...\}$.

2. The set of multiples of a number a is the set of counting numbers divisible by a. For example,

$$\{\text{multiples of } 5\} = \{5, 10, 15, 20, 25, ...\} = \{1\times5, 2\times5, 3\times5, 4\times5, 5\times5, ...\}.$$

3. The set of common multiples of two numbers a and b is the set of counting numbers divisible by a and by b. It is the intersection of the two sets:

$$\{\text{multiples of } a\} \quad \text{and} \quad \{\text{multiples of } b\}.$$

For example,

$$\{\text{common multiples of 3 and 5}\} = \{15, 30, 45, 60, ...\}$$
$$= \{3, 6, 9, 12, 15, ...\} \cap \{5, 10, 15, 20, ...\}.$$

4. The lowest common multiple (L.C.M.) of two numbers a and b is the smallest number of the set of common multiples of a and b. For example, the L.C.M. of 3 and 5 is 15.

Exercise C

In this exercise,

A = {multiples of 3 which are less than 100},
B = {multiples of 5 which are less than 100},
C = {multiples of 7 which are less than 100},
D = {multiples of 12 which are less than 100},
E = {even numbers less than 100},
F = {multiples of 16 which are less than 100}.

1. Make a copy of Figure 9 and put a tick in every square which contains a member of set E.

(*a*) What property do even numbers possess?
(*b*) How would you decide whether the number 2,354,678 is even or not?
(*c*) What do the unmarked squares represent?

2. Use the table you made for Question 1 and on it mark the squares representing the sets A, B and C with different coloured asterisks.

3. (*a*) List the members of the sets:

(i) $A \cap B$; (ii) $B \cap C$; (iii) $C \cap A$.

(*b*) What is the L.C.M. of:

(i) 3 and 5; (ii) 5 and 7; (iii) 7 and 3?

(*c*) List the members of the sets:

(i) P = {multiples of 15 less than 100};
(ii) Q = {multiples of 35 less than 100};
(iii) R = {multiples of 21 less than 100}.

(*d*) What is the connection between parts (*a*), (*b*) and (*c*) of this question?

4. Find the first three common multiples of the following, by using a copy of Figure 9:

(*a*) 3 and 4; (*b*) 2 and 7; (*c*) 6 and 15.

5. How many members has {common multiples of 4 and 5 less than 100}? Give an example of a common multiple of 4 and 5, which is greater than 1000.

6. (*a*) How many members has the set $D \cap F$?
(*b*) What is the L.C.M. of 12 and 16?

7. Make use of the diagram you used in the earlier questions to give four common multiples of 2, 3 and 4.

8. Find the L.C.M. of 10, 12 and 15.

9. (*a*) Make a new copy of Figure 9 and put an asterisk in all the squares which represent members of the sets A, B, C and E *except* 2, 3, 5 and 7.
(*b*) Do the blank squares form a regular pattern?
Make a list of all the numbers which correspond to blank squares.
Take any one of these numbers. Is it a multiple of any number other than 1 and itself?
These numbers, with the exception of 1, are all members of the set of *prime numbers*.

10. Copy the first eight rows and columns of Figure 9 (that is, omit the rows and columns labelled 8 and 9). This is now a number table in *octal*. Mark the numbers which are divisible by (*a*) 3, (*b*) 7. What do you notice about the sum of the digits of those numbers divisible by 7?