On the classification of flat commutative group schemes G with $V_G = 0$

von Thomas Zink

Introduction We fix a prime number p. Let R be an $\mathbb{F}_p$-algebra. Let $R\{\tau\}$ the non-commutative polynomial ring, such that $\tau r = r^p\tau$. Let M be a left $R\{\tau\}$-module. Let $D(M)$ the functor on the category of R-algebras S such that $D(M)(S) = \mathrm{Hom}_{R\{\tau\}}(M, S)$. Here S is regarded as a left $R\{\tau\}$-module by defining $\tau s = s^p$. Then $D(M)$ is an affine commutative group scheme. It is flat if M is projective as R-module.

D is a contravariant functor in M from the category $\mathbf{mod}_{R\{\tau\}}$ of left $R\{\tau\}$-modules to the category $\mathbf{ags}_R$ of affine commutative group schemes over R.

$$D : \mathbf{mod}_{R\{\tau\}} \overset{contra}{\longrightarrow} \mathbf{ags}_R \tag{1}$$

There is a contravariant adjoint functor C. Let $\mathbb{G}_{a,R} = \mathrm{Spec}\, R[T]$ be the additive group. We define the action $\tau : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ by $\tau^*T = T^p$. This gives an isomorphism $R\{\tau\} \to \mathrm{End}_{\mathrm{ags}/R}\, \mathbb{G}_{a,R}$. We define for $G \in \mathbf{ags}/R$

$$C(G) = \mathrm{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}). \tag{2}$$

It is a left $R\{\tau\}$-module via the action on $\mathbb{G}_{a,R}$. We call $C(G)$ the coordinate module of G. We have a functorial bijection

$$\mathrm{Hom}_{\mathrm{ags}/R}(G, D(M)) \to \mathrm{Hom}_{R\{\tau\}}(M, C(G)).$$

If $R = k$ is a field the functor D induces an antiequivalence between the category $\mathbf{mod}_{k\{\tau\}}$ and the full subcategory of objects $G \in \mathbf{ags}_k$ such that the Verschiebung $V_G : G^{(p)} \to G$ is zero, cf. [DG]. The contravariant functor C is a quasiinverse. Since both categories are abelian it follows that C and D are contravariant exact functors.

If we restrict D to the category of left $R\{\tau\}$-modules which are projective over R we obtain a fully faithful and exact contravariant functor with values in the category of flat affine group schemes over R. (See Proposition 23 for the meaning of exact in this context.)

$$D : \begin{pmatrix} \mathbf{mod}_{R\{\tau\}} \\ \text{projective}/R \end{pmatrix} \overset{contra}{\longrightarrow} \begin{pmatrix} \mathbf{ags}_R \\ \text{flat}/R \\ \text{Verschiebung } V = 0 \end{pmatrix} \tag{3}$$

Let $G \in \mathbf{ags}_R$ be a flat group scheme of finite presentation. We denote by ℓ_G the co-Lie complex as defined in [I]. It is an object in the derived category $D^b(\mathbf{mod}_R)$ of R-modules. There are at most to non-zero cohomology groups

$$\mathfrak{n}_G = H^{-1}(\ell_G), \quad \omega_G = H^0(\ell_G)$$

Definition 1. *Let $q = p^u$ a power of p. Let R be a $\mathbb{F}_q$-algebra. We consider $G \in \mathbf{ags}_R$ which is flat and of finite presentation. Let $\iota : \mathbb{F}_q \to \mathrm{End}_{ags/R}\, G$ be an action.*

We say that the action ι on G is strict *if the induced action $\mathbb{F}_q \to \mathrm{End}_R\, \omega_G$ coincides with the $\mathbb{F}_q$-module structure on ω_G obtained by restriction of scalars $\mathbb{F}_q \to R$.*

We say that the action ι is balanced *if it is strict and if the induced action $\mathbb{F}_q \to \operatorname{End}_R \mathfrak{n}_G$ coincides with the $\mathbb{F}_q$-module structure on $\mathfrak{n}_G$ obtained by restriction of scalars $\mathbb{F}_q \to R$.*

Instead of "balanced" we can say the the action of $\mathbb{F}_q$ on the cohomology of ℓ_G is strict.

Let $R\{\tau_q\}$ the non-commutative polynomial ring, such that $\tau r = r^q \tau$. We show the following variant of a result of T.Poguntke [P].

Theorem 2. *Let R be an $\mathbb{F}_q$-algebra.*

The category of pairs (G, ι) such that $G \in \mathbf{ags}_R$ is finite locally free with $V_G = 0$, and ι is a balanced action on G, is equivalent to the category of left $R\{\tau_q\}$-modules which are finite locally free as R-modules.

If $R = k$ is a field the category of pairs (G, ι) such that G is of finite type, $V_G = 0$, and ι is balanced is equivalent to the category of left $k\{\tau_q\}$-modules N which are finitely generated.

The proof of the first assertion is based on the fact that ω_{G^*} of the Cartier dual G^* of G is a locally free R-module. We include a proof of this fact mentioned in a footnote of [SGA3]. Thereby we relate ω_{G^*} to the crystal of G^* and dualy of G.

The connection with Poguntke's formulation is given by the following Proposition.

Proposition 3. *Let R be a $\mathbb{F}_p$-algebra. Let $M \in \mathbf{mod}_{R\{\tau\}}$ be of finite presentation such that M is projective as R-module. The group scheme $G = D(M)$ is of finite presentation and flat over R.*

Then the complex ℓ_G is quasiisomorphic to the complex

$$R \otimes_{\mathrm{Frob},R} M \xrightarrow{\tau^\sharp} M. \tag{4}$$

Poguntke's definition is based on the complex (4) and makes sense for each group scheme $G = D(M)$ in the essential image of (3). No finiteness condition is needed. One can classify all $\mathbb{F}_q$-actions on $G = D(M)$ which are balanced with respect to the complex above. One obtains a fully faithful exact contravariant functor from the category of left $R\{\tau_q\}$-modules to group schemes in $\mathbf{ags}_R$ with a balanced $\mathbb{F}_q$-action.

One can also classify all strict $\mathbb{F}_q$-action on $G = D(M)$. They correspond to $R\{\tau_q\}$-modules with an additional structure which we call a twisted filtration.

Crystals and group schemes annihilated by the Frobenius

Let R be a $\mathbb{F}_p$-algebra. We begin with finite flat group schemes G over R which are annihilated by p. (finite means that the affine algebra of G is a finitely presented R-module.)

Let us first consider the category of those G which are embedable in a p-divisible group X. Then we can find an exact sequence (of fppf sheaves)

$$0 \to G \to X \to Y \to 0, \tag{5}$$

such that X and Y are p-divisible groups. We denote by $\mathbb{D}_R(X)$ the Lie algebra of the universal extension of X. This is a locally free and finitely generated R-module. From (5) we obtain a map of R-modules

$$\mathbb{D}_R(X) \to \mathbb{D}_R(Y)$$

We denote the cokernel of this map by M_G. By standard arguments one shows that R-module M_G is independent of the resolution (5) and that it is functorial in G. We will denote the kernel of the multiplication $p^n : X \to X$ by $X[n]$. By our assumptions we have $M_{X[1]} = \mathbb{D}_R(X)$. In particular the last module is free and finitely generated. We can define M_G also for those G which are not embedable into a p-divisible group because we may take an affine covering $\cup U_i = \operatorname{Spec} R$, such that the restriction $G_{|U_i}$ is embedable in a p-divisible group over each U_i. We note that the functor $G \mapsto M_G$ commutes with arbitrary base change $R \to R'$:

$$M_{G_{R'}} = M_G \otimes_R R'.$$

Proposition 4. *The functor $G \mapsto M_G$ from the category of finite flat group schemes R, which are annihilated by p to the category of R-modules defined above has the following properties:*

(1) *M_G is a finitely generated locally free R-module.*
(2) *If G is a finite flat group scheme of order p^r, then M_G is locally free of rank r.*
(3) *If $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of sheaves such that G_i are finite flat and annihilated by p, the the induced sequence*

$$0 \to M_{G_1} \to M_{G_2} \to M_{G_3} \to 0 \tag{6}$$

Proof. In the case of a perfect field R this follows from Dieudonné theory. (We refer here to the covariant Dieudonné which is build on Cartier theory.) Let us denote by $X[1]$ the kernel of multiplication by p on X. If $G = X[1]$ then $M_G = \mathbb{D}_X(R)$. We already know that this is finitely generated and locally free and has the rank predicted by the Proposition.

In the general case we show first, that $G \mapsto M_G$ is a right exact functor. It is easy to embed the sequence of the G_i above into a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows an columns are short exact sequences of sheaves. The sheaves X_i and Y_i are p-divisible groups. Since $\mathbb{D}_R$ is exact on short exact sequences of p-divisible groups we obtain the asserted right exactness.

Now we embed G into X. Then we obtain an exact sequence

$$0 \to G \to X[1] \to H \to 0 \tag{7}$$

where H is a finite flat group scheme too. We obtain an exact sequence

$$M_G \to M_{X[1]} \to M_H \to 0. \tag{8}$$

We claim that this sequence is exact and all modules are projective. For this we may restrict to the case, where R is a local ring with residue class field k. We recall that M_G and M_H are finitely generated.

