Algebra II
6. Übungsblatt

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Abgabe: Bis zum 24.05.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 6.1. (1+1+1+1) Let R be a ring and let $x \in R$. We consider the following conditions:

(a) $xS = 0$ for all simple R-modules S.
(b) $x \in I$ for all maximal left ideals I in R.
(c) $1 - ax$ has a left inverse for all $a \in R$.
(d) $1 - ax$ is invertible for all $a \in R$.

(i) Prove (a)$\Rightarrow$(b) [Hint. If I is a maximal ideal, R/I is a simple module.]

(ii) Prove (b)$\Rightarrow$(c) [Hint. If $r \in R$ does not have a left inverse, then Rr is a proper left ideal, so contained in a maximal left ideal.]

(iii) Prove (c)$\Rightarrow$(d) [Hint. If y is a left inverse for $1 - ax$, then $y = 1 + yax$, and this also has a left inverse.]

(iv) Prove (d)$\Rightarrow$(a). [Hint. If S is a simple module and $xs \neq 0$ with $s \in S$, then $Rxs = S$, so $s \in Rxs$.]

Aufgabe 6.2. (2+2) The set $J(R)$ of all x satisfying the equivalent conditions in Aufgabe 6.1 is called the *Jacobson radical* of R.

(i) Show that $J(R)$ is an ideal in R (that is, it is both a left ideal and a right ideal).

(ii) Show that if R is a semisimple ring, then $J(R) = 0$.

Aufgabe 6.3. (2+2) Let R be a ring and let $x, a \in R$.

(i) If $1 - ax$ is invertible, with inverse y, show that $1 - xa$ is invertible, with inverse $1 + xya$. Deduce that $J(R) = J(R^{op})$.

(ii) If x is nilpotent, that is $x^n = 0$ for some $n > 0$, show that $1 - x$ is invertible. Deduce that if I is an ideal in R and every element of I is nilpotent, then $I \subseteq J(R)$.

Mehr...

Aufgabe 6.4. (1+1+2)

(i) Show that if I is an ideal in R and $I \subseteq J(R)$, then $J(R/I) = J(R)/I$. [Hint. Consider the maximal left ideals in R.]

(ii) Let R be a ring and I a nilpotent ideal in R. Show that if R/I is semisimple, then $J(R) = I$.

(iii) Let K be a field and let $T_n(K)$ be the subalgebra of $M_n(K)$ consisting of the upper triangular matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

with $a_{ij} \in K$. Show that $J(T_n(K))$ is the set of strictly upper triangular matrices (that is, upper triangular matrices with diagonal entries $a_{ii} = 0$).