Algebra II
2. Übungsblatt

William Crawley-Boevey

Abgabe: Bis zum 26.04.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 2.1. (2+2)

(i) Let G be an abelian group. Show that the set $t(G)$ of elements of G of finite order is a subgroup of G.

(ii) Show how to turn t into a functor $\mathrm{Ab} \to \mathrm{Ab}$, in a natural way, where Ab is the category of abelian groups.

Aufgabe 2.2. (2+2) Let K be a field, V a K-vector space and $\theta : V \to V$ a linear map.

(i) Show that V becomes a $K[X]$-module via the action

$$K[X] \times V \to V, \quad (p(X), v) \mapsto p(X)v := a_0 v + a_1\theta(v) + a_2\theta^2(v) + \dots$$

where $p(X) = a_0 + a_1 X + a_2 X^2 + \dots \in K[X]$.

(ii) Show that a subset W of V is a $K[X]$-submodule if and only if W is a subspace which is θ-invariant, meaning that $\theta(W) \subseteq W$.

Aufgabe 2.3. (2+2) Let K be a field and let $M_n(K)$ be the ring of $n \times n$ matrices with entries in K. Let K^n be the set of n-tuples of elements of K, written as column vectors. It is naturally a left $M_n(K)$-module, with the action given by the usual product of a matrix and a column vector.

(i) Show that the only $M_n(K)$-submodules of K^n are $\{0\}$ and K^n itself.

(ii) Show that the mapping sending a matrix to its transpose defines a ring isomorphism

$$M_n(K) \to M_n(K)^{op}, \quad A \mapsto A^{\mathrm{T}}.$$

Mehr...

Aufgabe 2.4. (1+1+2) Let R and S be rings with ones 1_R and 1_S, and consider $R \times S$ as a ring with the componentwise operations

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s)(r', s') = (rr', ss')$$

for $r, r' \in R$ and $s, s' \in S$.

(i) If V is an R-module and W is an S-module, show that $V \times W$ becomes an $R \times S$-module via the operations

$$(v, w) + (v', w') = (v + v', w + w'), \quad (r, s)(v, w) = (rv, sw)$$

for $v, v' \in V$, $w, w' \in W$, $r \in R$ and $s \in S$.

(ii) Let M be an $R \times S$-module and let

$$V = \{(1_R, 0)m : m \in M\} \subseteq M.$$

Show that V is an additive subgroup of M, and that it becomes an R-module with the action $*$ given by $r * v = (r, 0)v$.

(iii) Let M be an $R \times S$-module, let V be as in (ii) and, in the same way, let

$$W = \{(0, 1_S)m : m \in M\},$$

considered as an S-module with the action $s * w = (0, s)w$. Consider $V \times W$ as an $R \times S$-module as in (i). Show that mapping

$$\alpha : M \to V \times W, \quad m \mapsto ((1_R, 0)m, (0, 1_S)m)$$

is an isomorphism of $R \times S$-modules.