In the case where $R = k$ is a perfect field we have $\dim_k M_G + \dim_k M_H = \dim_k M_{X[1]}$ and therefore the sequence (8) is exact. For general R with residue class field k we deduce that $M_G \otimes_R k \to M_{X[1]} \otimes_R k$ is injective because the functor M commutes with base change. Therefore M_G is a direct summand of the free R-module $M_{X[1]}$. This shows the first property asserted in the Proposition. The second follows from base change to k and then to a perfect field. Since we already know that (6) is a right exact sequence of locally free R-modules the last assertion follows because

$$\operatorname{rank}_R M_{G_1} + \operatorname{rank}_R M_{G_3} = \operatorname{rank}_R M_{G_2}$$

holds by the second property. □

Corollary 5. *Let $\alpha : X \to Y$ be an isogeny of formal p-divisible groups whose kernel is annihilated by p. Then the kernel and the cokernel of the induced map $\mathbb{D}(\alpha) : \mathbb{D}_R(X) \to \mathbb{D}_R(Y)$ are locally free and finitely generated R-modules.*

Variant: Let R be a $\mathbb{F}_p$-algebra. Let G be a finite flat group scheme over R which is annihilated by p. Let $S \to R$ be an epimorphism of $\mathbb{F}_p$-algebras such that the kernel is endowed with nilpotent divided powers. We take a resolution (5). Then we define $M_{G,S}$ as the cokernel of the map

$$\mathbb{D}_S(X) \to \mathbb{D}_S(Y) \to M_{G,S} \to 0.$$

Because we have nilpotent divided powers $\mathbb{D}_S(X)$ is defined. The S-module $M_{G,S}$ clearly commutes with base change respect to morphisms of pd-thickenings $S' \to S$ of R. If $G = X[1]$ we obtain $M_{G,S} = \mathbb{D}_S(X)$ because the last module is annihilated by p. We see that $M_{G,S}$ is a locally free S-module. As in the proof of Proposition 4 we see that $G \mapsto M_{G,S}$ is a right exact functor.

Proposition 6. *We assume that $S \to R$ is a surjection of $\mathbb{F}_p$-algebras and that the kernel $\mathfrak{a}$ is endowed with nilpotent divided powers. Let G be a finite locally free group scheme over R which is annihilated by p.*

Then $M_{G,S}$ is a finite locally free S-module such that $p^{\operatorname{rank} M_{G,S}}$ is the order of G. The functor $G \mapsto M_{G,S}$ on the category of finite locally free group schemes over R which are annihilated by p is exact.

Proof. As before we consider an exact sequence (7). Then we obtain the right exact sequence

$$M_{G,S} \to M_{X[1],S} \to M_{H,S} \to 0.$$

If we tensor this by $S/\mathfrak{a}$ we obtain the exact sequence

$$0 \to M_G \to M_{X[1]} \to M_H \to 0.$$

of locally free R-modules. Localizing we may assume that we have free R-modules. We find a basis $\bar{u}_1, \ldots, \bar{u}_n$ of $M_{X[1]}$ such that $\bar{u}_1, \ldots, \bar{u}_r$ is a basis of M_G. We lift the elements $\bar{u}_i$ to elements $u_i \in M_{X[1],S}$ such that

$u_1, \dots, u_r \in M_{G,S}$. By the Lemma of Nakayma $u_1, \dots, u_n$ is a basis of the free module $M_{X[1],S}$ and $u_1, \dots, u_r$ is a set of generators of $M_{G,S}$. Therefore this set of generators is a basis. The rest follows exactly as in Proposition 4. □

We will recall an elementary definition. Let $G = \operatorname{Spec} A$ be an affine scheme over $\operatorname{Spec} R$ with a section $\varepsilon : \operatorname{Spec} R \to \operatorname{Spec} A$. We call (G, ε) a pointed scheme over $\operatorname{Spec} R$. Let I be the kernel of the comorphism $A \to R$. We write $\omega_G = I/I^2$ and view it as an R-module. This is the conormal sheaf (or module) of (G, ε). It is a contravariant functor. The formation of ω_G commutes with base change $R \to R'$.

Let $(G_2, \varepsilon_2) \to (G_3, \varepsilon_3)$ a morphism of pointed affine schemes over $\operatorname{Spec} R$. Let (G_1, ε_1) be the inverse image of the closed subscheme ε_3 by the morphism $G_2 \to G_3$ where ε_1 in induced from ε_2. Then we have the sequence

$$\omega_{G_3} \to \omega_{G_2} \to \omega_{G_1} \to 0 \tag{9}$$

is exact.

If (G_1, ε_1) and (G_3, ε_3) be two affine pointed schemes over R. The natural R-module homomorphism

$$\omega_{G_1} \oplus \omega_{G_3} \xrightarrow{+} \omega_{G_1 \times_R G_3}$$

is an isomorphism.

We write $\operatorname{Lie}(G, \varepsilon) = \operatorname{Hom}_R(\omega_G, R)$.

Proposition 7. *Let R be an $\mathbb{F}_p$-algebra. Let G be a commutative finite locally free group scheme over R such that the Frobenius $\operatorname{Fr}_G : G \to G^{(p)}$ is zero.*

Then ω_G is a finite locally free R-module. On the category of group schemes which satisfy the assumptions of this Proposition the contravariant functor $G \mapsto \omega_G$ is exact.

Proof. Since ω_G commutes with base change we may assume that R is a noetherian ring and then that R is an artinian local ring.

Let X be a p-divisible group and let H be the kernel of the isogeny $\operatorname{Fr}_X : X \to X^{(p)}$. Then $\omega_X = \omega_H$. Therefore the assertion is true for $G = H$.

For the general case we embed G in a p-divisible group X. Then we form the exact sequence of finite flat group schemes

$$0 \to G \to H \to I \to 0.$$

This induces an exact sequence

$$\omega_I \to \omega_H \to \omega_G \to 0. \tag{10}$$

Let k be a residue class field of R. We claim that the homomorphism

$$\omega_I \otimes_R k \to \omega_H \otimes_R k \tag{11}$$

is injective.

To show this we may assume that $R = k$. By Mumford AV or SGA 3 Exp. VII_B the group schemes I, H, G are described by their p-Lie algebras. In particular we have that $\operatorname{rank} G = p^{\dim \omega_G}$ and similiary for I and H. This implies that in the sequence (10) the first map is injective.

For arbitrary R the injectivity of (11) implies that ω_I is a direct summand of the free module ω_H. Therefore ω_G is free as desired. □

We write $\operatorname{Lie} G = \operatorname{Hom}_R(\omega_G, R)$ so that under the category of the Proposition $\operatorname{Lie} G$ is an exact functor.

Remark: In the Théorème 4.4. of SGA 3 Exp. VII_B the assumption that ω_G is locally free holds automatically if G is commutative. Therefore finite flat commutative group schemes G over an $\mathbb{F}_p$-algebra R which are annihilated by the Frobenius are classified by the p-Lie algebra $\operatorname{Lie} G$. Dually finite flat commutative group schemes G over an $\mathbb{F}_p$-algebra R which are annihilated by the Verschiebung V_G are classified by the p-Lie algebra $\operatorname{Lie} G^*$ of the Cartier dual. This is called by Genestier *la module coordonnée*, cf. Genestier, Espaces symmétriques de Drinfeld, Astérisque 234. We recall that for each finite locally free group scheme there is a canonical isomorphism

$$\operatorname{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}) = \operatorname{Hom}(\omega_{G^*}, R) = \operatorname{Lie} G^*. \tag{12}$$

Corollary 8. *With the assumptions of Proposition 7 there is a canonical isomorphism*

$$\operatorname{Lie} G \otimes_{R,\mathrm{Frob}} R \cong \mathbb{D}_R(G).$$

Proof. Since $\mathbb{D}_R(G)$ is canonically defined the question of the existence of such an isomorphism is local if we define it canonically.

Therefore we may assume that there is a resolution of G by p-divisible groups:

$$0 \to G \to X \to Y \to 0.$$

We apply the snake lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\mathrm{Fr}_G"'] & X \arrow[r] \arrow[d, "\mathrm{Fr}_X"'] & Y \arrow[r] \arrow[d, "\mathrm{Fr}_Y"'] & 0 \\
0 \arrow[r] & G^{(p)} \arrow[r] & X^{(p)} \arrow[r] & Y^{(p)} \arrow[r] & 0
\end{tikzcd}$$

The existence of the Verschiebung $\mathrm{Ver}_X : X^{(p)} \to X$ shows that $\mathrm{Fr}_X : X \to X^{(p)}$ is surjective because $\mathrm{Fr}_X \circ \mathrm{Ver}_X = p \, \mathrm{id}_{X^{(p)}}$. If we denote the kernel of Fr_X by $X[\mathrm{Fr}_X]$ we obtain an exact sequence

$$0 \to G \to X[\mathrm{Fr}_X] \to Y[\mathrm{Fr}_X] \to G^{(p)} \to 0.$$

By the proof of the Proposition this remains exact if we apply the functor Lie. In particular we find a surjection $\operatorname{Lie} Y \to \operatorname{Lie} G^{(p)}$. We consider the diagram

$$\begin{tikzcd}
\mathbb{D}_R(X) \arrow[r] \arrow[d] & \mathbb{D}_R(Y) \arrow[r] \arrow[d] & \mathbb{D}_R(G) \arrow[r] & 0 \\
\operatorname{Lie} X \arrow[r] & \operatorname{Lie} Y \arrow[r] & \operatorname{Lie} G^{(p)} \arrow[r] & 0.
\end{tikzcd}$$

Since the vertical arrows are surjective, we obtain a surjection $\mathbb{D}_R(G) \to \operatorname{Lie} G^{(p)}$. We know that the origin and the target of the last arrow are finitely generated projective R-modules of the same rank. We conclude that the arrow is an isomorphism. This is equivalent to our assertion. □

Let R be a $\mathbb{F}_p$-algebra. We consider the non-commutative polynomial ring

$$R\{\tau\} = \{\textstyle\sum_{i=0}^{N} a_i\tau^i \mid a_i \in R,\ N \in \mathbb{N}\} \quad \text{with}$$
$$\tau a = a^p\tau \quad a \in R.$$

This ring acts on $\operatorname{End}\mathbb{G}_{a,R}$. We fix once for all and isomorphism $\mathbb{G}_{a,R} = \operatorname{Spec} R[T]$ such that the group struture is given by $T \mapsto T\otimes 1 + 1\otimes T$. Then the action is given as follows

$$\begin{array}{ll} \tau : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R} & \tau^*T := T^p \\ a : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R} & a^*T := aT \end{array}$$

This gives an isomorphism of rings, c.f. Laumon, [DG].

$$R\{\tau\} \to \operatorname{End}\mathbb{G}_{a,R}. \tag{13}$$

In the ring $R\{\tau\}$ we have the Euclidean division. Let

$$g(\tau) = \tau^d + a_{d-1}\tau^{d-1} + \ldots + a_1\tau + a_0 \tag{14}$$

be a unitary polynomial. Then any $f(\tau) \in R\{\tau\}$ has a unique expressiion

$$f(\tau) = q(\tau)g(\tau) + r(\tau),$$

where $q(\tau), r(\tau) \in R\{\tau\}$ and $\deg r(\tau) < d$. If $R = k$ is a field this implies that each left ideal $\mathfrak{a} \subset k\{\tau\}$ is principal, i.e. $\mathfrak{a} = k\{\tau\}g(\tau)$. More generally a submodule of a free finitely generated left $k\{\tau\}$-module is free. We remark that there are right ideals in $k\{\tau\}$ which are not finitely generated unless the field k is perfect.

We will define two contravariant functors, cf. Introduction

$$\mathbf{mod}_{R\{\tau\}} \ \underset{C}{\overset{D}{\rightleftarrows}} \ \mathbf{ags}_R$$

The are considered in (cf. [DG] IV §3.6) if R is the field but for the definitions R can be arbitrary.

Definition 9. *Let M be a left $R\{\tau\}$-module. We define $D(M)$ as a functor on the category of R-algebras S with values in the category of abelian groups. We set*

$$D(M)(S) = \{\phi \in \operatorname{Hom}_R(M, S) \mid \phi(\tau m) = \phi(m)^p\}$$

$D(M)(S)$ is an abelian group with respect to the addition of homomorphisms ϕ.

We consider S as a left $R\{\tau\}$-module by setting $\tau s = s^p$, $s \in S$. Then we may write $D(M)(S) = \operatorname{Hom}_{R\{\tau\}}(M, S)$. We note that $D(M)$ is left exact in the argument M.

The functor $D(M)$ commutes with base change. More precisely we write $D_R = D$ to indicate that we are over that base ring R. Let $R \to R'$ be a ring homomorphism. $R' \otimes_R M = R'\{\tau\} \otimes_{R\{\tau\}} M$ is an $R'\{\tau\}$-module. There is a canonical isomorphism

$$\operatorname{Spec} R' \times_{\operatorname{Spec} R} D_R(M) \cong D_{R'}(R' \otimes_R M).$$

Proposition 10. *The functor $D(M)$ is an affine group scheme over* $\operatorname{Spec} R$.

Proof. Let $\mathcal{S}_R(M)$ be the symmetric algebra of the R-module M. Let $\mathfrak{u}^{[p]}$ be the ideal of $\mathcal{S}_R(M)$ which is generated by all elements $m^p - \tau m$, $m \in M$. We set $U^{[p]}(M) = \mathcal{S}_R(M)/\mathfrak{u}^{[p]}$. Clearly we have $D(M) = \operatorname{Spec} U^{[p]}(M)$. $\square$

We note that the comorphism $\Delta : U^{[p]}(M) \to U^{[p]}(M) \otimes_R U^{[p]}(M)$ of the group structure of $D(M)$ is given by $\Delta(m) = m \otimes 1 + 1 \otimes m$ for $m \in M$. Elements of $U^{[p]}(M)$ which satisfy the last equation are called primitive.

We follow the recent literature and call D the Drinfeld functor.

Definition 11. *Let $G = \operatorname{Spec} A$ be an affine group scheme over R. We set*

$$C(G) = \operatorname{Hom}_{ags/R}(G, \mathbb{G}_{a,R}).$$

The homomorphism are homomorphisms of group schemes over R. C_G is a let $R\{\tau\}$-module via the action of $R\{\tau\}$ on $\mathbb{G}_{a,R}$, cf. (13). We call C_G the coordinate module.

Let $\Delta : A \to A \otimes_R A$ be the comorphism of the group law on G. The we have

$$C(G) = \{x \in A \,|\Delta x = x \otimes 1 + 1 \otimes x \in A \otimes_R A\,\}. \tag{15}$$

The element x corresponds to the homomorphism $T \mapsto x$ of $C(G)$. The set on the right hand side of (15) is called the set of primitive elements of A.

Definition 12. *Let G be a commutative group scheme over* $\operatorname{Spec} R$. *We will say that G has the base change property (BCM) if for each R-algebra R' the canonical morphism*

$$R' \otimes_R \operatorname{Hom}_{ags/R}(G, \mathbb{G}_{a,R}) \to \operatorname{Hom}_{ags/R'}(G_{R'}, \mathbb{G}_{a,R'}) \tag{16}$$

is an isomorphism.

We remark that the canonical morphism is always an isomorphism if affine and $R \to R'$ is flat. Indeed, we have $G = \operatorname{Spec} B$. Indeed by (15) we may write C_G as a kernel of an exact sequence of R-modules

$$\begin{array}{ccccc} 0 \to C_G \to & A & \to & A \otimes_R A, \\ & x & \mapsto & \Delta_A x - x \otimes 1 - 1 \otimes x \end{array}$$

If we tensor with the flat R-algebra R' we see that base change (16 holds. From this we easily obtain that the Zariski-sheaf $\underline{\operatorname{Hom}}(G, \mathbb{G}_{a,R})$ on $X = \operatorname{Spec} R$ is the quasicoherent $\mathcal{O}_X$-module $C_G^{\sim}$ which is associated to the R-module C_G. If we want to work over a general base scheme X we can define C_G as a quasicoherent $\mathcal{O}_X$-module.

For each left $R\{\tau\}$-module M we have a canonical homomorphism

$$M \to C(D(M)) = \operatorname{Hom}_{\mathrm{ags}/R}(D(M), \mathbb{G}_{a,R}). \tag{17}$$

To define it we must associate each $m \in M$ a morphism of functors $D(M)(S) \to \mathbb{G}_{a,R}(S) = S$. This morphism is simply the evaluation at m of homomorphisms in $D(M)(S) = \operatorname{Hom}_{R\{\tau\}}(M, S)$. Equivalently we can use that $C(D(M))$ consists of the primitive elements of $U^{[p]}(M)$. Then we can say that the map (17) is given by the natural map $M \to U^{[p]}(M)$.

Lemma 13. *Let M be a left $R\{\tau\}$-module which is projective as R-module.*

Then $U^{[p]}(M)$ is a projective R-module and $D(M)$ is a flat group scheme over R. Moreover the map the map $M \to C(D(M))$ (17) is bijective.

Proof. This is stated for $R = k$ a field in [DG] IV, §3,6 Lemma. If M is free as R-module the proof of [DG] works because for the basis of $U^{[p]}(M)$ used there, the reference is to a section of the book where k is not a field (sic). In particular this shows that $U^{[p]}(M)$ is a free R-module and therefore flat.

In general we find a projective R-module N, such that $M \oplus N$ is a free module. We endow N with a $R\{\tau\}$-module structure such that $\tau(N) = 0$. With this structure we have $D(M \oplus N) = D(M) \times_{\operatorname{Spec} R} D(N)$. This implies that $U^{[p]}(M) \otimes_R U^{[p]}(N) = U^{[p]}(M \oplus N)$ is a free R-module. The natural augmentation $U^{[p]}(N) \to R$ which gives the unit section shows that R is a direct summand of the R-module $U^{[p]}(N)$. Therefore $U^{[p]}(M)$ is a direct summand of the free R-module M.

Let $G = D(M)$ and $H = D(N)$. We have a split sequence $0 \to G \to G \times_R H \to H \to 0$. If we apply the functor C this gives again a split sequence. We obtain a commutative diagram where the vertical maps are the adjunction morphisms

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M \oplus N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(H) \arrow[r] & C(G \times H) \arrow[r] & C(G) \arrow[r] & 0
\end{tikzcd}$$

We know that the vertical arrow in the middle is bijective. Therefore the first vertical arrow is injective and the last vertical arrow is surjective. Interchanging the roles of M and N we conclude that $M \to C(G)$ is bijective. □

Lemma 14. *Let G be a finite locally free commutative group scheme over R such that $V_G = 0$ or let $G = D(M)$ where M is projective as an R-module. Then G has the property (BCM), cf. Definition 12.*

Proof. Let $G = D_R(M)$. It follows from Lemma 13 that $\operatorname{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}) = M$ Since the construction of $D_R(M)$ commutes with base change we obtain $G_{R'} = D_{R'}(R' \otimes_R M)$ and therefore $\operatorname{Hom}_{\mathrm{ags}/R}(G_{R'}, \mathbb{G}_{a,R'}) = R' \otimes_R M$. This proves the assertion for $G = D(M)$.

Now assume the G is finite and locally free. If $V_G = 0$ the R-module ω_{G^*} is by Proposition 7 a locally free. Therefore we obtain

$$\operatorname{Hom}_R(\omega_{G^*}, R) \otimes_R R' \cong \operatorname{Hom}_{R'}(\omega_{G^*_{R'}}, R').$$

Because taking the Cartier dual commutes with base change and because of (12) we obtain (BCM) in this case. □

We remark that a finite group scheme $G = \operatorname{Spec} A$ is locally free over R iff A is a finitely generated projective module.

Proposition 15. *Let G be an affine commutative group scheme over R and let M be an $R\{\tau\}$-module. Let $\alpha : G \to D(M)$ be a homomorphism of groups. The contravariant functor C and (17) gives a homomorphism of left $R\{\tau\}$-modules*

$$M \to C(D(M)) \xrightarrow{C(\alpha)} C(G).$$

We obtain a functorial homomorphism

$$\operatorname{Hom}_{ags/R}(G, D(M)) \to \operatorname{Hom}_{R\{\tau\}}(M, C(G)), \tag{18}$$

which is bijective.

Proof. We set $G = \operatorname{Spec} A$ and we denote the comorphism of group law by $\Delta : A \to A \otimes_R A$. By definition a morphism of schemes $\phi : G \to D_M$ is an element of $D_M(A) = \operatorname{Hom}_{R\{\tau\}}(M, A)$. The morphism ϕ is a homomorphism of group schemes iff the following diagram is commutative

$$\begin{tikzcd} G \times_{\operatorname{Spec} R} G \arrow[r, "\phi\times\phi"] \arrow[d] & D(M) \times_{\operatorname{Spec} R} D(M) \arrow[d] \\ G \arrow[r, "\phi"] & D(M) \end{tikzcd} \tag{19}$$

where the vertical arrows are the group laws. We note that $D(M) \times_{\operatorname{Spec} R} D(M) \cong D(M \oplus M)$ and that the group law is induced by the diagonal $M \to M \oplus M$. The morphism $G \times_{\operatorname{Spec} R} G \to D(M)$ given by the upper way in the diagram (19) corresponds to the map

$$\begin{array}{ccccc} M & \to & M \oplus M & \longrightarrow & A \otimes_R A. \\ m & \mapsto & (m, m) & \mapsto & \phi(m) \otimes 1 + 1 \otimes \phi(m) \end{array}$$

The lower way in (19) corresponds to the map

$$M \xrightarrow{\phi} A \xrightarrow{\Delta} A \otimes_R A.$$

Therefore $\phi \in \operatorname{Hom}_{R\{\tau\}}(M, A)$ defines a homorphism of group schemes $G \to D(M)$ if

$$\Delta(\phi(m)) = \phi(m) \otimes 1 + 1 \otimes \phi(m), \quad \text{for each } m \in M.$$

In other words all elements $\phi(m)$, $m \in M$ must be primitve elements of A. Because of (15) we the that ϕ induces a group homomorphism if an only if it lies in $\operatorname{Hom}_{R\{\tau\}}(M, C_G)$. This shows that (18) in bijective. $\square$

From (18) we deduce a canonical adjunction homomorphism

$$\kappa : G \to D(C(G)). \tag{20}$$

We can also decribe this adjunction morphism explicitly. Let S be an R-algebra. Let $\xi \in G(S)$. We must associate to ξ an element

$$\xi_{ad} \in \operatorname{Hom}_{R\{\tau\}}(\operatorname{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}), S).$$

Let $\alpha \in \operatorname{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R})$. Then we define $\xi_{ad}(\alpha) = \alpha_S(\xi)$ where $\alpha_S : G(S) \to \mathbb{G}_{a,R}(S) = S$ is induced from α.

We cite without proof the following result:

Proposition 16. *(*[DG]*, Chapt IV, §3, 6.6) Let $R = k$ be a field. Let G be a commutative affine group scheme such that the Verschiebung $V_G : G^{(p)} \to G$ is zero.*

Then the adjunction homomorphism $\kappa : G \to D(C(G))$, cf. (20) is an isomorphism.

The functors D and C are antiequivalences between the category $\mathbf{mod}_{k\{\tau\}}$ and the full subcategory of $\mathbf{ags}_k$ of objects G with $V_G = 0$. The modules of finite type coorespond under this antiequivalence to affine group schemes of finite type over k.

One knows that both categories are abelian. Therefore for a field $R = k$ both contravariant functors D and C are exact.

Corollary 17. *If there monomorphism $G \to \mathbb{G}_{a,R}^I$ for some set I then $\kappa : G \to D(C(G))$ is a monomorphism.*

If $C(G)$ is projective as R-module the homomorphism deduced from (20)

$$C(\kappa) : C(D(C(G))) \to C(G).$$

is an isomorphism of $R\{\tau\}$-modules.

Let $H = D(M)$. Then the functor C induces an injection

$$\mathrm{Hom}_{ags/R}(G, H) \longrightarrow \mathrm{Hom}_{R\{\tau\}}(C(H), C(G)).$$

If M is projective as R-module the last map is bijective.

Assume that M is projective as R-module. Then the functor D induces a bijection

$$\mathrm{Hom}_{R\{\tau\}}(N, M) \xrightarrow{\sim} \mathrm{Hom}_{ags/R}(D(M), D(N)).$$

Proof. Assume we have a monomorphism $G \to \mathbb{G}_{a,R}^I$. Let $\xi \in G(S)$ be in the kernel of κ_S. Then for each $\alpha \in \mathrm{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R})$ we have $\alpha_S(\xi) = 0$. Let $\xi' \in \mathbb{G}_{a,R}^I(S)$ be the image of ξ. Then it follows that for each projection $\pi : \mathbb{G}_{a,R}^I \to \mathbb{G}_{a,R}$ the element $\pi_S(\xi') = 0$. This shows the first assertion.

If we apply to $G \to D(C(G))$ the adjunction (18) we obtain by definition $\mathrm{id}_{C(G)}$. Therefore the composite of the following maps is the identity

$$C(G) \to C(D(C(G))) \xrightarrow{C(\kappa)} C(G).$$

The first arrow is (17) applied to $M = C(G)$ and therefore by Lemma 13 an isomorphism. Hence $C(\kappa)$ is an isomorphism.

For the third assertion we write the definition of (18)

$$\mathrm{Hom}_{\mathrm{ags}/R}(G, D(M)) \to \mathrm{Hom}_{R\{\tau\}}(C(D(M)), C(G)) \to \mathrm{Hom}_{R\{\tau\}}(M, C(G)).$$

By the Proposition the composite of these arrows in bijective. If M is projective as R-module the second arrow is by Lemma 13 bijective. Therefore we obtain the second assertion of the Proposition.

Finally we find by the same arguments bijections

$$\mathrm{Hom}_{\mathrm{ags}/R}(D(M), D(N)) \cong \mathrm{Hom}_{R\{\tau\}}(N, C(D(M))) \cong \mathrm{Hom}_{R\{\tau\}}(N, M).$$

$\square$

Proposition 18. *Let G be a locally free and finite commutative group scheme over R. Then the adjunction map $G \to D(C(G))$ is an isomorphism.*

The functor D is an antiequivalence of categories

$$D : \begin{pmatrix} \mathbf{mod}_{R\{\tau\}} \\ \textit{finite projective}/R \end{pmatrix} \xrightarrow{contra} \begin{pmatrix} \mathbf{ags}_R \\ \textit{finite locally free}/R \\ \textit{Verschiebung}\, V = 0. \end{pmatrix} \tag{21}$$

The contravariant functor C is a quasiinverse.

Proof. We note that an R-module is finite and locally free iff it is finite and projective. We know by Proposition 7 that $C(G) = \operatorname{Lie} G^*$ is a finite locally free R-module. Then $D(C(G))$ is by Lemma 13 a finite locally free group scheme. The Verschiebung of $D(C(G))$ is zero. To see this we chose a surjection of $R\{\tau\}$-modules $R\{\tau\}^I \to C(G)$, where I is a finite set. This gives a monomorphism of schemes $D(C(G)) \to \mathbb{G}_{a,R}^I$. Since the Verschiebung of $G_{a,R}$ is zero the same holds for $D(C(G))$.

If we apply C to the adjunction map we obtain by Corallary 17 an isomorphism

$$C(D(C(G))) \to C(G).$$

It induces an isomorphism

$$\operatorname{Lie} D(C(G))^* \to \operatorname{Lie} G^*. \tag{22}$$

By [SGA3] this implies that $D(C(G))^* \to G^*$ is an isomorphism. Hence $G \to D(C(G))$ is an isomorphism. (We remark that we need by base change the classifiction of finite locally free group schemes with $F = 0$ only in the case of a base field to see that (22) is an isomorphism.

This shows that the functor D of (21) is essentially surjective. On the other hand it is fully faithful by Corollary 17. $\square$

We need sheaves in the fpqc-topology. This is the topology on the category of affine schemes which has as covering families finite families of flat morphisms $\{u_i : \operatorname{Spec} S_i \to \operatorname{Spec} S\}_{i\in I}$ such that $\cup_{i\in I} \operatorname{Im} u_i = \operatorname{Spec} S$, cf [DG] III, §1, 3.2.

Proposition 19. *Let M be an left $R\{\tau\}$-module which is locally free an an R-module. We consider $R\{\tau\} \otimes_R M$ as a left $R\{\tau\}$-module via the first factor. The multiplication $gm \in M$ for $g \in R\{\tau\}$ and $m \in M$ induces a morphism of left $R\{\tau\}$-modules $R\{\tau\} \otimes_R M \to M$.*

Then the following sequence of left $R\{\tau\}$-module is exact.

$$\begin{array}{ccccc} 0 \to & R\{\tau\} \otimes_{\tau,R} M & \to & R\{\tau\} \otimes_R M & \to M \to 0 \\ & g \otimes m & \mapsto & g\tau \otimes m - g \otimes \tau m & \end{array} \tag{23}$$

In particular we have proj.dim$_{R\{\tau\}-}M \leq 1$.

If we apply to the sequence above the functor D we obtain an exact sequence in the category of fpqc sheaves

$$0 \to D(M) \to D(R\{\tau\} \otimes_R M) \to D(R\{\tau\} \otimes_{\tau,R} M) \to 0. \tag{24}$$

The next to last arrow is a faithfully flat morphism of affine group schemes. We call (24) the canonical resolution *of $D(M)$.*

Proof. We begin with the case where M is a free as an R-module. We choose a basis $e_i \in M$, $i \in I$. Then we obtain an isomorphism $R\{\tau\}\otimes_R M \cong R\{\tau\}^{(I)}$ such that $f_i = 1\otimes e_i$ is the standard basis of the last module. We find $a_{ij} \in R$, such that

$$\tau e_i = \sum_{j\in I} a_{ij} e_j, \quad \text{for all } i \in I.$$

We may rewrite the sequence of the Proposition

$$\begin{array}{ccccc} 0 \to & R\{\tau\}^{(I)} & \to & R\{\tau\}^{(I)} & \to M \to 0 \\ & f_i & \mapsto & \tau f_i - \sum_{j\in I} a_{ij} f_j & \end{array} \tag{25}$$

where the next to last arrow maps f_i to e_i. For $\mathbb{G}_{a,R}$ we have $C_{\mathbb{G}_{a,R}} = R\{\tau\}$. Since the elements $\tau^n f_i$, $n \in \mathbb{N}$, $i \in I$ are a basis of the R-module $R\{\tau\}^{(I)}$ it follows immediatly that the elements

$$\tau^n f_i - \sum_{j\in I} a_{ij}^{p^{n-1}} \tau^{n-1} f_j, \text{ with } i \in I,\ n > 1, \quad \text{and} \quad f_i, \text{ with } i \in I$$

generate the R-module $R\{\tau\}^{(I)}$ and are linearly independent. The exactness of (25) follows. In the case where M is not necessarily a free R-module we proceed as in the proof of Lemma 13. Let N be a projective R-module such that $M \oplus N$ is a free R-module. We regard N as an $R\{\tau\}$-module with $\tau(N) = 0$. We denote the sequence (23) for M resp. N by $E_2 \to E_1 \to M$ resp. $F_2 \to F_1 \to N$. By the case of a free module the direct sum of these to sequences

$$0 \to E_2 \oplus F_2 \to E_1 \oplus F_1 \to M \oplus N \to 0$$

is exact. Therefore each of the former sequences is exact. This shows the first part of the Proposition.

Clearly the functor D is left exact. Therefore for the last assertion of the Proposition we need only to check that D applied to the second arrow of (25) gives a faithfully flat map of affine group schemes. Again we begin with the case where M is free. We make the identification of functors on R-algebras S:

$$D(R\{\tau\}^{(I)})(S) = \mathrm{Hom}_{R\{\tau\}}(R\{\tau\}^{(I)}, S) = S^I = \mathbb{G}_{a,R}^I(S).$$

The evaluation at $f_i \in R\{\tau\}^{(I)}$ corresponds to the projection to the i-th factor $\mathbb{G}_{a,R}^I \to \mathbb{G}_{a,R}$. We see that D applied to the second arrow gives

$$\begin{array}{ccc} \mathbb{G}_{a,R}^I & \longrightarrow & \mathbb{G}_{a,R}^I \\ (\xi_i) & \longmapsto & (\eta_i) \end{array} \text{ with } \eta_i = \xi_i^p - \sum_{j\in I} a_{ij}\xi_j. \tag{26}$$

Let $\underline{X} = \{X_i\}_{i\in I}$. The affine scheme $\mathbb{G}_{a,R}^I$ is represented by the polynomial ring $R[\underline{X}]$ since a morphism $\xi : R[\underline{X}] \to S$ is given by $\xi_i = \xi(X_i)$. It follows that the comorphism of (26) is

$$\begin{array}{ccc} R[\underline{X}] & \to & R[\underline{X}] \\ X_i & \mapsto & X_i^p - \sum_{j\in I} a_{ij} X_j \end{array}$$

Therefore for M free the last assertion is a consequence of the following Lemma 20.

Now we treat the case where M is not necessarily a free R-module. We keep the notations N, E_i, F_i used above in this proof. Since D is left exact we already have a left exact sequence of group schemes $0 \to D(M) \to D(E_1) \to D(E_2)$. We have to show that the last arrow is a faithfully flat morphism of group schemes. The natural inclusion $M \to M \oplus N$ induces a morphism of the canonical resolutions

$$\begin{tikzcd} 0 \arrow[r] & D(M) \arrow[r] & D(E_1) \arrow[r] & D(E_2) \\ 0 \arrow[r] & D(M\oplus N) \arrow[r] \arrow[u] & D(E_1)\times_R D(F_1) \arrow[r, "\pi"] \arrow[u] & D(E_2)\times_R D(F_2) \arrow[u] \end{tikzcd}$$

We know that the last horizontal arrow π below is faithfully flat. We consider the morphism $D(E_2) \to D(E_2) \times_R D(F_2)$ induced by the unit section of $D(F_2)$. If we take the fibre product with π. We obtain a faithfully flat morphism $D(E_1) \times_R D(N) \to D(E_2)$. Since $D(N) = \operatorname{Spec} U^{[p]}(N)$ is by Lemma 13 faithfully flat over R we conclude by the following Lemma 21. $\square$

Lemma 20. *Let R be commutative ring. Let I be a set. Let $\underline{X} = \{X_i\}_{i \in I}$ and $\underline{Y} = \{Y_i\}_{i \in I}$ be to sets of variables. Let n_i, $i \in I$ be natural numbers such that $n_i \geq 2$ for all $i \in I$. Let $h_i(\underline{X}) \in R[\underline{X}]$ for $i \in I$ be polynomials of total degree* $\operatorname{totdeg} h_i < n_i$. *We consider the homomorphism of R-algebras*

$$R[\underline{Y}] \to R[\underline{X}], \quad Y_i \mapsto X_i^{n_i} + h_i(\underline{X}). \tag{27}$$

Then the monomials

$$X_{i_1}^{u_{i_1}} \cdot X_{i_2}^{u_{i_2}} \cdot \ldots \cdot X_{i_d}^{u_{i_d}}, \qquad \begin{array}{l} \{i_1, \ldots, i_d\} \subset I, \\ 0 \leq u_j < n_j, \ \text{for } j \in \{i_1, \ldots, i_d\}. \end{array} \tag{28}$$

are a basis of the $R[\underline{Y}]$-module $R[\underline{X}]$. In particular this module is free.

Proof. In the case I finite see [Z]. The general case may be reduced to the finite case. $\square$

Lemma 21. *Let C be a faithfully flat R-algebra. Let $\alpha : U_2 \to U_1$ be a homomorphism of R-algebras such that the homomorphism $U_2 \to U_1 \otimes_R C$ is faithflat.*

Then the homomorphism $\alpha : U_2 \to U_1$ is faithfully flat.

Proof. Let $M_1 \to M_2$ be a monomorphism of U_2 modules. We have to show that $M_1 \otimes_{U_2} U_1 \to M_2 \otimes_{U_2} U_1$ is a monomorphism. It suffices to prove the injectivity after tensoring with $\otimes_R C$ because C is faithfully flat. But then we obtain $M_1 \otimes_{U_2} (U_1 \otimes_R C) \to M_1 \otimes_{U_2} (U_1 \otimes_R C)$ which is injective by assumption. Finally for any U_1 module M we obtain from $M_1 \otimes_{U_2} U_1 = 0$ that $M_1 \otimes_{U_2} (U_1 \otimes_R C) = 0$ which implies $M_1 = 0$. $\square$

Corollary 22. *Let M be as in Proposition 19. Assume that*

$$0 \to P_2 \to P_1 \to M \to 0$$

is an exact sequence of left $R\{\tau\}$-modules such that P_1 is a projective module.

Then P_2 is projective. The sequence of fpqc-sheaves

$$0 \to D(M) \to D(P_1) \to D(P_2) \to 0$$

is exact. The next to last arrow is a faithfully flat morphism of affine group schemes.

If M is a finitely generated $R\{\tau\}$-module the affine scheme $D(M)$ is flat and of finite type over $\operatorname{Spec} R$.

Proof. We consider the standard resolution $0 \to L_2 \to L_1 \to M \to 0$ of M, cf. Proposition 19. We find a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_2 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It defines a homotopy equivalence of the complexes $[L_2 \to L_1] \cong [P_2 \to P_1]$. If we apply the functor D we obtain a homotopy equivalence of complexes of sheaves $[D(L_1) \to D(L_2)] \cong [D(P_1) \to D(P_2)]$. Because these complexes have the same cohomology groups we obtain a diagram of sheaves with exact rows

$$\begin{tikzcd} 0 \arrow[r] & D(M) \arrow[r] & D(L_1) \arrow[r] & D(L_2) \arrow[r] & 0 \\ 0 \arrow[r] & D(M) \arrow[r] \arrow[u, "="'] & D(P_1) \arrow[r] \arrow[u] & D(P_2) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

It follows formally that the second square in this diagram is a fibre product of sheaves and hence of schemes. Since we know that by the Proposition that $D(L_1) \to D(L_2)$ is faithfully flat the same is true for the arrow $D(P_1) \to D(P_2)$ obtained by base change.

If M is finitely generated as an $R\{\tau\}$-module we can take for P_1 a finitely generated free module. We obtain a closed immersion $D(M) \to \mathbb{G}_{a,R}^N$ for some natural number N. We know the flatness of $D(M)$ by Lemma 13. $\square$

We generalize the last Corollary.

Proposition 23. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of left $R\{\tau\}$-modules which are projective as R-modules.*

Then by applying D we obtain an exact sequence of sheaves

(29) $$0 \to D(M_3) \to D(M_2) \to D(M_1) \to 0,$$

such that the next to last arrow is a faithfully flat map of affine schemes.

Proof. Our assumption implies that the sequence of $R\{\tau\}$-modules

$$0 \to R\{\tau\} \otimes_R M_1 \to R\{\tau\} \otimes_R M_2 \to R\{\tau\} \otimes_R M_3 \to 0$$

is split exact. It is therefore mapped by D to an exact sequence of sheaves. Therefore the exactness of (29) in the category of sheaves follows from the Proposition by using the canonical resolutions of the $D(M_i)$ and the snake Lemma.

To prove the last assertion we proceed as in the proof of the last Corollary. We find a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \\ 0 \arrow[r] & P_1 \arrow[r] \arrow[u] & P_2 \arrow[r] \arrow[u] & M_3 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where P_1 and P_2 are left projective $R\{\tau\}$-modules. By the last Corollary we know that $\phi : D(P_2) \to D(P_1)$ is a faithfully flat morphism of affine schemes. Applying D to the diagram we see that $D(M_2) \to D(M_1)$ is obtained by base change from ϕ and therefore a faithfully flat morphism. $\square$

We have a canonical isomorphism

$$\omega_{\mathbb{G}_{a,R}} \cong TR[T]/T^2R[T] \cong R, \quad T \mapsto 1.$$

This induces a canonical homomorphism of R-modules

(30) $$\mathfrak{d}_G : \mathrm{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}) \to \mathrm{Hom}_R(\omega_{\mathbb{G}_{a,R}}, \omega_G) = \omega_G.$$

The morphism $\tau : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ induces the zero map on $\omega_{\mathbb{G}_{a,R}}$. The multiplication by τ on the coordinate module induces a Frobenius linear map

$$(31) \qquad R \otimes_{Frob,R} \mathrm{Hom}(G, \mathbb{G}_{a,R}) \xrightarrow{\tau^\sharp} \mathrm{Hom}_{\mathrm{ags}/R}(G, \mathbb{G}_{a,R}) \to \omega_G$$

The composite of these maps is zero by what we said above.

In the case $G = \mathbb{G}_{a,R}$ the map (30) identifies with

$$R\{\tau\} \to R, \quad f \mapsto f(0)$$

which maps a polynomial in $f \in R\{\tau\}$ to its constant term. For $G = \mathbb{G}_{a,R}$ (**??**) is an exact sequence. Indeed, we have an isomorphism $R \otimes_{Frob,R} R\{\tau\} \cong R\{\tau\}$ which sends $r \otimes f(\tau)$ to $rf^{(p)}(\tau)$ where $f^{(p)}(\tau)$ is obtained from $f(\tau)$ by raising the coefficients to the p-th power. This identifies (**??**) with the exact sequence

$$0 \to R\{\tau\} \xrightarrow{\times\tau} R\{\tau\} \xrightarrow{\mathfrak{d}} R \to 0,$$

where $\times\tau$ denotes the right multiplication by τ. Now we consider the case $G = G^I_{a,R} = \operatorname{Spec} R[\underline{X}]$ where I is any set. Then we obtain

$$\mathrm{Hom}_{\mathrm{ags}/R}(\mathbb{G}^{(I)}_{a,R}, \mathbb{G}_{a,R}) \cong R\{\tau\}^{(I)}.$$

This is a special case of Lemma 13. But we could also remark that each morphism $G^I_{a,R} \to \mathbb{G}_{a,R}$ factors through the projection $G^I_{a,R} \to G^J_{a,R}$ where $J \subset I$ is a finite subset. Indeed, such a morphism is given by a polynomial in $R[\underline{X}]$ which depends only on finitely many variables. Therefore the map

$$\mathfrak{d}_{G^I_{a,R}} : \mathrm{Hom}_{\mathrm{ags}/R}(\mathbb{G}^I_{a,R}, \mathbb{G}_{a,R}) \to \omega_{G^I_{a,R}}$$

identifies with the R-module homomorphism $R\{\tau\}^{(I)} \to R^{(I)}$ which takes the constant term component wise. The sequence (31) becomes for $G = \mathbb{G}^I_{a,R}$ an exact sequence

$$0 \to R \otimes_{Frob,R} R\{\tau\}^{(I)} \to R\{\tau\}^{(I)} \to R^{(I)} \to 0.$$

Proposition 24. *Let M be an $R\{\tau\}$-module which is projective as R-module. Let $G = D(M)$.*

Then the sequence (31) is right exact:

$$(32) \qquad R \otimes_{Frob,R} \mathrm{Hom}(G, \mathbb{G}_{a,R}) \xrightarrow{\tau^\sharp} \mathrm{Hom}_{ags/R}(G, \mathbb{G}_{a,R}) \xrightarrow{\mathfrak{d}} \omega_G \to 0.$$

In other words

$$\omega_G \cong C(G)/R\tau(C(G)) \cong M/R\tau(M),$$

cf. Lemma 13. If M is a projective $R\{\tau\}$-module the sequence (32) is also left exact.

Proof. We begin with the case where M is free. We choose a basis e_i, $i \in I$ and use the exact sequence (25). This sequence remains exact if we tensor with $R\otimes_{Frob,R}$. We obtain a commutative diagram whose rows and first two columns are exact

$$\begin{tikzcd}
0 \to R \otimes_{Frob,R} R\{\tau\}^{(I)} \arrow[r] \arrow[d, "\tau^\sharp"'] & R \otimes_{Frob,R} R\{\tau\}^{(I)} \arrow[r] \arrow[d, "\tau^\sharp"'] & R \otimes_{Frob,R} M \to 0 \arrow[d, "\tau^\sharp"'] \\
0 \to R\{\tau\}^{(I)} \arrow[r] \arrow[d, "\eth"'] & R\{\tau\}^{(I)} \arrow[r] \arrow[d, "\eth"'] & M \to 0 \arrow[d, "\eth"'] \\
0 \to \omega_{\mathbb{G}^I_{a,R}} \arrow[r] \arrow[d] & \omega_{\mathbb{G}^I_{a,R}} \arrow[r] \arrow[d] & \omega_G \to 0 \\
0 & 0 &
\end{tikzcd}$$

It is easily seen that the last column must be also right exact sequence. For arbitrary M we take an $R\{\tau\}$-module N which is projective as R-module and such that $M \oplus N$ is a free R-module. We noted that $\omega_{D(M \oplus N)} \cong \omega_{D(M)} \oplus \omega_{D(N)}$. By the free case we obtain a right exact sequence

$$(R \otimes_{Frob,R} M) \oplus (R \otimes_{Frob,R} N) \to M \oplus N \to \omega_{D(M)} \oplus \omega_{D(N)} \to 0$$

Since this is the direct sum of the two complexes (31) for $G = D(M)$ and $G = D(N)$ we see that each of these two complexes is right exact.

The last assertion follows immediately from the case where M is a free $R\{\tau\}$-module which we already know. $\square$

Definition 25. *Let M be an $R\{\tau\}$-module which is projective as R-module and let $G = D(M)$. Then we call*

$$(33) \qquad R \otimes_{Frob,R} \operatorname{Hom}(G, \mathbb{G}_{a,R}) \xrightarrow{\tau^\sharp} \operatorname{Hom}_{ags/R}(G, \mathbb{G}_{a,R})$$

the co-Lie complex of G. We denote the kernel of (33) by $\mathfrak{n}_G$. By Proposition the cokernel in ω_G.

If M is of finite presentation this complex is isomorphic to the co-Lie complex of G as defined by [I] Chapt. VII. To see this we begin with a finitely generated projective $R\{\tau\}$-module P. Then $D(P)$ is smooth over R, because it is flat and of finite type and the geometric fibres of $D(P) \to \operatorname{Spec} R$ are isomorphic to $\mathbb{G}_a^n$, cf. below ??? or [DG]. In this case the co-Lie complex of Illusie coincides with $\omega_{D(P)}$ which is quasiisomorphic to the complex (33) for $G = D(P)$. Assume that M is of finite presentation. Then we find a resolution $0 \to P_2 \to P_1 \to M \to 0$ where P_1 and P_2 are finitely generated projective left $R\{\tau\}$-modules. We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & R \otimes_{Frob,R} P_2 \arrow[r] \arrow[d, "\tau^\sharp"'] & R \otimes_{Frob,R} P_1 \arrow[r] \arrow[d, "\tau^\sharp"'] & R \otimes_{Frob,R} M \arrow[r] \arrow[d, "\tau^\sharp"'] & 0 \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d, "\mathfrak{d}"'] & P_1 \arrow[r] \arrow[d, "\mathfrak{d}"'] & M \arrow[r] \arrow[d, "\mathfrak{d}"'] & 0 \\
0 \arrow[r] & \omega_{D(P_2)} \arrow[r] \arrow[d] & \omega_{D(P_1)} \arrow[r] \arrow[d] & \omega_{D(M)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This shows that the complexes $R \otimes_{Frob,R} M \xrightarrow{\tau^\sharp} M$ and $\omega_{D(P_2)} \to \omega_{D(P_1)}$ are isomorphic in the derived category. By [I] Chapt.VII, Prop. 3.1.1.5 applied to the exact sequence $0 \to D(M) \to D(P_1) \to D(P_2) \to 0$ we see that $\omega_{D(P_2)} \to \omega_{D(P_1)}$ is the co-Lie complex of $D(M)$.

Let $q = p^u$. We classify group schemes in the essential image of D with an action of $\mathbb{F}_q$ over an $\mathbb{F}_q$-algebra R.

Definition 26. *Let G a commutative affine group scheme over an $\mathbb{F}_q$-algebra R endowed with an action*

$$\iota : \mathbb{F}_q \to \operatorname{End}_{\operatorname{Spec} R} G.$$

We say that the action ι is strict if the induced action $\mathbb{F}_q \to \operatorname{End}_R \omega_G$ coincides with the $\mathbb{F}_q$-module structure on ω_G obtained by restriction of scalars $\mathbb{F}_q \to R$.

Let M be a left $R\{\tau\}$-module which is projective over R. An action $i : \mathbb{F}_q \to \operatorname{End}_{R\{\tau\}} M$ defines an action on the group scheme $D(M)$. Since $\omega_{D(M)} \cong M/R\tau M$. We call i strict if the induced action on the R-module $M/R\tau M$ is strict, i.e. it coincides with the action given by $\mathbb{F}_q \to R$.

Definition 27. *Let (M, ι) as above. We call the action i* balanced *(cf.* [P]*) if it is strict and if the induced action on the R-module $\mathfrak{n}_{D(M)}$ coincides with the action given by $\mathbb{F}_q \to R$.*

For the coordinate module M this Definition means the following. M becomes via i an $\mathbb{F}_q \otimes R$-module. We consider the decomposition

$$\mathbb{F}_q \otimes R \cong \prod_{k \in \mathbb{Z}/u\mathbb{Z}} R, \quad a \otimes r \mapsto \prod_i a^{p^k} r.$$

The product $a^{p^i} r$ makes sense because R is an $\mathbb{F}_q$-algebra. For the $R\{\tau\}$-module M we deduce a decomposition

$$M = \oplus_{k \in \mathbb{Z}/u\mathbb{Z}} M_k,$$

where $M_k = \{m \in M \mid \iota(a)m = a^{p^k} r,\ a \in \mathbb{F}_q\}$. the acation of τ on M is graded of degree 1, i.e. $\tau(M_k) \subset M_{k+1}$. Clearly an $\mathbb{F}_q$-action is the same thing as such a graduation of this type. We obtain similiar decompositions

$$\omega_{D(M)} = \oplus_{k\in\mathbb{Z}/u\mathbb{Z}}\, \omega_{D(M),k}, \quad \mathfrak{n}_{D(M),k} = \oplus_{k\in\mathbb{Z}/u\mathbb{Z}}\, \mathfrak{n}_{D(M),k}.$$

Strict means that $\omega_{D(M),k} = 0$ for $k \neq 0$ and balanced if moreover $\mathfrak{n}_{D(M),k} = 0$.

The homomorphism

$$\tau^\sharp : R \otimes_{Frob,R} M_{k-1} \to M_k \tag{34}$$

has kernel $\mathfrak{n}_{D(M),k}$ and cokernel $\omega_{D(M),k}$. Therefore strict means that (34) is surjective for $i \neq 0$ and balanced means the it is bijective for $k \neq 0$.

For a comfortable construction of modules (M, i) we make two definitions. We write $R\{\tau_q\}$ for the non-commutative polynomial ring such that $\tau_q r = r^q \tau_q$. Let N be a left $R\{\tau_q\}$-module. A twisted filtration on N consists of R-submodules $E_k \subset R \otimes_{Frob^k,R} N$ $k = 0, \dots u-1$ which are direct summands such that $E_0 = 0$ and such that for $k = 1, \dots u-1$

$$R \otimes_{Frob,R} E_{k-1} \subset E_k \subset R \otimes_{Frob^k,R} N.$$

Proposition 28. *The category pairs (M, i) consisting of a left $R\{\tau\}$-module M which is projective over R and a strict $\mathbb{F}_q$-action i is equivalent to the category of pairs $(N, E.)$ where N is a left $R\{\tau_q\}$-module which is projective over R and $E.$ is a twisted filtration such that $R \otimes_{Frob,R} E_{u-1} \subset R \otimes_{Frob^u,R} N$ is contained in the kernel of the map $\tau_q^\sharp : R \otimes_{Frob^u,R} N \to N$.*

The balanced pairs (M, i) correspond to pairs $(N, E.)$ with $E. = 0$, i.e. the category of balanced pairs is equivalent to the category of $R\{\tau_q\}$-modules N which are projective as R-modules.

Proof. We start with (M, i) with i strict. Then the maps

$$(\tau^k)^\sharp : R \otimes_{Frob^k,R} M_0 \to M_k. \tag{35}$$

are for $k = 0, 1, \dots u-1$ surjective. Since M_k is projective as R-module the kernel E_k is a direct summand. We set $N = M_0$ and

$$\tau_q = \tau^u : N \to N.$$

In the other direction we set $M_k = (R \otimes_{Frob^k,R} N)/E_k$. For $k = 0, \dots u-2$ we define $\tau : M_k \to M_{k+1}$ by the natural surjection

$$\tau^\sharp : R \otimes_{Frob,R} (R \otimes_{Frob^k,R} N/E_k) \to R \otimes_{Frob^{k+1},R} N/E_{k+1}$$

and

$$\tau^\sharp := \tau_q^\sharp : R \otimes_{Frob,R} (R \otimes_{Frob^{u-1},R} N/E_{u-1}) \to N$$

□

We can now proof Theorem 2. It follows from Proposition 18, Proposition 16, the remark after Definition 25 and from the last Proposition 28.

ZUR EINLEITUNG It is more convenient to arbitrary finitely generated projective $R\{\tau\}$-modules in place of free modules $R\{\tau\}^I$. An important example is obtained as follows. Let V be a finitely generated projective R-module. Then

$$R\{\tau\} \otimes_R V \tag{36}$$

is a finitely generated projective left $R\{\tau\}$-module. Note that we take the action of R on $R\{\tau\}$ from the right to form this tensor product. Therefore

(36) inherits a left $R\{\tau\}$-module structure via the first factor. Since V is a direct summand of a free R-module R^n we conclude that (36) is a finitely generated projective $R\{\tau\}$-module.

The case of a ground field

We discuss the the ring $R\{\tau\}$ in more detail if $R = k$ is a field. By Euclidean division (14) $k\{\tau\}$ is left noetherian and any left ideal is principal.

Let M be a left $k\{\tau\}$-module. An element $m \in M$ is called torsion if there is an element $f(\tau) \in k\{\tau\}$, $f(\tau) \neq 0$ such that $f(\tau)m = 0$. Equivalently we may ask, that m is contained in a $k\{\tau\}$-submodule $N \subset M$ which is finite dimensional over k. The set of all torsion elements $T \subset M$ is a $k\{\tau\}$-submodule. We call T this the torsion submodule of M. If M is a finitely generated $k\{\tau\}$-module then T is a finite dimensional k-vector space because M is noetherian.

Lemma 29. *Let k be a field of characterictic p. Let M be a finitely generated $k\{\tau\}$-module. Then there is a finitely generated free $k\{\tau\}$-submodule $F \subset M$ such that M/F is a finite dimensional k-vector space.*

Proof. Let $m_1, \ldots, m_r \in M$ a minimal set of generators of the $k\{\tau\}$-module M. We make induction on r. If the m_i are a basis of M there is nothing to prove. In the other case there is a nontrivial relation

$$f_1 m_1 + \ldots + f_r m_r = 0, \quad f_i \in k\{\tau\}.$$

We may assume that $f_1 \neq 0$. Let $M_1 \subset M$ the submodule generated by $m_2, \ldots, m_r$. It is clear that M/M_1 is a finite dimensional k-vector space. By assumption of the induction we find a free $k\{\tau\}$-module $F \subset M_1$, such that M_1/F is finite dimensional. But then M/F is finite dimensional too. $\square$

Proposition 30. *Let R be a local ring such that the maximal ideal $\mathfrak{m}$ of R is nilpotent. Let M be a finitely generated $R\{\tau\}$-module which is free as R-module.*

Then there exists a free and finitely generated $R\{\tau\}$-submodule $F \subset M$, such that M/F is a finite and free R-module. We note that M is an $R\{\tau\}$-module of finite presentation.

Proof. We consider the $k\{\tau\}$-module

$$\bar{M} = k\{\tau\} \otimes_{R\{\tau\}} M = k \otimes_R M.$$

We may choose a free and finitely generated submodule $\bar{F} \in \bar{M}$ as in Lemma 29. Let $\bar{f}_1, \ldots, \bar{f}_d$ be a basis of the $k\{\tau\}$-module $\bar{F}$. By Lemma 29 there are elements $\bar{t}_1, \ldots, \bar{t}_e \in \bar{M}$ such that the elements

$$\tau^n \bar{f}_i,\ \bar{t}_j \quad \text{for } n \geq 0,\ i = 1, \ldots, d,\ j = 1, \ldots, e$$

are a basis of the k-vector space $\bar{M}$. We lift the elements $\bar{f}_1, \ldots, \bar{f}_d, \bar{t}_1, \ldots, \bar{t}_e$ arbitrarily to elements $f_1, \ldots, f_d, t_1, \ldots, t_e \in M$. By the Lemma of Nakayama the elements

$$\tau^n f_i,\ t_j \quad \text{for } n \geq 0,\ i = 1, \ldots, d,\ j = 1, \ldots, e$$

are a basis of the R-module M. Therefore the element $f_1, \ldots, f_d$ generate a free $R\{\tau\}$-submodule $F \subset M$ with the desired properties. $\square$

Proposition 31. *Let k be a perfect field of characteristic p. Let M be a finitely generated $k\{\tau\}$-module. Then M is isomorphic to a direct sum of $k\{\tau\}$-modules*

$$M = T \oplus k\{\tau\}^m$$

where T is finite dimensional as k-vector space and $m \geq 0$ is some integer.

Proof. Let $T \subset M$ be the torsion submodule. Then M/T has no torsion, i.e. there is no $k\{\tau\}$-submodule $\bar{N} \subset M/T$, $N \neq 0$ which is finite dimensional over k. Indeed, the preimage $N \subset M$ of $\bar{N}$ would be also finite dimensional and therefore $N = T$ by definition.

Therefore it suffices to show that a finitely generated torsion free $k\{\tau\}$-module M is free.

We will show that for torsion free $M \neq 0$

$$\operatorname{Hom}_{k\{\tau\}}(M, k\{\tau\}) \neq 0. \tag{37}$$

Note this is trivial if M is a submodule of a free module and therefore such modules M are free.

To show (37) we take by Lemma 29 a free $k\{\tau\}$-submodule $F \subset M$ such that M/F is a finite dimensional k-vector space. The group $\operatorname{Ext}^1_{k\{\tau\}}(M/F, k\{\tau\})$ has a $k\{\tau\}$-right module structure via the second factor and in particular a k-vector space.

The assertion (37) follows if we show that

$$\dim_k Ext^1_{k\{\tau\}}(M/F, k\{\tau\}) < \infty. \tag{38}$$

Indeed, from the exact sequence $0 \to F \to M \to M/F \to 0$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_{k\{\tau\}}(M, k\{\tau\}) \to \operatorname{Hom}_{k\{\tau\}}(F, k\{\tau\}) \to Ext^1_{k\{\tau\}}(M/F, k\{\tau\}),$$

because $k\{\tau\}$ contains no τ-invariant finite dimensional (left) k-vector subspace and therefore $\operatorname{Hom}_{k\{\tau\}}(M/F, k\{\tau\}) = 0$. Since M is torsionfree we have $U \neq 0$ and then $\operatorname{Hom}_{k\{\tau\}}(F, k\{\tau\})$ is an infinite dimensional k-vector space. The assertion (38) implies therefore (37).

Now we show (38). A monogen left $k\{\tau\}$-module which is torsion is of the form $k\{\tau\}/k\{\tau\}f$ for a polynomial $f = \tau^n + a_{n-1}\tau^{n-1} + \ldots + a_0 \neq 0$, $a_i \in k$. If suffices to show that $\operatorname{Ext}^1_{k\{\tau\}}(k\{\tau\}/k\{\tau\}f, k\{\tau\})$ is a finte dimensional k-vector space. We easily find an isomorphism of right $k\{\tau\}$-modules

$$\operatorname{Ext}^1_{k\{\tau\}}(k\{\tau\}/k\{\tau\}f, k\{\tau\}) \cong k\{\tau\}/fk\{\tau\}.$$

Because k is perfect the dimension of the last vector space in n. This proves (37).

Finally we show that the torsionfree M is free. Indeed, there exists a non-zero $k\{\tau\}$-module isomorphism $\varphi : M \to k\{\tau\}$. Since the image of φ is a principal ideal we obtain a decomposition

$$M = M' \oplus k\{\tau\}. \tag{39}$$

We note that $M/k\{\tau\}\tau M$ is a finite dimensional (left) k-vector space. From (39) we obtain

$$\dim_k M/k\{\tau\}\tau M = 1 + \dim_k M'/k\{\tau\}\tau M'.$$

We see that an induction applied to (39) shows that M is a free $k\{\tau\}$-module. The proposition follows. $\square$

Let R be a local ring such that the maximal ideal $\mathfrak{m}$ of R is nilpotent. Let $\mathbb{M}_R$ be the category of all finitely generated $R\{\tau\}$-modules M which are free as R-module as in Proposition 30.

Then we define a contravariant functor

$$D : \mathbb{M}_R \to \{\text{flat group schemes over } R\}. \tag{40}$$

For the definition of D_M we take a resolution

$$0 \to L_2 \to L_1 \to M \to 0 \tag{41}$$

where L_1 is a free finite $R\{\tau\}$-module. Then $k \otimes_R L_2$ is a free $k\{\tau\}$-module because it is a submodule of the free $k\{\tau\}$-module $k \otimes_R L_1$. Therefore L_2 is a free $R\{\tau\}$-module. Because two resolutions are homotopically equivalent we may define D_M as the kernel

$$0 \to D_M \to D_{L_1} \to D_{L_2}.$$

We claim that the last arrow is a surjection of sheaves. It is enough to show that the last arrow is a faithfully flat morphism. By EGA IV, Corollaire (11.3.11) we are reduced to the case where $R = k$ is a field. By base change we may assume that k is perfect. In this case we have by Proposition 31 that $M \cong T \oplus k\{\tau\}^m$. We choose a resolution $0 \to P_2 \to P_1 \to T \to 0$. Then

$$0 \to P_2 \to P_1 \oplus k\{\tau\}^m \to T \oplus k\{\tau\}^m \to 0$$

is a resolution for M and the map $D_{P_1} \oplus D_{k\{\tau\}^m} \to D_{P_2}$ is faithfully flat by Corollary **??** and in particular a surjection of sheaves. Hence for any other resolution (41) we have an surjection of sheaves. The argument of Corollary **??** shows then that $D_{L_1} \to D_{L_2}$ is a faithfully flat morphism.

Proposition 32. *Let R be a local ring such that the maximal ideal $\mathfrak{m}$ of R is nilpotent.*

Then the functor D (40) is fully faithful and exact. The coordinate module is a quasiinverse functor.

Let $M \in \mathbb{M}_R$, i.e. M is a finitely generated $R\{\tau\}$-module which is free as R-module. Let $L_1 \to M$ a surjection from a free finitely generated $R\{\tau\}$-module. Its kernel L_2 is a finitely generated free module. The induced sequence

$$0 \to D_M \to D_{L_1} \to D_{L_2} \to 0 \tag{42}$$

is an exact sequence of sheaves and the next-to-last arrow is a faithfully flat morphism of schemes. There is a natural isomorphism

$$M \to \operatorname{Hom}_{gs/R}(D_M, \mathbb{G}_{a,R}). \tag{43}$$

Proof. We already proved the assertions about (42). This implies that the functor D is eaxact. Indeed, let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of modules in $\mathbb{M}_R$. Then we find a diagram of $R\{\tau\}$-modules, where all modules named L are free and such that all rows and columns are exact:

$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_3 \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & L_{11} \arrow[r] \arrow[u] & L_{12} \arrow[r] \arrow[u] & L_{13} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & L_{21} \arrow[r] \arrow[u] & L_{22} \arrow[r] \arrow[u] & L_{23} \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

If we apply the functor D the exactness follows from (42).

From (**??**) and (42) one obtains a natural map

$$\kappa_M : M \to \mathrm{Hom}_{\mathrm{gs}}(D_M, \mathbb{G}_a)$$

which is injective, cf. (**??**). To prove that κ_M is an isomorphism we use an exact sequence

$$0 \to F \to M \to M/F \to 0$$

as in Proposition 30. We write $T = M/F$. Then we find a commutative diagram

$$\begin{tikzcd} 0 \to \mathrm{Hom}_{gs}(D_F, \mathbb{G}_a) \arrow[r] & \mathrm{Hom}_{gs}(D_M, \mathbb{G}_a) \arrow[r] & \mathrm{Hom}_{gs}(D_T, \mathbb{G}_a) \\ 0 \to F \arrow[r] \arrow[u, "\kappa_F"] & M \arrow[r] \arrow[u, "\kappa_M"'] & T \to 0 \arrow[u, "\kappa_{M/F}"'] \end{tikzcd}$$

Since T is a finitely generated free R-module we know by Proposition **??** that κ_T is an isomorphism. Since κ_F is also an isomorphism we conclude that κ_M is an isomorphism.

We now show that the functor D is fully faithful. Let G be a group scheme over R and let $N = \mathrm{Hom}_{\mathrm{gs}/R}(G, \mathbb{G}_{a,R})$ be the coordinate module. Clearly we have for the free module $F = R\{\tau\}$ that

$$\mathrm{Hom}_{\mathrm{gs}/R}(G, D_F) \cong \mathrm{Hom}_{R\{\tau\}}(F, N). \tag{44}$$

Therefore this equation holds for each free $R\{\tau\}$-module F. We consider the commutative diagram

$$\begin{tikzcd} 0 \to \mathrm{Hom}_{gs}(G, D_M) \arrow[r] & \mathrm{Hom}_{gs}(G, D_{L_1}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{gs}(G, D_{L_2}) \to 0 \arrow[d, "\cong"'] \\ 0 \to \mathrm{Hom}_{R\{\tau\}}(M, N) \arrow[r] & \mathrm{Hom}_{R\{\tau\}}(L_1, N) \arrow[r] & \mathrm{Hom}_{R\{\tau\}}(M, L_2) \to 0 \end{tikzcd}$$

Therefore we obtain a natural isomorphism

$$\mathrm{Hom}_{gs}(G, D_M) \xrightarrow{\sim} \mathrm{Hom}_{R\{\tau\}}(M, N) \tag{45}$$

for each group scheme G over R and each $M \in \mathbb{M}_R$. This shows the fully faithfulness and also that the coordinate module C is a quasiinverse of D. $\square$

Corollary 33. *Let R as in the Proposition. For $M \in \mathbb{M}_R$ the group scheme D_M has the property (BCM), cf. Definition 12. If G is a flat group which has the property (BCM) and such that $C_G \in \mathbb{M}_R$ then $G = C_M$ for $M = C_G$.*

Proof. For the first assertion we have to show that the coordinate module of $\operatorname{Spec} S \times_{\operatorname{Spec} R} D_M$ is $S \otimes_R M$. We consider the resolution ???. We see easily that $\operatorname{Spec} S \times_{\operatorname{Spec} R} D_{L_i} \cong D_{S \otimes_R L_i}$. Therefore the first assertion follows from the exact sequence

$$o \to S \otimes_R L_2 \to S \otimes_R L_1 \to S \otimes_R M \to 0.$$

$\square$

References

[BBM] P.Berthelot, L.Breen, W.Messing, *Théorie de Dieudonné II*, Springer Lecture Notes 930, 1982.

[DG] M.Demazure, P.Gabriel, *Groupes Algébriques*, Paris, Amsterdam 1970.

[I] L.Illusie, Complexe cotangent II, SLN 283.

[P] T.Poguntke, Group schemes with $\mathbb{F}_q$-action, Bulletin de la Société mathématique de France 145(2):345-380, (2017).

[SGA3] SGA 3 Exp. VII_B.

[Z] *Übungen zur Algebra, Übung 9*, zink*at*math.uni-bielefeld.de/∼zink